SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 1 To be submitted until: Thursday, 12.11.2020, before the lecture

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Exercise 1 (3 Points)

Let (X, d) be a metric space. Show that, for $x, x', y, y' \in X$,

$$\left|d(x,y) - d(x',y')\right| \leq d(x,x') + d(y,y').$$

Exercise 2 (1+3+2=6 Points)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing with $\lim_{x\to\infty} f(x) = c$ and let $d(x,y) = |x-y|$ be the usual metric on $\mathbb{R}$. Show that

(a) $d_f(x,y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$.

(b) $(\mathbb{R}, d)$ and $(\mathbb{R}, d_f)$ possess the same open sets and the same convergent sequences.

(c) $(\mathbb{R}, d_f)$ is not complete.

Exercise 3 (2*+2=2*+2 Points)

Let $(X, \mathfrak{M}, \mu)$ be a measure space with $\mu(X) < \infty$ and let $1 \leq p < q < \infty$ be real numbers. Show that

(a) $\mathscr{L}^q(\mu) \subset \mathscr{L}^p(\mu)$ and $\|f\|_p \leq \mu(X)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q$ for all $f \in \mathscr{L}^q(\mu)$.

(b) $\ell^p \subset \ell^q$ and $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.

(*Hint : Use Hölder's inequality in (a) and consider the unit balls in (b).*)

(please turn over)

Exercise 4 **(4 Points)**

Show that

$$c = \left\{ (x_n) \in \ell^\infty;\ \lim_{n\to\infty} x_n \text{ exists in } \mathbb{C} \right\}$$

equipped with the metric $d((x_n),(y_n)) = \sup_{n\in\mathbb{N}} |x_n - y_n|$ is complete.

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***Only two of the four exercises are corrected: This time Exercise 4 and a second exercise chosen randomly.***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre