UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology

Winter term 2017/18

Sheet 5 Deadline: Wednesday, 29/11/2017, just before the lecture

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Special Exercise **(0 points)**

Please give anononymous feedback for the lecture and the exercise group on

www.math.uni-sb.de/ag/eschmeier/lehre/WS1718/top/evaluation.html

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Exercise 20 **(1+1+2+2*=4 + 2* points)**

Let X be a non-empty set and let

$$\tau = \{U \subset X;\ U = \varnothing \text{ or } X \setminus U \text{ finite}\}$$

be the topology from Exercise 16.

(a) Let A be a subset of X. What is the closure of A with respect to τ?

(b) Which sets are compact in (X, τ)?

(c) Show that a function $f : (X, \tau) \to (X, \tau)$ is continuous if and only if it is constant or $f^{-1}(\{x\})$ is finite for all $x \in X$.

(d) * Let $(x_n)_n$ be a sequence in X and let

$$\Gamma = \{x \in X;\ x = x_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

Show: If $\Gamma = \varnothing$ the sequence $(x_n)_n$ converges to every point $x \in X$. If Γ consists of exactly one point $x \in X$ then the sequence $(x_n)_n$ converges (only) to this point x. In any other case, the sequence $(x_n)_n$ does not converge.

Exercise 21 **(3 points)**

Let $X \neq \varnothing$ be a set. Show that a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X is the basis of a topology τ on X if and only if the following conditions hold:

(i) For every $x \in X$ there is $B \in \mathcal{B}$ with $x \in B$.

(ii) For every $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$ there is $B_0 \in \mathcal{B}$ such that $x \in B_0 \subset B_1 \cap B_2$.

Show also that in this case, the following holds:

$$\begin{aligned} \tau &= \{U \subset X;\ \forall x \in U\ \exists B \in \mathcal{B} \text{ with } x \in B \subset U\} \\ &= \bigcap_{t \text{ topology on } X \text{ with } \mathcal{B} \subseteq t} t. \end{aligned}$$

(bitte wenden)

Exercise 22 **(1,5+1,5+1 = 4 points)**

(a) Let $X \neq \emptyset$ be a set and let τ_1, τ_2 be topologies on X. Show:

(i) If $\mathcal{B}_1, \mathcal{B}_2$ are bases of τ_1 respectively τ_2, we have:

$$\tau_1 \subseteq \tau_2 \Leftrightarrow \forall B_1 \in \mathcal{B}_1 : \forall x \in B_1 : \exists B_2 \in \mathcal{B}_2 : x \in B_2 \subseteq B_1.$$

(ii) If $\mathcal{S}_1, \mathcal{S}_2$ are subbases of τ_1 respectively τ_2, we have:

$$\tau_1 \subseteq \tau_2 \Leftrightarrow \forall S \in \mathcal{S}_1 : \forall x \in S : \exists S_1, ..., S_n \in \mathcal{S}_2 : x \in \bigcap_{i=1}^{n} S_i \subseteq S.$$

(b) Let (X, τ_1), (Y, τ_2) be topological spaces, $\mathcal{S}_2$ a subbase of τ_2 and $f : X \to Y$ a function. Show that f is continuous if and only if $f^{-1}(S) \in \tau_1$ holds for all $S \in \mathcal{S}_2$.

Exercise 23 **(1+1+1+1+1= 5 points)**

Let

$$\mathfrak{B} = \{[a, b);\ a, b \in \mathbb{R} \text{ mit } a < b\}$$

be the collection of half-open intervals. Show:

(a) $\mathfrak{B}$ is a basis of a Hausdorff topology τ on $\mathbb{R}$.

(b) We have $\tau_{|\cdot|} \subsetneq \tau$.

(c) A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converges with respect to τ to an $x \in \mathbb{R}$, if and only if it converges in $(\mathbb{R}, \tau_{|\cdot|})$ to x and we additionally have $x_n \geq x$ for almost every $n \in \mathbb{N}$.

(d) $(\mathbb{R}, \tau)$ is separable and first countable.

(e) $(\mathbb{R}, \tau)$ is not metrizable (i.e. there is no metric d on $\mathbb{R}$ such that $\tau = \tau_d$).

Exercise 24* **(2*+1* + 1* = 4* points)**

For $a, b, x \in \mathbb{Z}$, we write as usual

$$x \equiv b \bmod a,$$

if there is $k \in \mathbb{Z}$ such that $x - b = ka$. For all $b \in \mathbb{Z}$, $a \in \mathbb{Z}\backslash\{0\}$ we set

$$V(a, b) = \{x \in \mathbb{Z}, x \equiv b \bmod a\}.$$

(a) Show that

$$\mathcal{B} = \{V(a, b); b \in \mathbb{Z}, a \in \mathbb{Z}\backslash\{0\}\}$$

is a base of a topology τ on $\mathbb{Z}$.

(b) Show that the sets $V(a, b)$ $(b \in \mathbb{Z}$, $a \in \mathbb{Z}\backslash\{0\})$ are closed in $(\mathbb{Z}, \tau)$.

(c) Use $a)$ and $b)$ to show that there are infinitely many prime numbers.