SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
Prof. Dr. Jörg Eschmeier
M. Sc. Sebastian Toth

Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 2 To be submitted until: Thursday, 19.11.2020, before the lecture

---

Exercise 5 (1+1+2=4 Points)

Let (X, t) be a topological space and $\emptyset \neq Y \subset X$ a subset equipped with the relative topology $t|_Y$. Show:

(a) A set $A \subset Y$ is closed in $(Y, t|_Y)$ if and only if there is a closed set F in (X, t) with $A = F \cap Y$.

(b) A set $A \subset Y$ is compact in $(Y, t|_Y)$ if and only if A is compact in (X, t).

(c) A set $K \subset X$ is compact if and only if, for each family $(F_i)_{i \in I}$ of closed sets F_i in $(K, t|_K)$ with finite intersection property, the intersection $\bigcap_{i \in I} F_i$ is non-empty.

Exercise 6 (4 Points)

Let $U \subset \mathbb{R}^n$ be open and convex. Show that each function $f \in C^1(U)$ with:

$$\left\| \frac{\partial f}{\partial x_i} \right\|_U < \infty \quad (i = 1, \ldots, n)$$

admits a continuous extension $F \in C(\overline{U})$.

(*Hint : Use Theorem 1.16.*)

Exercise 7 (1+2=3 Points)

Let (X, d) be a metric space. Given a subset $\emptyset \neq Y \subset X$ show that:

(a) The topology on Y induced by the restricted metric $d|_{Y \times Y}$ coincides with the relative topology $t|_Y$ of the topology t induced by d on X.

(b) If (X, d) is complete, then $Y \subset X$ is closed if and only if $(Y, d|_{Y \times Y})$ is complete.

(please turn over)

Exercise 8 **(1+1+2=4 Points)**

Let (X_i, t_i) be topological spaces $(i \in I)$. Equip $X = \prod_{i \in I} X_i$ with its product topology. For $i \in I$, let $A_i \subset X_i$ be subsets. Define $A = \prod_{i \in I} A_i$. Show that:

(a) If all $A_i \subset X_i$ $(i \in I)$ are closed, then $A \subset X$ is closed.

(b) $\overline{A} = \prod_{i \in I} \overline{A_i}$,

(c) $\mathrm{Int(A)} \subset \prod_{\mathrm{i} \in \mathrm{I}} \mathrm{Int(A_i)}$.

Decide whether equality holds in (c)! (Proof or counterexample.)

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***Only two of the four exercises are corrected: This time Exercise 6 and a second exercise chosen randomly.***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre