SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 10 To be submitted until: Thursday, 28.01.2021, before the lecture

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Exercise 41 (3 Points)

Let H be a Hilbert space, $x \in H$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H. Show that the following are equivalent:

(i) $x_n \xrightarrow{n} x$ with respect to the norm topology.

(ii) $x_n \xrightarrow{n} x$ with respect to the weak topology τ_w and $\|x_n\| \xrightarrow{n} \|x\|$.

Exercise 42 (4 Points)

Let H be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle \colon H \times H \to K$ and let $\|\cdot\|$ be the induced norm on H. Further let $(\tilde{H}, \|\cdot\|^\sim)$ be the completion of $(H, \|\cdot\|)$ (cf. 4.8). Show: The scalar product on H extends to a scalar product on $\tilde{H}$ that generates the norm $\|\cdot\|^\sim$ of $\tilde{H}$.
(Hint : Define $\langle \cdot, \cdot \rangle^\sim$ by $\langle x, y \rangle^\sim = \lim_{n\to\infty} \langle x_n, y_n \rangle$ with suitable sequences $(x_n)_{n\in\mathbb{N}}$,$(y_n)_{n\in\mathbb{N}}$ in H or use Exercise 44.)

Exercise 43 (2+2* Points)

Let H be a Hilbert space and let $(\cdot, \cdot) \colon H \times H \to K$ be a sesquilinear form such that there exists a real number $c > 0$ with

$$|(x, y)| \leq c\|x\|\|y\| \text{ for all } x, y \in H.$$

Show:

(a) There exists a unique operator $T \in \mathcal{L}(H)$ with

$$(x, y) = \langle x, Ty \rangle \text{ for all } x, y \in H.$$

(b) If there exists a real number $d > 0$ with $|(x, x)| \geq d\|x\|^2$ for all $x \in H$, then the operator T from part (a) is bijective with $\|T^{-1}\| \leq \frac{1}{d}$.
(Hint : The operator T has dense range and is bounded below.)

Exercise 44* (4* Points)

Let E be a normed space over $K = \mathbb{R}$ or $\mathbb{C}$ such that its norm admits the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\| + 2\|y\|^2$$

for all $x, y \in E$. Show, there exists an inner product $\langle \cdot, \cdot \rangle$ on E with $\|x\| = \langle x, x \rangle$ for all $x \in E$.

(please turn over)

Exercise 45* (2*+2*=4* Points)

Let $(H_i, \langle \cdot, \cdot \rangle_i)_{i \in I}$ be a family of Hilbert spaces and let

$$H = \{(x_i)_{i \in I} \in \prod_{i \in I} H_i;\ \sum_{i \in I} \langle x_i, x_i \rangle_i < \infty\}.$$

Show:

(a) $H \subset \prod_{i \in I} H_i$ is a subspace and

$$\langle (x_i)_{i \in I}, (y_i)_{i \in I} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle_i$$

defines a well-defined inner product on H.

(b) Show the vector space $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

People often use the notation $H = \bigoplus_{i \in I} H_i$ for the Hilbert space H.

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***This time all Exercises will be corrected!***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre