SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
Prof. Dr. Jörg Eschmeier
M. Sc. Sebastian Toth

Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 3 To be submitted until: Thursday, 25.11.2020, before the lecture

---

Exercise 9 (4 Points)

Let (X_n, t_n) $(n \in \mathbb{N})$ be separable topological spaces. Show that the product topology on $X = \prod_{n\in\mathbb{N}} X_n$ is separable.

Exercise 10 (1+1+2=4 Points)

Let X be a normed space, $S_X = \{x \in X;\ \|x\| = 1\}$ the unit sphere in X and $u\colon X \setminus \{0\} \to S_X,\ x \to \frac{x}{\|x\|}$. Show that:

(a) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and $(\alpha_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, then $(\alpha_n x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

(b) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X \setminus \{0\}$, which does not converge to 0, then $(u(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.

(c) The set S_X is complete with respect to the metric $d(x, y) = \|x - y\|$ if and only if X is a Banach space.

For a set $\emptyset \neq A$ in a metric space (X, d) and $x \in X$, define

$$d(x, A) = \inf_{a\in A} d(x, a).$$

Exercise 11 (2+2=4 Points)

Let X be a normed space, $\emptyset \neq A \subset X$ and $B_X = \{x \in X;\ \|x\| \leq 1\}$. Show that:

(a) $\overline{A + B_X} = \{x \in X;\ d(x, A) \leq 1\}$,

(b) $A + B_X \subset X$ is closed if and only if, for each $x \in X$ with $d(x, A) = 1$, there is an $a \in A$ with $\|x - a\| = 1$.

(please turn over)

Exercise 12 **(1+2+2=5 Points)**

Let $a, b \in \mathbb{R}$ with $a < b$. For $f\colon [a,b] \to \mathbb{C}$, define its variation by

$$V(f) = \sup\{\sum_{\nu=1}^{n} |f(t_\nu) - f(t_{\nu-1})|;\ n \in \mathbb{N}^* \text{ and } a = t_0 < t_1 < \ldots < t_n = b\}.$$

Set $\mathrm{BV[a,b]} = \{\mathrm{f\colon [a,b] \to \mathbb{C};\ V(f) < \infty}\}$ and

$$\|f\|_{\mathrm{BV}} = |f(a)| + V(f)$$

for $f \in \mathrm{BV[a,b]}$. Show that:

(a) $\|f\|_{[a,b]} \leq \|f\|_{\mathrm{BV}}$ for all $f \in \mathrm{BV[a,b]}$.

(b) $(\mathrm{BV[a,b]}, \|\cdot\|_{\mathrm{BV}})$ is a normed space.

(c) $(\mathrm{BV[a,b]}, \|\cdot\|_{\mathrm{BV}})$ is complete.

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***Only two of the four exercises are corrected: This time Exercise 10 and a second exercise chosen randomly.***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre