UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Übungen zur Vorlesung Topologie

Wintersemester 2017/18

Blatt 9 Abgabetermin: Wednesday, 3/1/2018, just before the lecture

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Exercise 39 (4 points)

(Uniform limits of nets of continuous functions are continuous)

Let (X, τ) be a topological space, (Y, d) a metric space and $(f_\alpha)_{\alpha\in A}$ a net in

$$C(X,Y) = \{f : X \to Y; f \text{ is continuous }\}.$$

Show: If $(f_\alpha)_{\alpha\in A}$ converges uniformly to a function $f : X \to Y$, i.e. if

$$\sup_{x\in X} d(f(x), f_\alpha(x)) \xrightarrow{\alpha} 0$$

holds, f is continuous

For topological spaces X, Y, a continuous mapping $f : X \to Y$ is called proper*, if the preimage $f^{-1}(K) \subset X$ of every comapct set $K \subset Y$ is also compact.*

Exercise 40 (3+3+2+2*=8+2* points)

(Connection of the properties 'proper', 'closed', 'open', 'continuous')

(a) Find a subset $A \subseteq \mathbb{R}$ and a function $f : (A, \tau_{|\cdot|}) \to (\mathbb{R}, \tau_{|\cdot|})$ such that:

  (i) f is continuous, but not proper

  (ii) We have that $f^{-1}(K)$ is compact for all compact sets $K \subseteq \mathbb{R}$, but f is not proper,

  (iii) f is closed, but the preimage $f^{-1}(K)$ is not compact for all compact sets $K \subseteq \mathbb{R}$.

Now, let X, Y be locally compact Hausdorff spaces with one-point compactifications $\hat{X} = X \cup \{\infty\}$ and $\hat{Y} = Y \cup \{\infty\}$.

(b) Let $f : X \to Y$ be continuous. Show that f is proper if and only if its extension $\hat{f} : \hat{X} \to \hat{Y}$ given by

$$\hat{f}(x) = \begin{cases} f(x) & \text{, if } x \in X \\ \infty & \text{, if } x = \infty \end{cases}$$

is continuous.

(c) Show that every proper function $f : X \to Y$ is closed.

(d) * Find a mapping between topological spaces which is proper, but not closed.

(please turn the page)

A topological space is called σ-compact if it is a countable union of compact sets.

Exercise 41 (3+1 = 4 points)

(Compact exhaustion of σ-compact spaces)

Let X be a locally compact space with one-point compactification $\hat{X} = X \cup \{\infty\}$. Show:

(a) X is σ-compact if there is a sequence $(U_n)_{n\in\mathbb{N}}$ of sets open in X such that

  (i) $\overline{U_n}$ is compact for all $n \in \mathbb{N}$.

  (ii) $\overline{U_n} \subset U_{n+1}$ holds for all $n \in \mathbb{N}$.

  (iii) $X = \bigcup_{n\in\mathbb{N}} U_n$.

  (*Hint: Corollary 7.13*)

(b) X is σ-compact if and only if ∞ has a countable neighbourhood base in $\hat{X}$.
