UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Additional notes supplementing the exercises for the lecture topology

Winter term 2017/18

Sheet 6 Deadline: -

Exercise 25

A metric space which fulfills the condition

$$\forall \epsilon > 0 : \exists n \in \mathbb{N} : \exists x_1, ..., x_n \in X : X = B_\epsilon(x_1) \cup ... \cup B_\epsilon(x_n)$$

is called totally bounded. One can show that a metric space is compact (and thus sequentially compact) if and only if it is totally bounded and complete.

We use this exercise in the proof of Theorem 7.2, thus we should not use that (X, d) is compact to solve this problem.

Exercise 26

A name for the weak topology (defined in the lecture) which sounds closer to the dual situation with respect to the final topology defined here is 'initial topology'.

Note that the coarsest topology on X which makes all the f_i $(i \in I)$ continuous is simply $\{\emptyset, X\}$ and thus not very interesting.

Basic and useful examples of final topologies include the quotient topology (cf. Exercise 35), the final topology induced by the inclusion mappings $X_i \hookrightarrow \sqcup_{i\in I} X_i, x \mapsto (x, i)$ to the disjoint union $\sqcup_{i\in I} X_i = \cup_{i\in I}\{(x, i), x \in X_i\}$ as well as direct limits of topological spaces.

Exercise 27

As is implied by the caption, the box topology is quite pathological and is used mainly for obscure examples. This is my main motivation in posing this exercise since it shows that the intuitively easier definition of a topology on a product introduced here is not convenient at all.

It is easy to show that the product topology equals the box topology if and only if the product is finite. Note that the product topology is always coarser almost by definition.

Part d) intends to show that no version of Tychonoff's Theorem holds for the box topology (not even in the countable case). Note that all the topological spaces appearing there are metrizable.

Exercise 28

Note that our proofs of a) and b) hold also for the box topology and that in c), even equality holds for the box topology.

Also note that unions are not compatible with products!

In our proof of b), the axiom of choice was vital. Actually, the statement in b) is even equivalent to the axiom of choice in ZF.

Exercise 29

$(\mathbb{R}^2, \sigma)$ is called the Sorgenfrey plane. It is used for more counterexamples, cf. the corresponding Wikipedia article.

Note that $D \subseteq (\mathbb{R}^2, \sigma)$ is closed (use e.g. nets) and thus separability is not even inherited by closed subsets (like e.g. completeness or compactness).