SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 4 To be submitted until: Thursday, 03.12.2020, before the lecture

Exercise 13 (2+2=4 Points)

Let $A, B \subset \ell^1$ be defined by

$$A = \left\{(a_n)_{n\in\mathbb{N}} \in \ell^1;\ a_{2n} = 0 \text{ for all } n \in \mathbb{N}\right\},$$

$$B = \left\{(b_n)_{n\in\mathbb{N}} \in \ell^1;\ b_{2n} = 2^{-n} b_{2n+1} \text{ for all } n \in \mathbb{N}\right\}.$$

Show that:

(a) A and B are closed linear subspaces of ℓ^1.

(b) $A + B \subset \ell^1$ is not closed.
(*Hint : Consider the sequences* $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ *in* ℓ^1 *defined as* $x_k = e_1 + e_3 + \ldots + e_{2k+1}$ *and* $y_k = e_0 + 2^{-1}e_2 + \ldots + 2^{-k}e_{2k} + x_k$ *for* $k \in \mathbb{N}$, *where* $e_i = (\delta_{i,n})_{n\in\mathbb{N}}$ $(i \in \mathbb{N})$.)

Exercise 14 (1+2+2=5 Points)

Let $(E, \|\cdot\|)$ be a normed space and let $E_0 \subset E$ be a closed linear subspace. Let E/E_0 be the quotient vector space of E modulo E_0. Write the elements of E/E_0 as $[x] = x + E_0 \in E/E_0$. Show that:

(a) $\|[x]\| = \inf\{\|x - y\|;\ y \in E_0\}$ defines a norm on E/E_0.

(b) The quotient map $q\colon E \to E/E_0$, $q(x) = [x]$ is continuous linear and open.

(c) If E is complete, then E/E_0 is complete.
(*Hint : Show that each absolutely convergent series in* E/E_0 *converges.*)

Exercise 15 (4 Points)

Let E be an infinite-dimensional vector space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Show that there is a discontinuous linear map $u\colon E \to K$.

(please turn over)

Exercise 16 **(2+2+2=6 Points)**

Let $E = C[0,1]$ be equipped with the norm $\|f\|_\infty = \sup\{|f(t)|;\ t \in [0,1]\}$ and $F = C[0,1]$ equipped with the norm $\|f\|_1 = \int_0^1 |f(t)|dt$. Decide whether the following linear maps are continuous. If they are continuous, calculate their norm.

(a) $T\colon E \to E,\ (Tf)(t) = \int_0^1 e^{st} f(s)ds,$

(b) $T\colon F \to F,\ (Tf)(t) = f(t^2),$

(c) $T\colon E \to F,\ (Tf)(t) = \int_0^t sf(s)ds.$

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***Only two of the four exercises are corrected: This time Exercise 14 and a second exercise chosen randomly.***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre