UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology

Winter term 2017/18

Sheet 7 Deadline: Wednesday, 13/12/2017, just before the lecture

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Definition: *A function $f : X \to Y$ between topological spaces is called open (closed) if $f(N) \subseteq Y$ is open (closed) for every $M \subseteq X$ open (closed).*

Exercise 30 (2+2=4 points)

(Hausdorff proberty is conserved by products; projections are open)

Let (X_i, τ_i) $(i \in I)$ be topological spaces and let $X = \prod_{i\in I} X_i$ be equipped with the product topology τ. Show:

(a) (X, t) is Hausdorff if and only if all the spaces (X_i, t_i) $(i \in I)$ are Hausdorff.

(b) The canonical projections

$$\pi_j : X \to X_j,\ (x_i)_{i\in I} \mapsto x_j \quad (j \in I)$$

are open, but not closed in general.

Definition: *Let $X \neq \emptyset$. A set $\mathcal{F} \subseteq \mathcal{P}(X)$ is called filter if*

(i) $\emptyset \notin \mathcal{F}, X \in \mathcal{F}$,

(ii) $F_1, F_2 \in \mathcal{F} \implies F_1 \cap F_2 \in \mathcal{F}$,

(iii) $F \in \mathcal{F}, F' \in \mathcal{P}(X)$ *with* $F \subseteq F' \implies F' \in \mathcal{F}$.

A subset $\mathcal{F}_0 \subseteq \mathcal{F}$ is called filter base for $\mathcal{F}$, if for all $F \in \mathcal{F}$, there is a $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq F$.

Definition: *Let (X, τ) be a topological space, $\mathcal{F} \subseteq \mathcal{P}(X)$ a filter, $x \in X$.*
We call $\mathcal{F}$ convergent to x if $\mathcal{U}(x) \subseteq \mathcal{F}$ holds. In that case, we write $\mathcal{F} \to x$ and call x limit of $\mathcal{F}$. We call x cluster point of $\mathcal{F}$ if $x \in \bigcap_{F\in\mathcal{F}} \overline{F}$ holds.

Exercise 31 (1 + 1 + 1 + 1 = 4 points)

(Filters can also be used to describe topological concepts)

Let (X, τ) be a topological space. Show:

(a) A collection of sets $\emptyset \neq \mathcal{F}_0 \subseteq \mathcal{P}(X)$ with $\emptyset \notin \mathcal{F}_0$ is a filter base of a filter $\mathcal{F} \subseteq \mathcal{P}(X)$ if and only if for all $F_1, F_2 \in \mathcal{F}_0$, there is $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq F_1 \cap F_2$ holds. In that case,

$$\mathcal{F} = \{F \subseteq X; \exists F_0 \in \mathcal{F}_0 : F_0 \subseteq F\} = \bigcap_{\mathcal{F}' \supseteq \mathcal{F}_0 \text{ filter}} \mathcal{F}'.$$

(please turn the page)

Now, let in addition $\mathcal{F} \subseteq \mathcal{P}(X)$ be a filter, $x \in X$. Show:

(b) x is a cluster point of $\mathcal{F}$ if and only if there is a filter $\mathcal{G} \subseteq \mathcal{P}(X)$ with $\mathcal{F} \subseteq \mathcal{G}$ such that $\mathcal{G} \to x$.

(c) For $A \subseteq X$, we have

$$\overline{A} = \{x \in X, \text{ there is a filter} \mathcal{F} \subseteq \mathcal{P}(X) \text{ with } A \in \mathcal{F} \text{ such that } \mathcal{F} \to x\}.$$

Now, let (Y, t) be another topological space and let $f : X \to Y$ be a function. Let $f(\mathcal{F}) \subseteq \mathcal{P}(X)$ be the filter with filter base $\{f(F), F \in \mathcal{F}\}$. Show:

(d) f is continuous in x if and only if: For every filter $\mathcal{F} \subseteq \mathcal{P}(X)$ with $\mathcal{F} \to x$, we have $f(\mathcal{F}) \to f(x)$.

Notation: For a net $(x_\alpha)_{\alpha \in A}$ in a topological space (X, τ) and $\alpha \in A$, we write $\alpha^+ = \{x_\beta, \beta \geq \alpha\}$.

Exercise 32 **(0,5+0,5+1*+1*+3 = 4+2* points)**

(Connection between filters and nets)

Let (X, τ) be a topological space. Show:

(a) Let $\mathbf{x} = (x_\alpha)_{\alpha \in A}$ be a net in A. Then, $\mathrm{Filt}_0(\mathbf{x}) = \{\alpha^+, \alpha \in A\}$ is a filter base of a filter $\mathrm{Filt}(\mathbf{x}) \subseteq \mathcal{P}(X)$.

(b) Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a filter. For every $F \in \mathcal{F}$, choose $x_F \in F$. Then, $(x_F)_{F \in \mathcal{F}}$ is a net where $\mathcal{F}$ is quasi-ordered by the relation

$$F_1 \leq F_2 :\Leftrightarrow F_1 \supseteq F_2 \quad (F_1, F_2 \in \mathcal{F}).$$

We call such a net a net associated with $\mathcal{F}$.
With $A(\mathcal{F}) = \{(x, F) \in X \times \mathcal{F}, x \in F\}$ quasi-ordered by the relation

$$(x_1, F_1) \leq (x_2, F_2) :\Leftrightarrow F_1 \supseteq F_2 \quad ((x_1, F_1), (x_2, F_2) \in A(\mathcal{F})),$$

the family/function $A(\mathcal{F}) \to X, (x, F) \mapsto x$ is also a net. We write $\mathrm{Net}(\mathcal{F})$ for this net $(x)_{(x,F) \in A(\mathcal{F})}$.

(c) * For every filter $\mathcal{F} \subseteq \mathcal{P}(X)$, we have $\mathrm{Filt}(\mathrm{Net}(\mathcal{F})) = \mathcal{F}$.

(d) * For a net $\mathbf{x} = (x_\alpha)_{\alpha \in A}$ in X, the equality $\mathrm{Net}(\mathrm{Filt}(\mathbf{x})) = \mathbf{x}$ does not need to hold.
(*Hint: Use that for each set X, we have $|\mathcal{P}(\mathcal{P}(X) \times X)| > |\mathcal{P}(X) \times X|$.*)

(e) Let $\mathbf{x} = (x_\alpha)_{\alpha \in A}$ be a net in X, $\mathcal{F} \subseteq \mathcal{P}(X)$ a filter and $x_0 \in X$. Show:

(i) We have $\mathcal{F} \to x_0$ if and only if $\mathrm{Net}(\mathcal{F})$ converges to x_0. This also holds if and only if every net associated with $\mathcal{F}$ converges to x_0.

(ii) We have $x_\alpha \xrightarrow{\alpha} x_0$ if and only if $\mathrm{Filt}(\mathbf{x}) \to x_0$ holds.

(iii) For a subnet $\mathbf{y} = (y_i)_{i \in I}$ of $\mathbf{x}$, we have $\mathrm{Filt}(\mathbf{x}) \subseteq \mathrm{Filt}(\mathbf{y})$.

(please turn the page)

Definition: *Let (X, τ) be a topological space and let $(x_\alpha)_{\alpha\in A}$ be a net in X. We call a net $(y_i)_{i\in I}$ in X a cofinal subset of $(x_\alpha)_{\alpha\in A}$ if $I \subseteq A$, $y_i = x_i$ holds for all $i \in I$ and for every $\alpha \in A$, there is $i \in I$ such that $i \geq \alpha$.*

Exercise 33 (0,5+ 1,5 + 1* + 2* + 2 + 2* + 1* = 4 + 6* points)

(Is it possible to define subnets more easily?)

Let (X, τ) be a topological space.

(a) Show that every cofinal subnet of a net is also a subnet of the same net.

(b) Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X. Show that there is a cofinal subnet $\mathbf{y} = (y_i)_{i\in I}$ of $\mathbf{x}$ such that $\mathrm{Filt}((x_{n_k})_{k\in\mathbb{N}}) = \mathrm{Filt}(\mathbf{y})$ holds.

(c) * Let $(y_i)_{i\in I}$ be a cofinal subnet of $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$. Show that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $\mathbf{x}$ such that $\mathrm{Filt}((x_{n_k})_{k\in\mathbb{N}}) = \mathrm{Filt}(\mathbf{y})$ holds.

(d) * Find a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}, \tau_{|\cdot|})$ and a subnet $\mathbf{y}$ such that there is no subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which fulfills $\mathrm{Filt}((x_{n_k})_{k\in\mathbb{N}}) = \mathrm{Filt}(\mathbf{y})$.

(e) Let $\mathbf{x} = (x_\alpha)_{\alpha\in A}$ be a net in X, $x_0 \in X$. Show that x_0 is a cluster point of $\mathbf{x}$ if and only if there is a subnet $\mathbf{y}$ of $\mathbf{x}$ which converges to x_0.

(f) * Now, let $(X, \tau) = (l^1, \tau_{\|\cdot\|_1})$ and $(l^1)' = \{u : l^1 \to \mathbb{C}, u \text{ is continuously linear}\}$ its dual space. Define τ_w on l^1 as the weak topology generated by the functions $u : l^1 \to \mathbb{C}$ ($u \in (l^1)'$). You may assume that l^1 has the Schur proberty, i.e. that a sequence $(x^{(n)})_{n\in\mathbb{N}}$ in l^1 converges in τ_w to $x \in l^1$ if and only if it converges in $\tau_{\|\cdot\|_1}$ to x. Use this to deduce that there is a sequence $(x^{(n)})_{n\in\mathbb{N}}$ in (l^1, τ_w) which has 0 as a cluster point, but which does not have a subsequence converging to 0.

(*Hint: Solving this exercise is considerably easier if you have some experience with weak topologies from functional analysis.*)

(g) * Show that there is a topological space (Y, t) and a net $\mathbf{x} = (x_\alpha)_{\alpha\in A}$ in Y such that there is a cluster point x_0 of $\mathbf{x}$ for which there is no cofinal subnet $(y_i)_{i\in I}$ of $\mathbf{x}$ such that $y_i \xrightarrow{i} x_0$.