**UNIVERSITÄT DES SAARLANDES**
**FACHRICHTUNG 6.1 – MATHEMATIK**
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**Additional notes supplementing the exercises for the lecture topology**
Winter term 2017/18

**Sheet 7** Deadline: -

## Exercise 30

Some authors (especially in older French texts) use the notion 'compact' for spaces which we call 'compact and Hausdorff'. Part a) of this exercise shows that Tychonoff's Theorem remains valid with this definition of compactness. However, this version (i.e. products of compact Hausdorff spaces are compact Hausdorff spaces) is not equivalent to the axiom of choice anymore (it is weaker).

## Exercise 31

Part b) shows that superfilters of a given filter $\mathcal{F}$ (i.e. filters $\mathcal{G}$ with $\mathcal{G} \supseteq \mathcal{F}$) should be seen as analogues to subnets/subsequences.

Some intuitive background for the definition of filters can be given by interpreting filters as 'locating schemes', cf. the English wikipedia article *Filter(mathematics)*.

## Exercise 32

In part b), the main exercise is to understand what you have to show. Note that the index set $A(\mathcal{F})$ is not partially ordered. This is the reason for which I allowed index sets of nets to be quasi-ordered (many authors demand that they are partially ordered).

For me, the problem in part d) is caused by the fact that we allow index sets which are 'too large' in some sense as index sets of nets.

In part e), I could also have added the additional exercise '$x_0$ is a cluster point of $\mathbf{x}$ if and only if it is a cluster point of $\mathrm{Filt}(\mathbf{x})$' and vice versa for $\mathcal{F}$, $\mathrm{Net}(\mathcal{F})$. Solving such an exercise would also provide further motivation for the definition of a cluster point of a filter which might look peculiar at first.

However, part e), iii) cannot be appended by the inverse implication. For me, the reason for this is that the usual definition of subnets is not the right one, at least not in a context where filters are involved. Also compare Exercsie 34.

**Exercise 33**

This exercise examines whether it is really neccessary to define subnets in such a general way as done in the lecture, in paricular if it is neccessary to allow for arbitrary index sets, even ones that are bigger than the index set of the original net.

While the first results (Parts b) and c)) look promising at first glance (they show that cofinal subnets of subsequences are 'essentially' just the subsequences), even those results prove rather dubious on closer examination. For, if subents of sequences where 'essentially' just the subsequences, every compact topological space would be seqeuentially compact. This is not true as can be seen in Exercise 37.

The further situation is even worse: One of the most important features of subsequences is that their limits are exactly the cluster points of the original sequences. Parts f) and g) show that this need not be true for the easier but impractical definition of subnets provided here. It fails even in the rather concrete setting of $l^1$ (Other, more abstract settings where it fails include the long line (cf. Wikipedia) whose definition involves a theory of cardinal numberes).

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