Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 1 Deadline: 10.22.2019

Remark: Please deliver your solution of the exercise sheet on Tuesday just before the lecture.

Exercise 1 (1+2+2=5 Points)

Let (X, d) be a metric space and let $p \in X$ be an element. For $x, y \in X$ we define

$$\tilde{d}(x,y) = \begin{cases} d(x,p) + d(p,y), & x \neq y \\ 0, & x = y \end{cases}$$

(a) Show that $\tilde{d}$ defines an additional metric on X.

(b) Calculate for $x \in X$ and $\varepsilon > 0$ the (open) ε-ball around x corresponding to $\tilde{d}$. Hereby distinguish the cases $\varepsilon \leq d(x,p)$ and $\varepsilon > d(x,p)$.

(c) Characterize the open sets corresponding to $\tilde{d}$.

For a metric space (X, d) and a nonempty subset $A \subset X$ we denote by

$$d(x, A) = \inf\{d(x,y);\, y \in A\}$$

the **distance** *between the element $x \in X$ and the set A.*

Exercise 2 (2+1+1+2=6 Points)

Let (X, d) be a be a metric space and $\emptyset \neq A \subset X$.

(a) Show that the equation

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

holds for all $x, y \in X$.

(b) Show that the function

$$d_A : X \to \mathbb{R}, \quad x \mapsto d(x,A)$$

is continuous.

(c) Show that we have $d(x,A) = 0$ if and only if x is an element of $\overline{A}$.

(d) Let $A_1, A_2 \subset X$ be closed and disjoint sets. Show that there exists a continuous function $f : X \to [0,1]$ with $f|_{A_1} \equiv 0$ and $f|_{A_2} \equiv 1$. (*Hint : Use (b) and (c).*)

(please turn over)

A metric space (X, d) is called **complete**, *if every Cauchy sequence in X converges.*

Exercise 3 **(1+2+2=5 Points)**

Consider the following metric on the real line $\mathbb{R}$:

$$d(x,y) = |x-y| \quad \text{und} \quad \tilde{d}(x,y) = |\arctan(x) - \arctan(y)|.$$

(a) Prove that $\tilde{d}$ defines a metric on $\mathbb{R}$.

(b) Show that $(\mathbb{R}, d)$ and $(\mathbb{R}, \tilde{d})$ have the same open sets.

(c) Show that $(\mathbb{R}, \tilde{d})$ is not complete.

Exercise 4 **(2+2*=4 Points)**

Let $((X_n, d_n))_{n\in\mathbb{N}}$ be a sequence of metric spaces and let $X = \prod_{n=0}^{\infty} X_n$ be the Cartesian product of the sets X_n.

(a) Prove that

$$d : X \times X \to [0, \infty), \quad ((x_n)_n, (y_n)_n) \mapsto \sum_{n=0}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

defines a metric on X.

(b*) Show that the metric space (X, d) is complete if and only if all (X_n, d_n) are complete.

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html