Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 11 Deadline: 01.14.2019

Exercise 41 (4 Points)

Let X be a topological space and let (Y, d) be a metric space. Consider a net $(f_\alpha)_{\alpha \in A}$ in $C(X, Y) = \{g : X \to Y;\ g \text{ continuous}\}$ and a function $f : X \to Y$ such that

$$\sup_{x \in X} d(f_\alpha(x), f(x)) \xrightarrow{\alpha} 0.$$

Show that $f \in C(X, Y)$.

Exercise 42 (4 Points)

Let X be a locally compact Hausdorff space, $K \subset X$ compact and $U \subset X$ an open set such that $K \subset U$. Show that there is a continuous function $f : X \to [0, 1]$ with compact support $\mathrm{supp}(f) \subset X$ such that $f|_K \equiv 1$ and $\mathrm{supp}(f) \subset U$.

(*Hint : Choose an open set V with compact closure such that $K \subset V \subset \bar{V} \subset U$ and apply Urysohn's lemma to $K \cup \partial V \subset \bar{V}$.*)

A subset A of a topological space is called a G_δ-set if A is a countable intersection of open sets.

Exercise 43 (4 Points)

Let X be a normal topological space and let $A \subset X$ be a subset. Show that there is a continuous function $f : X \to [0, 1]$ with $f^{-1}(\{0\}) = A$ if and only if $A \subset X$ is a closed G_δ-set.

(*Hint : Define $f = \sum_{n=1}^{\infty} 1/2^n\ f_n$ with functions f_n constructed with the help of Urysohn's lemma.*)

Let $A \subset X$ be a closed subset of a topological space X. Denote by X/A the set $\{[x];\ x \in X\}$ of all equivalence classes with respect to the equivalence relation

$$x \sim y \quad :\Longleftrightarrow \quad x, y \in A \text{ or } x = y.$$

and define $q : X \to X/A$, $x \to [x]$. The topology (Exercise 27) on X/A defined by

$$\tau = \{U \subset X/A;\ q^{-1}(U) \subset X \text{ is open}\}$$

is called the quotient topology of X/A.

Exercise 44* (1*+1*+1*+2*=5* Points)

Let $A \subset X$ be a closed subset of a Hausdorff topological space (X, t). Let X/A be equipped with its quotient topology. With the notation from above show that:

(please turn over)

(a) Each one-point set $\{[x]\} \subset X/A$ $(x \in X)$ is closed.

(b) If $M \subset X$ is a set with $A \subset M$ or $M \cap A = \emptyset$, then $M = q^{-1}(q(M))$.

(c) If $F \subset X/A$, then $A \subset q^{-1}(F)$ or $q^{-1}(F) \cap A = \emptyset$.

(d) X is normal, then X/A is normal.

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html