SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
Prof. Dr. Jörg Eschmeier
M. Sc. Sebastian Toth

Exercises for the lecture Functional Analysis I

Winter term 2020/2021

Sheet 7 — To be submitted until: Thursday, 07.01.2021, before the lecture

---

Exercise 25 (4 Points)

Let E be a locally convex space and $M \subset E$ a closed subspace. Show: For each vector $y \in E \setminus M$, there exists a continuous linear form $u\colon E \to K$ with $u|_M \equiv 0$ and $u(y) \neq 0$.
(*Hint : Use a suitable separation theorem.*)

Exercise 26 (1+3=4 Points)

Let E be a normed space. Show for $M \subset E$ and $N \subset E'$

(a) $\overline{\mathrm{LH}(M)} = {}^{\perp}(M^{\perp})$.

(b) $\overline{\mathrm{LH}(N)}^{\tau_{w^*}} = ({}^{\perp}N)^{\perp}$.
(*Hint : Use Exercise 25.*)

Exercise 27 (2+2=4 Points)

Let X, Y be normed spaces and $T \in \mathcal{L}(X, Y)$. Show:

(a) $\ker T' = (\operatorname{Im} T)^{\perp}$ and $\overline{\operatorname{Im} T} = {}^{\perp}(\ker T')$.

(b) $\ker T = {}^{\perp}(\operatorname{Im} T')$ and $\overline{\operatorname{Im} T'}^{\tau_{w^*}} = (\ker T)^{\perp}$.

(*Hint : Use Exercise 26.*)

Exercise 28 (4 Points)

Show that

$$\langle \cdot, \cdot \rangle\colon c_0 \times \ell^1 \to \mathbb{C},\ \langle (x_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}} \rangle = \sum_{n=0}^{\infty} x_n u_n$$

defines a bilinear map that induces an isometric isomorphism

$$\Phi\colon \ell^1 \to c_0',\ \Phi(u) = \langle \cdot, u \rangle.$$

(please turn over)

Exercise 29* **(2*+2*=4* Points)**

For $f \in L^1[0,1]$, define a sequence $f^\# = (f_n^\#)_{n\in\mathbb{N}}$ in $\mathbb{C}$ by

$$f_n^\# = \int_0^1 f(t)t^n dt \quad (n \in \mathbb{N}).$$

(a) Show that $T\colon L^1[0,1] \to c_0,\ f \to f^\#$ is a well- defined continuous linear operator.

(b) Calculate the adjoint operator $T'\colon \ell^1 \to L^\infty[0,1]$.

Exercise 30* **(4* Points)**

Let X be a normed space and $M \subset X$ a subset with $\overline{\mathrm{LH}}(M) = X$. Let $(u_\alpha)_{\alpha\in A}$ be a bounded net in X'. Show that $(u_\alpha)_{\alpha\in A}$ is τ_{w^*} -convergent to some functional $u \in X'$ if and only if the limit $\lim_\alpha u_\alpha(x)$ exists for each $x \in M$.

Exercise 31* **(4* Points)**

Let X be a separable normed space. Show that

$$B_{X'} = \{x' \in X'; \|x'\| \leq 1\}$$

equipped with the relative topology of the weak* topology τ_{w^*} is a compact metric space.
(*Hint : Modify the proof of Theorem 7.5 and use (without proof) that countable topological products of metrizable topological spaces are metrizable.*)

We wish you a Merry Christmas all the best, happiness and good health for the New Year 2021!

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***This time all Exercises will be corrected!***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre