UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology

Winter term 2017/18

Sheet 6 Deadline: Wednesday, 6/12/2017, just before the lecture

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***Reminder**: A topological space (X, τ) is called sequentially compact if every sequence in X has a convergent subsequence.*

Exercise 25 (2 + 2 = 4 points)

(Sequentially compact metric spaces are separable)

Let (X, d) be a sequentially compact metric space. Show:

(a) For all $\epsilon > 0$, there are $x_1, ...x_n \in X$ such that $X = B_\epsilon(x_1) \cup ... \cup B_\epsilon(x_n)$.

(b) (X, d) is separable.
(*Hint: Use part a) for all $k \in \mathbb{N}^*$ with $\epsilon = \frac{1}{k}$.*)

Excercise 26 (2+1+1 = 4 points)

(Definition and elementary properties of the final topology)

Let $X \neq \emptyset$ be a set and I an arbitrary index set. For $i \in I$, let (X_i, t_i) be a topological space and $f_i : X_i \to X$ a function. Show:

(a) The collection of sets
$$\tau = \{U \subset X;\ f_i^{-1}(U) \in t_i \text{ for all } i \in I\}$$
defines a topology on X.
(This topology is called the final topology generated by the f_i ($i \in I$).)

(b) τ is the finest topology on X with respect to which the functions $f_i : (X_i, t_i) \to (X, \tau)$ are continuous for all $i \in I$.

(c) If (Y, t) is a further topological space, a function $g : (X, \tau) \to (Y, t)$ is continuous if and only if the function $g \circ f_i : (X_i, t_i) \to (Y, t)$ is continuous for all $i \in I$.

(please turn the page)

Exercise 27 **(1+2+1+2* = 4 + 2* points)**

(Is it possible to define the product topology in an easier way?)

Let $(X_i, \tau_i)_{i \in I}$ be a family of topological spaces and let $X = \prod_{i \in I} X_i$. Furthermore, let

$$\mathcal{B} = \{\prod_{i \in I} U_i; U_i \subseteq X_i \text{ open for all} i \in I\}$$

be the collection of products of open sets. Show:

(a) $\mathcal{B}$ is base of a topology τ on X.

(b) * Now, assume in addition that the spaces (X_i, τ_i) $(i \in I)$ are Hausdorff. A sequence $(x^{(n)})_{n \in \mathbb{N}} = ((x_i^{(n)})_{i \in I})_{n \in \mathbb{N}}$ converges in (X, τ) to an $x = (x_i)_{i \in I} \in X$ if and only if $x_i^{(n)} \xrightarrow{n \to \infty} x_i$ in (X_i, τ_i) for all $i \in I$ and there is a finite set $J \subseteq I$ and an $N \in \mathbb{N}$ such that $x_i^{(n)} = x_i^{(m)}$ for all $i \in I \backslash J$ and all $n, m \geq N$.

(c) Now, let in particular $I = \mathbb{N}$ und $(X_i, \tau_i) = (\mathbb{R}, \tau_{|\cdot|})$ for all $i \in \mathbb{N}$. Then,

$$f : \mathbb{R} \to \prod_{i \in \mathbb{N}} \mathbb{R}, x \mapsto (x)_{i \in \mathbb{N}}$$

is not continuous.

(d) Now, let in particular $I = \mathbb{N}$ and $(X_i, \tau_i) = (\{0, 1\}, \mathcal{P}(\{0, 1\}))$ for all $i \in \mathbb{N}$. Then, the sequence $(x^{(n)})_{n \in \mathbb{N}} = ((x_m^{(n)})_{m \in \mathbb{N}})_{n \in \mathbb{N}}$ in X defined by

$$x_m^{(n)} = \begin{cases} 0, & \text{, falls } m < n \\ 1 & \text{, falls } m \geq n \end{cases}$$

for all $n, m \in \mathbb{N}$ does not have a convergent subsequence.

Exercise 28 **(1+1,5+1,5= 4 points)**

(Compatibility of products with closures and interiors)

Let (X_i, τ_i) be topological spaces $(i \in I)$ and let $X = \prod_{i \in I} X_i$ be equipped with the product topology τ. For $i \in I$, let $A_i \subset X_i$. Show:

(a) If all $A_i \subset X_i$ $(i \in I)$ are closed, $\prod_{i \in I} A_i \subseteq X$ is also closed.

(b) We have $\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}$.

(c) We have $\operatorname{Int}(\prod_{i \in I} A_i) \subset \prod_{i \in I} \operatorname{Int}(A_i)$. Does equality hold in general? (Find a proof or provide a counterexample)

Exercise 29* **(3* points)**

(Relative topologies of separable topologies need not be separable)

Let σ be the topology generated on $\mathbb{R}$ by the half-open intervals $[a, b)$ $(a, b \in \mathbb{R},\ a < b)$ (cf. exercise 23) and let

$$(\mathbb{R}^2, \tau) = (\mathbb{R}, \sigma) \times (\mathbb{R}, \sigma)$$

be the topological product (i.e. $\mathbb{R}^2$ equipped with the product topology). Show that $(\mathbb{R}^2, \tau)$ is a separable topological space while the set

$$D = \{(x, -x);\ x \in \mathbb{R}\} \subset \mathbb{R}^2$$

equipped with the relative toplogy $\tau|_D$ is not separable.
