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Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 11 To be submitted until: /

Exercise 46 (4 Points)

Let X be a Banach space and let $(S_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ be sequences in $L(X)$ such that the limits $S(x) := \lim_{n\to\infty} S_n x$ and $T(x) := \lim_{n\to\infty} T_n x$ exist for all $x \in X$. Show:

$$S(Tx) = \lim_{n\to\infty} S_n T_n x$$

for all $x \in X$.

Exercise 47 (4 Points)

Let E, F be Banach spaces and let $T : E \to F$, $S : F' \to E'$ be linear operators with

$$\langle Tx, y' \rangle = \langle x, Sy' \rangle$$

for all $x \in E$ and $y' \in F'$. Show that T is continuous.

Exercise 48 (2+2=4 Points)

(a) Let X be a Banach space and let $X_1, X_2 \subseteq X$ be closed subspaces with $X_1 + X_2 = X$. Show:

$$X' = X_1^\perp + X_2^\perp \Leftrightarrow X_1 \cap X_2 = \{0\}.$$

(b) Let $H_1, H_2 \subseteq H$ be closed subspaces of a Hilbert space H. Show:

$$(H_1 \cap H_2)^\perp = \overline{H_1^\perp + H_2^\perp}.$$

Exercise 49 (4 Points)

Let E be a Banach space, F a normed space and $A \in L(E, F)$, $K \in K(E, F)$ such that $\mathrm{Im} A \subseteq \mathrm{Im} K$. Show that $A \in K(E, F)$.
(*Hint : Try to use a factorization of A with the operator $\hat{K} : E/\ker K \to F, \hat{K}([y]) = Ky$.*)

(please turn over)

Let H be a Hilbert space. A self-adjoint operator $T \in L(H)$ is called positive, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. If $S, T \in L(H)$ are self-adjoint, then we write $S \leq T$, if $T - S$ is positive.

Exercise 50 **(2+2+2=6 Points)**

Let $T_n, A \in L(H)$ $(n \in N)$ be self-adjoint operators and let $S \in L(H)$ be a positive operator on a Hilbert space H. Show:

(a)
$$|\langle Sx, y\rangle|^2 \leq \langle Sx, x\rangle\langle Sy, y\rangle.$$
for all $x, y \in H$ and hence $\|Sx\|^2 \leq \langle Sx, x\rangle\|S\|$ for all $x \in H$.

(b) If $(\langle T_n x, x\rangle)_{n\in\mathbb{N}}$ converges for all $x \in H$, then there exists a self- adjoint operator $T \in L(H)$ with $\langle Tx, y\rangle = \lim_{n\to\infty}\langle T_n x, y\rangle$ for all $x, y \in H$.

(c) If $T_n \leq T_{n+1} \leq A$, then there exists a self-adjoint operator $T \in L(H)$ with $Tx = \lim_{n\to\infty} T_n x$ for all $x \in H$.
(*Hint : Show that* $(\|T_n\|)_{n\in\mathbb{N}}$ *is bounded and use part (a).*)

Exercise 51 **(4 Points)**

Let X be a normed space. Show: If there exists a norm on X that induces a topology, which coincides with the weak topology τ_w on X, then X is finite dimensional.
(*Hint : The unit-ball generated by the norm contains no non-trivial-subspace.*)
One can show, but you don't have to prove, that the same result holds for the weak-*-topology on X'.

This exercise sheet will not be corrected, but should help you to prepare for the exam. The first three exercises are typically for written exams. The last three exercises are more difficult but can also help you on with the preparation.

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre