SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 6 To be submitted until: Thursday, 17.12.2020, before the lecture

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Exercise 21 (1+1+2=4 Points)

Let X be a normed space. For $M \subset X$ and $N \subset X'$, define

$$M^{\perp} = \{x' \in X';\ x'(x) = 0 \text{ for all } x \in M\} \subset X'$$

and

$$^{\perp}N = \{x \in X;\ x'(x) = 0 \text{ for all } x' \in N\} \subset X.$$

For $M \subset X$ arbitrary, show that

(a) $M^{\perp}$ is a closed linear subspace of X'.

(b) $M \subset {}^{\perp}(M^{\perp})$

(c) If $M \subset X$ is a linear subspace, then $\overline{M} = {}^{\perp}(M^{\perp})$

Exercise 22 (1+2+1=4 Points)

Let X be a normed space and let $A \in \mathcal{L}(X)$. Show that:

(a) The mapping $\mathcal{L}(X) \to \mathcal{L}(X')$, $T \mapsto T'$, is linear and satisfies $(TS)' = S'T'$ for all $S, T \in \mathcal{L}(X)$.

(b) A' is injective $\Leftrightarrow$ $\overline{AX} = X$.

(c) If A is invertible in $\mathcal{L}(X)$, then A' is invertible in $\mathcal{L}(X')$ and $(A')^{-1} = (A^{-1})'$.

(please turn over)

Let $U \subset \mathbb{C}$ be open and let E be a normed space over $\mathbb{C}$. A function $f: U \to E$ is called

- *weakly holomorphic if $u \circ f: U \to \mathbb{C}$ is holomorphic for all $u \in E'$,*
- *holomorphic if the limit $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists in E for all $z_0 \in U$.*

Exercise 23 **(2+2=4 Points)**

Let $U \subset \mathbb{C}$ be open, E a normed complex vector space and $f: U \to E$ a function. Show:

(a) f holomorphic $\Rightarrow$ f weakly holomorphic.

(b) If $U = \mathbb{C}$ and f is holomorphic and bounded, then f is constant.

If E is a Banach space it is possible to prove that f is weakly holomorphic if and only if f is holomorphic. You don't have to show this to achieve all points in the exercise!

Exercise 24 **(2+2=4 Points)**

Let E, F be Banach spaces, $E_0 \subset E$ a closed subspace and $S \in \mathcal{L}(E_0, F)$. Show

(a) If $F = \ell^\infty$, then there is an operator $T \in \mathcal{L}(E, F)$ with $S = T|_{E_0}$ and $\|S\| = \|T\|$.

(b) If $E_0 = F = c_0$, $E = c$ and $S = I_{c_0}$, then each continuous linear extension $T \in \mathcal{L}(E, F)$ of S satisfies $\|T\| \geq 2$.
(*Hint : Let $e = (1, 1, 1, \ldots) \in c$. Then $\|T\| \geq \sup\{\|T(e) - x\|;\ x \in c_0$ and $\|x - e\| \leq 1\}$.*)

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***This time all Exercises will be corrected!***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre