Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 10 Deadline: 01.07.2019

Exercise 37 (4 Points)

For $i \in I$, let (X_i, t_i) be a topological space and let $\emptyset \neq Y_i \subset X_i$ be subsets. Let t be the product topology on $X = \prod_{i \in I} X_i$. Show that the relative topology $t|_Y$ of t on $Y = \prod_{i \in I} Y_i \subset X$ coincides with the product topology of the relative topologies $t_i|_{Y_i}$.

Exercise 38 (1+1+2=4 Points)

(a) Show that subspaces of topological Hausdorff spaces (equipped with the relative topology) are always again Hausdorff spaces.

Let now X be a locally compact Hausdorff space and let the subset $\emptyset \neq Y \subset X$ be equipped with the relative topology. Show:

(b) If Y is open or closed, then Y is locally compact.

(c) The topological space Y is locally compact if and only if there is an open subset $U \subset X$ and a closed subset $F \subset X$ such that $Y = U \cap F$ holds. (*Hint : If Y is a locally compact topological space, one can show that $Y \subset \overline{Y}$ is open.*)

For locally compact Hausdorff spaces X, Y we call $f : X \to Y$ a proper map if and only if for every compact set $K \subset Y$ the preimage $f^{-1}(K) \subset X$ is also compact.

Exercise 39 (2+2=4 Points)

Let X, Y be locally compact Hausdorff spaces with one-point compactifications $\hat{X} = X \cup \{\infty\}$ and $\hat{Y} = Y \cup \{\infty\}$.

(a) Let $f : X \to Y$ be a continuous map. Show that f is proper if and only if the continuation $\hat{f} : \hat{X} \to \hat{Y}$ of f defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{,falls } x \in X \\ \infty & \text{,falls } x = \infty \end{cases}$$

is continuous.

(b) Show that every proper map $f : X \to Y$ is closed.

(please turn over)

A locally compact Hausdorff space is called countable at infinity if it is the union of countable many compact subsets.

Exercise 40* (3*+1*=4* Points)

Let X be a locally compact Hausdorff space with one-point compactification $\hat{X} = X \cup \{\infty\}$. Show:

(a) X is countable at infinity if and only if there is a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets in X such that:

  (i) $\overline{U_n}$ is compact for all $n \in \mathbb{N}$.

  (ii) $\overline{U_n} \subset U_{n+1}$ for all $n \in \mathbb{N}$.

  (iii) $X = \bigcup_{n\in\mathbb{N}} U_n$.

  (*Hint : Choose the sets U_n recursively using Corollary 7.10*)

(b) X is countable at infinity if and only if the point ∞ has a countable neighbourhood base in $\hat{X}$.

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html