Saarland University
FACULTY 6.1 – MATH
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Exercises for the lecture topology

Winter term 2019/2020

Sheet 7 Deadline: 12.03.2019

A set $K \subset X$ in a topological space (X,t) is called sequentially compact if each sequence $(x_n)_{n\in\mathbb{N}}$ in K has a convergent subsequence with limit in K.

Exercise 25 (2+2=4 Points)

Let (X,t) be a first countable topological space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$ an arbitrary point. Show that:

(a) If $(x_n)_{n\in\mathbb{N}}$ has no subsequence that converges to x, than there is a neighborhood $U \in \mathfrak{U}(x)$ of x such that the set $\{n \in \mathbb{N};\ x_n \in U\}$ is finite.

(b) Each compact set $K \subset X$ is sequentially compact.

Exercise 26 (4 Points)

Let (X,t) be a second countable topological space. Show that each sequentially compact set $K \subset X$ is compact. (*Hint : Use Lindelöf's theorem.*)

Exercise 27 (3x2=6 Points)

Let $X \neq \emptyset$ be a set and let I be an arbitrary index set. For $i \in I$ let (X_i, t_i) be a topological space and let $f_i : X_i \to X$ be a map. Show that:

(a) The system $t = \{U \subset X;\ f_i^{-1}(U) \in t_i\ \forall i \in I\}$ defines a topology on X.

(b) The topology t is the strongest topology on X such that the mappings $f_i : (X_i, t_i) \to (X,t)$ are continuous for every $i \in I$.

(c) A mapping $g : (X,t) \to (Y,\tau)$ into a topological space (Y,τ) is continuous if and only if $g \circ f_i : (X_i, t_i) \to (Y,\tau)$ is continuous for each $i \in I$.

(please turn over)

Exercise 28 **(2+2=4 Points)**

Let σ be the topology on $\mathbb{R}$ defined in Exercise 24 and let $\tau = \sigma \times \sigma$ be the product topology on $\mathbb{R}^2$. Show that:

(a) $(\mathbb{R}^2, \tau)$ is separable.

(b) The set $D = \{(x, -x);\ x \in \mathbb{R}\} \subset \mathbb{R}^2$ equipped with the relative topology $\tau|_D$ is not separable.
(Hint : Consider the sets $([x, x+1) \times [-x, -x+1)) \cap D$*.)*

You can also find the exercise sheets on our homepage:

https://www.math.uni-sb.de/ag/eschmeier/lehre/WS1920/top/index.html