UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Additional notes supplementing the exercises for the lecture topology
Winter term 2017/18

Sheet 9 Deadline: -

Exercise 39

This is a quite general version of the corresponding result in Analysis I. It can be generalised even further if one knows about uniform spaces (which generalize metric spaces).

Note that

$$\tilde{d}(f,g) = \sup_{x\in X} d(f(x), g(x)) \quad (f, g \in C(X,Y))$$

does not define a metric on $C(X,Y)$ since it need not be finite for all $f, g \in C(X,Y)$.

Exercise 40

Note that no implications hold between the properties 'preimage of compact sets is compact', 'closed', 'open', 'continuous':
preimage of compact sets compact $\not\Rightarrow$ closed: cf. part d)
preimage of compact sets compact $\not\Rightarrow$ open: cf. part d)
preimage of compact sets compact $\not\Rightarrow$ open: cf. part a) ii)
closed $\not\Rightarrow$ preimage of compact sets compact: cf. part a) iii)
closed $\not\Rightarrow$ open: $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$, since $f(\mathbb{R}) = [0,\infty)$, but $\sqrt{\cdot} : [0,\infty), x \mapsto \sqrt{x}$ is a continuous inverse function of $\tilde{f} : \mathbb{R} \to [0,\infty), x \mapsto f(x)$.
closed $\not\Rightarrow$ continuous: $f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 0, & \text{falls } x < 0 \\ 1, & \text{falls } x \geq 0 \end{cases}$, since $f(A)$ is finite, hence compact in $\mathbb{R}$ for all $A \subseteq \mathbb{R}$.
open $\not\Rightarrow$ preimage of compact sets compact: $\pi_1 : \mathbb{R}^2 \to R, (x,y) \mapsto x$, since $\pi_1^{-1}(\{0\}) = \{0\}\times\mathbb{R}$.
open $\not\Rightarrow$ closed $\pi_1 : \mathbb{R}^2 \to R, (x,y) \mapsto x$, since $\pi_1(\{(x,y) \in \mathbb{R}^2; xy = 1\}) = \mathbb{R}\backslash\{0\}$.
open $\not\Rightarrow$ continuous: $f : S_1 = \{(\cos(x), \sin(x)) \in \mathbb{R}^2, x \in (-\pi,\pi]\} \to]-\pi,\pi], (\cos(x), \sin(x)) \mapsto x$, which is open since its inverse function is continuous, but not continuous in $(-1,0)$ since $(\cos(-\pi+\frac{1}{n}), \sin(-\pi+\frac{1}{n})) \xrightarrow{n\to\infty} (-1,0) = (\cos(\pi), \sin(\pi))$, but $-\pi + \frac{1}{n} \xrightarrow{n\to\infty} -\pi \neq \pi$.
continuous $\not\Rightarrow$ preimage of compact sets is compact: constant functions $\mathbb{R} \to \mathbb{R}$.
continuous $\not\Rightarrow$ closed: $f : \mathbb{R} \to \mathbb{R}, x \mapsto e^{-x}$ since $f([0,\infty)) = [0,1)$.
contunuous $\not\Rightarrow$ open: $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ since $f((-1,1)) = [0,1)$.
I tried to give counterexamples which are as basic, but also as non-pathological as possible. Hence only very few apperances of discrete /indiscrete topology which provide much more counterexamples.

(please turn the page)

Exercise 36

Note that $(\mathbb{R}, \tau_{|\cdot|})$ and $(\mathbb{C}, \tau_{|\cdot|})$ are σ-compact and sequences $(U_n)_{n\in\mathbb{N}}$ as in part b) are often used in applications.