SAARLAND UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Exercises for the lecture Functional Analysis I
Winter term 2020/2021

Sheet 8 To be submitted until: Thursday, 14.01.2021, before the lecture

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Exercise 32 **(4 Points)**

Let X be a Banach space and $T \in \mathcal{L}(X)$. Show that $\sigma(T) = \sigma(T')$.

Exercise 33 **(2+2=4 Points)**

(a) Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be surjective. Show that there is a constant $c > 0$ such that, for each $y \in Y$, there is a vector $x \in X$ with $Tx = y$ and $\|x\| \leq c\|y\|$.

(b) Let $Y, Z \subset X$ be closed subspaces of a Banach space X such that $X = Y + Z$. Show that there is a constant $c > 0$ such that, for all $x \in X$, there are $y, z \in X$ with $x = y + z$ and $\|y\| + \|z\| \leq c\|x\|$.
(*Hint : Equip $Y \oplus Z$ with a suitable norm.*)

Exercise 34 **(4 Points)**

Let $T\colon X \to Y$ be a linear map between Banach spaces X and Y. Show that $T\colon X \to Y$ is norm-continuous if and only if $T\colon (X, \tau_w) \to (Y, \tau_w)$ is continuous.

Exercise 35 **(2+2=4 Points)**

Let E be a normed space and let $\emptyset \neq M \subset E$ be a non-empty subset. Show:

(a) The set

$$\begin{aligned} C(M) :&= \left\{ \sum_{i=1}^{n} t_i x_i;\ n \in \mathbb{N}^*, x_1, \ldots, x_n \in M, t_1, \ldots, t_n \geq 0 \text{ with } \sum_{i=1}^{n} t_i = 1 \right\} \\ &= \bigcap (A \subset E;\ A \text{ convex with } M \subset A) \end{aligned}$$

is the smallest convex subset of E that contains M.

(b) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E such that $(x_n) \xrightarrow{n} x$ with respect to the weak topology τ_w on E, then there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $C(\{x_n;\ n \in \mathbb{N}\})$ such that $(y_n) \xrightarrow{n} x$ with respect to the norm-topology on E.
(*Hint : Lemma 7.17*)

(please turn over)

Let E be a normed space. We write $B_E = \{x \in E;\ \|x\| \leq 1\}$ for the closed unit ball and $S_E = \{x \in E;\ \|x\| = 1\}$ for the unit sphere in E. A subset $K \subset X$ of a topological space (X, t) is called sequentially compact if each sequence $(x_k)_{k\in\mathbb{N}}$ in K possesses a convergent subsequence with limit in K.

Exercise 36* **(2*+1*+3*=6* Points)**

Let X be a normed space. Show:

(a) If X' is separable, then so is X.
(Hint : Choose a dense subset $\{x_n';\ n \in \mathbb{N}\}$ of $S_{X'}$ and $x_n \in S_X$ such that $|\langle x_n, x_n'\rangle| \geq \frac{1}{2}$ for all n.)

(b) If X is reflexive, then X is separable if and only if X' is separable.

(c) If X is reflexive, then $B_X \subset X$ is τ_w-sequentially compact.
(Hint : Regard a sequence $(x_k)_{k\in\mathbb{N}}$ in B_X as a sequence in the closed unit ball of the dual space $\overline{\text{span}}\{x_k;\ k \in \mathbb{N}\} \cong (\overline{\text{span}}\{x_k;\ k \in \mathbb{N}\}')'$ and use the result of Exercise 31.)

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. " But please avoid to meet in person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Homework marked with a star allows you to achieve additional points. ***This time all Exercises will be corrected!***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre