Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 3 Deadline: 11.05.2019

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*A map $\Phi : X_1 \to X_2$ between metric spaces $(X_1, d_1), (X_2, d_2)$ is called **isometric** if $d_2(\Phi(x), \Phi(y)) = d_1(x, y)$ for all $x, y \in X_1$.*

Exercise 9 (4 Points)

Let $(X_1, d_1), (X_2, d_2)$ be two complete metric spaces and let $i_1 : X \to X_1$, $i_2 : X \to X_2$ be isometric maps with dense range from a metric space (X, d) into X_1 and X_2. Show that there is a unique isometric and bijective map $\Phi : X_1 \to X_2$ with $i_2 = \Phi \circ i_1$.

Exercise 10 (2+2=4 Points)

Let (X, d) be a metric space and let $x_0 \in X$ be arbitrary. For every $x \in X$, denote by $f_x : X \to \mathbb{R}$ the mapping defined by

$$f_x(t) = d(x, t) - d(x_0, t).$$

Show that:

(a) $j : X \to l^\infty(X)$, $j(x) = f_x$, defines an isometric map between metric spaces.

(b) Use part (a) to give an alternative proof for the existence of the completion of X.

Exercise 11 (4 Points)

Let (X, d) be a metric space. Suppose that, for each sequence of closed subsets $\emptyset \neq A_k \subset X$ with $A_{k+1} \subset A_k$ for all $k \in \mathbb{N}$ and

$$\operatorname{diam}(A_k) := \sup\{d(x, y);\ x, y \in A_k\} \overset{k\to\infty}{\longrightarrow} 0,$$

there is an element $x \in X$ with $\bigcap_{k\in\mathbb{N}} A_k = \{x\}$. Show that (X, d) is complete.

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*Recall that a set $K \subset X$ in a metric space (X, d) is called **compact** if, for each family $(U_i)_{i\in I}$ of open sets $U_i \subset X$ with $K \subset \bigcup_{i\in I} U_i$, there are finitely many indices $i_1, \dots, i_r \in I$ with $K \subset U_{i_1} \cup \dots \cup U_{i_r}$.*

Exercise 12 (1+1+2=4 Points)

Let (X, d) be a metric space and let $\emptyset \neq Y \subset X$ be a subset equipped with the relative metric d_Y. Show that:

(please turn over)

(a) A set $U \subset Y$ is open in (Y, d_Y) if and only if there is an open set $V \subset X$ such that $U = V \cap Y$.

(b) A set $F \subset Y$ is closed in (Y, d_Y) if and only if there is a closed set $A \subset X$ such that $F = A \cap Y$.

(c) A set $K \subset Y$ is compact in (Y, d_Y) if and only if it is compact in (X, d).

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html