Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 14 Discussion: February 11 from 10-12 in SR6

Exercise 53

(i) Let (X, d) be a metric space and $f : X \to (\mathbb{R}, |\cdot|)$, $g : X \to (\mathbb{R}, |\cdot|)$ be two uniformly continuous functions. Show that the function $h : X \to (\mathbb{R}^2, \|\cdot\|_1)$ defined by $h(x) = (f(x), g(x))$ is also uniformly continuous.

(ii) Let (X, d) be a metric space. Show that (X, d) is complete if and only if every sequence $(x_n)_{n \in N}$ in X, for which the series $\sum_{n=0}^{\infty} d(x_n, x_{n+1})$ is convergent, is itself convergent in X.

Exercise 54

1. Let (X, τ) be a topological space and $A \subset X$ a subset. Show that A is the intersection of an open set in (X, τ) and a closed set in (X, τ) if and only for every $x \in A$ there is a neighbourhood $U_x \in \mathcal{U}(x)$ such that $U_x \cap A \subset U_x$ is closed.
2. Let (X, τ) be a Hausdorff topological space and $F \subset X$ be a finite set. Show that $F \subset X$ is closed.
3. Let (X, τ) be a topological space and let $A, B \subset X$ be two compact subsets. Show that $A \cup B$ is compact in (X, τ).
4. Let (X, τ) be a Hausdorff topological space and let $A, B \subset X$ be two compact subsets. Show that $A \cap B$ is compact in (X, τ).
5. Let (X, τ), (Y, t) be two topological spaces, $\mathcal{B} \subset \tau$ a base of τ and $f : X \to X$ a surjective, continuous, open map. Show that $f(\mathcal{B})$ is a base of t.

Exercise 55

1. Let (X, τ) be a topological space. Show that (X, τ) is a Hausdorff space if and only if for all $x \in X$ the identity $\{x\} = \bigcap_{U \in \mathcal{U}(x)} \overline{U}$ holds.
2. Let (X, τ) be a topological space such that for every $x \in X$ there is a continuous function $f : X \to (\mathbb{R}, |\cdot|)$ with $f^{-1}(\{0\}) = \{x\}$. Show that (X, τ) is a Hausdorff space.

(please turn over)

Exercise 56

Let (X, τ) be a compact topological space and (Y, t) a topological space. Equip the cartesian product $X \times Y$ with the product topology. Show that the projection $\pi_2 : X \times Y \to Y,\ (x, y) \mapsto y$ is closed.

Exercise 57

(i) Let (X, τ) be a connected topological space and let $f : X \to \mathbb{R}$ be a locally constant function, i.e. for all $x \in X$ there is a neighbourhood $U \in \mathcal{U}(x)$ of x and an element $c \in \mathbb{R}$ such that $f(y) = c$ for all $y \in U$. Show that f is constant.

(ii) Find a topological space (X, τ) and a locally constant function $f : X \to \mathbb{R}$ which is not constant.

A Hausdorff topological space X is called completely regular if, for each closed set $F \subset X$ and each point $x \in X \backslash F$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_F \equiv 1$.

Exercise 58

Show that each locally compact Hausdorff space (X, t) is completely regular.

Exercise 59

Let t_1 and t_2 be topologies on a set X such that (X, t_1) and (X, t_2) are both completely regular and such that each continuous function $f : (X, t_1) \to \mathbb{R}$ is also continuous with respect to t_2. Show that $\mathrm{id} : (X, t_2) \to (X, t_1)$ is continuous.

Exercise 60

Let X be a completely regular topological space. Show that there is a topological embedding $j : X \to \prod_{i \in I} X_i$ into a topological product of compact Hausdorff spaces X_i.

(*Hint : For* $I = C(X, [0, 1])$, *consider the map* $j : X \to \prod_I [0, 1],\ x \to (f(x))_{f \in I}$.)

There will be a discussion of the exercises on Tuesday, February 11 from 10-12 in SR 6.

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html