SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 5 To be submitted until: Thursday, 10.12.2020, before the lecture

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Exercise 17 (4 Points)

Let $\|\cdot\|_{\ell^\infty/c_0}$ be the quotient norm on ℓ^∞/c_0 (cf. Exercise 14). Show that, for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$,

$$\|x + c_0\|_{\ell^\infty/c_0} = \limsup_{n\to\infty} |x_n|$$

(*Hint : Show that* $\limsup_{n\to\infty} |x_n - y_n| = \limsup_{n\to\infty} |x_n|$ *for* $(x_n)_{n\in\mathbb{N}} \in \ell^\infty$ *and* $(y_n)_{n\in\mathbb{N}} \in c_0$.)

Exercise 18 (4 Points)

Let E be a normed space. Show: If $E_0 \subset E_1 \subset E$ are linear subspaces of E such that $E_0 \subset E$ is closed and $\dim(E_1/E_0) < \infty$, then $E_1 \subset E$ is closed.
(*Hint : Use the quotient map* $q\colon E \to E/E_0$ *from Exercise 14.*)

Exercise 19 (6+2=8 Points)

Given $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty$, consider the operators $R, L, M_\alpha\colon \ell^1 \to \ell^1$ defined by

$$R(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots)$$

$$L(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$$

$$M_\alpha(x_0, x_1, x_2, \ldots) = (\alpha_0 x_0, \alpha_1 x_1, \alpha_2 x_2, \ldots).$$

Convince yourself that these operators are well defined and continuous linear and determine their norms. (For this part no proofs are necessary!)

(a) Calculate $\sigma(T)$ and $\sigma_p(T)$ for $T \in \{R, L, M_\alpha\}$.

(b) Show that M_α is compact if and only if $\alpha \in c_0$.

(please turn over)

Let X, Y be topological spaces. A map $f\colon X \to Y$ is called open if $f(U) \subset Y$ is open for each open set $U \subset X$.

Exercise 20 **(2+2=4 Points)**

Let E be a normed space and $T \in \mathcal{K}(E)$. Show:

(a) $\operatorname{Ker}(I - T)$ is finite-dimensional.

(b) If T is open, then $\dim E < \infty$.

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***This time all Exercises will be corrected!***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre