Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 4 Deadline: 11.12.2019

Exercise 13 (2+2=4 Points)

Let K be a compact metric space and let $\varrho : K \to K$ be a mapping with

$$d(\varrho(x), \varrho(y)) < d(x, y)$$

for all $x, y \in K$ with $x \neq y$. Show that:

(a) Each sequence in K has a convergent subsequence.

(b) There is a unique point $x_0 \in K$ with $\varrho(x_0) = x_0$.

(*Hint : Approximate* $\inf\{d(\varrho(x), x);\ x \in K\}$ *by a sequence.*)

Exercise 14 (4 Points)

Show that there is no sequence $(U_n)_{n\in\mathbb{N}}$ of open sets $U_n \subset \mathbb{R}$ with

$$\mathbb{Q} = \bigcap_{n\in\mathbb{N}} U_n.$$

(*Hint : Use also the sets* $V_q = \mathbb{R} \setminus \{q\}$ $(q \in \mathbb{Q})$ *and apply Baire's theorem.*)

Exercise 15 (3+1=4 Points)

(a) Let $f : X \to X$ be a function on a metric space X. Define

$$U_n = \bigcup (U \subset X \text{ open};\ \operatorname{diam} f(U) < \frac{1}{n})\ (n \in \mathbb{N}^*).$$

Show that $\bigcap_{n=1}^{\infty} U_n$ is the set of all points $x \in X$ such that f is continuous in x.

(b) Show that there is no function $f : \mathbb{R} \to \mathbb{R}$, which is continuous in each $x \in \mathbb{Q}$ and discontinuous in each $x \in \mathbb{R} \setminus \mathbb{Q}$.

Exercise 16 (4 Points)

Let $K = \bigcup_{i\in I} U_i$ be an open cover of a compact metric space (K, d). Show that there is a real number $\delta > 0$ such that each set $A \subset K$ with $\operatorname{diam}(A) < \delta$ is completely contained in one of the sets U_i $(i \in I)$.

(*Hint : Otherwise there were sets* $A_n \subset K$ *with* $\operatorname{diam}(A_n) < 1/n$ *and* $A_n \cap U_i^c \neq \emptyset$ *for all* $i \in I$. *Choose points* $x_n \in A_n$ *and apply Exercise 13(a).*)

(please turn over)

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html