Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 6 Deadline: 11.26.2019

Exercise 21 (3 Points)

Let $(X_n, d_n)_{n=1}^{\infty}$ be a sequence of separable metric spaces. Show that the cartesian product $X = \prod_{n=1}^{\infty} X_n$ equipped with the metric d from Exercise 4 is separable.

Exercise 22 (1+1+1+1=4 Points)

Let (X, t) be a topological space and let $\emptyset \neq Y \subset X$ be a set equipped with the relative topology $t|_Y$. Show:

(a) If the topology t is generated by a metric d on X, then the topology $t|_Y$ is generated by the relative metric d_Y.

(b) If $\mathcal{B}$ is a base of t, then the collection of subsets $\mathcal{B}|_Y = \{B \cap Y;\ B \in \mathcal{B}\}$ is a base of $t|_Y$.

(c) In the case of $Y \in t$ we have $t|_Y = \{U \in t;\ U \subset Y\}$.

(d) If t is generated by a metric d and (X, t) is separable, then $(Y, t|_Y)$ is also separable.

Exercise 23 (4 Points)

Let $X \neq \emptyset$ be a set. Show the following assertions:

A system $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X defines a base of a topology t on X if and only if the following two conditions are fulfilled:

1. For every $x \in X$ there is a set $B \in \mathcal{B}$ with $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and all $x \in B_1 \cap B_2$ there is a $B_0 \in \mathcal{B}$ with $x \in B_0 \subset B_1 \cap B_2$.

In this case we get

$$\begin{aligned} t &= \{U \subset X;\ \forall x \in U\ \exists B \in \mathcal{B}:\ x \in B \subset U\} \\ &= \bigcap(\mathcal{O};\ \mathcal{O} \text{ is a topology on } X \text{ with } \mathcal{B} \subset \mathcal{O}\}. \end{aligned}$$

(please turn over)

Exercise 24 **(1+2+2+2=7 Points)**

Let

$$\mathcal{B} = \{[a, b);\ a, b \in \mathbb{R} \text{ with } a < b\}.$$

(a) Show that $\mathcal{B}$ is the base of a Hausdorff topology σ on $\mathbb{R}$.

(b) Show that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers converges in $(\mathbb{R}, \sigma)$ to a real number $x \in \mathbb{R}$ if and only if it converges to x in $\mathbb{R}$ equipped with the standard topology and additionally $x_n \geq x$ holds for almost all $n \in \mathbb{N}$.

(c) Show that $(\mathbb{R}, \sigma)$ is separable and that the first countability axiom is fulfilled.

(d) Does the second countability axiom hold in (R, σ)? Is $(\mathbb{R}, \sigma)$ metrizable?

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html