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Master's Thesis

Universal Operator Algebras for Commuting Row Contractions

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I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken,

Contents

Introduction

Suppose that $p_1, \dots, p_r$ are homogeneous polynomials in d variables with complex coefficients, and consider the system of equations

$$p_i(a_1, \dots, a_d) = 0 \quad (i = 1, \dots, r). \tag{1}$$

In (projective) complex algebraic geometry, the set of all complex scalars $a_1, \dots, a_d$ which satisfy (1) is studied. The problem, however, makes sense whenever the a_i are assumed to be pairwise commuting elements of a unital complex algebra. For example, one can seek for solutions where the a_i are pairwise commuting matrices, or, more generally, pairwise commuting bounded linear operators on a complex Hilbert space. It is a remarkable fact that in the realm of operators on a Hilbert space, (1) admits a universal solution in some sense, provided one adds a certain contractivity condition.

To explain this phenomenon, it is helpful to consider for the moment the most basic case, where $d = 1$ and the only polynomial in (1) is the zero polynomial. For $d = 1$, our contractivity condition is just that the solution is a contraction. So the solutions we are interested in are precisely all contractions. A famous result concerning contractions on a Hilbert space is von Neumann's inequality [vN51].

Theorem 1. *Let H be a Hilbert space, and let T be a contraction on H. Then the inequality*

$$||p(T)|| \leq ||p||_{\overline{\mathbb{D}}}$$

holds for all polynomials in one variable, where the right-hand side denotes the supremum norm of p on the closed unit disk.

Observe that the unilateral shift M_z on the Hardy space $H^2(\mathbb{D})$ is a contraction which achieves equality in von Neumann's inequality for all polynomials. If $\mathcal{A}_1$ denotes the unital norm-closed non-selfadjoint subalgebra of $\mathcal{L}(H^2(\mathbb{D}))$ generated by M_z, we can therefore restate von Neumann's inequality as follows:

Theorem 2. *For any contraction T on a Hilbert space H, there is a (necessarily unique) unital contractive algebra homomorphism*

$$\Phi : \mathcal{A}_1 \to \mathcal{L}(H) \quad \textit{with} \quad \Phi(M_z) = T.$$

In this sense, the unilateral shift M_z can be thought of as the universal contractive solution for (1) when $d = 1$ and there are no relations. We call $\mathcal{A}_1$ the universal operator algebra generated by a contraction. Note that the algebra $\mathcal{A}_1$ can be naturally identified with the disk algebra $A(\mathbb{D})$.

Assume now that $d \geq 1$. Let $T = (T_1, \ldots, T_d)$ be a d-tuple of pairwise commuting bounded linear operators on a Hilbert space H. For $d \geq 1$, the contractivity condition we will impose is that T is a row contraction, that is, that the row operator

$$H^d \to H, \quad (x_i)_{i=1}^d \mapsto \sum_{i=1}^d T_i x_i$$

is a contraction, or, equivalently, that $\sum_{i=1}^d T_i T_i^* \leq 1$. The correct analogue of von Neumann's inequality for commuting row contractions was discovered by Drury [Dru78] and Arveson [Arv98]. In the multivariate setting, the role of the Hardy space $H^2(\mathbb{D})$ is played by the Drury-Arveson space, also known as symmetric Fock space. It is the reproducing kernel Hilbert space on the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ with reproducing kernel

$$K(z, w) = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}} \quad (z, w \in \mathbb{B}_d).$$

The coordinate functions z_i induce multiplication operators

$$M_{z_i} : H_d^2 \to H_d^2, \quad f \mapsto z_i f,$$

and the commuting tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ is a row contraction, which is called the *d-shift*. Note that for $d = 1$, the Drury-Arveson space is just the Hardy space $H^2(\mathbb{D})$ and the d-shift is the unilateral shift. If $\mathcal{A}_d$ denotes the unital norm-closed non-selfadjoint subalgebra of $\mathcal{L}(H_d^2)$ generated by $M_{z_1}, \ldots, M_{z_d}$, the multivariate analogue of Theorem 2 is the following result.

Theorem 3. *For any commuting row contraction $T = (T_1, \ldots, T_d)$ on a Hilbert space H, there is a (necessarily unique) unital completely contractive algebra homomorphism*

$$\Phi : \mathcal{A}_d \to \mathcal{L}(H) \quad \textit{with} \quad \Phi(M_{z_i}) = T_i \quad \textit{for } i = 1, \ldots, d.$$

Consequently, we can think of M_z as the universal solution for (1) within the class of all row contractions when there are no relations. We call $\mathcal{A}_d$ the universal operator algebra generated by a commuting row contraction.

Let us now consider the case when there are actual relations in (1). The existence of a universal operator algebra when there are no relations easily implies the existence of a universal object in the case of general homogeneous polynomial relations.

If I is the ideal generated by the polynomials p_i from (1), let us denote the norm-closure of I in $\mathcal{A}_d$ by $\widetilde{I}$, where we identify a polynomial p with the operator $p(M_z)$. Then the residue classes of M_{z_i} in $\mathcal{A}_d/\widetilde{I}$ satisfy the relations in I, and Theorem 3 has the following consequence.

Corollary 4. *Let $I \subset \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous ideal. Suppose that $T = (T_1, \dots, T_d)$ is a commuting row contraction on a Hilbert space H satisfying $p(T) = 0$ for all $p \in I$. Then there is a (necessarily unique) unital completely contractive algebra homomorphism*

$$\Phi : \mathcal{A}_d/\widetilde{I} \to \mathcal{L}(H) \quad \textit{with} \quad \Phi([M_{z_i}]) = T_i \quad \textit{for } i = 1, \dots, d,$$

where $\widetilde{I}$ denotes the closure of I in $\mathcal{A}_d$.

From a certain point of view, this result is not completely satisfactory, since the algebra $\mathcal{A}_d/\widetilde{I}$ is not an algebra of operators on a Hilbert space. In particular, the equivalence class of the tuple M_z, which is the universal solution of the equations in I in the above sense, is not an operator tuple. The algebra $\mathcal{A}_d/\widetilde{I}$ is, however, an abstract operator algebra in the sense of Blecher, Ruan and Sinclair (see, for example, [ER00, Chapter 17]), hence it is completely isometrically isomorphic to a concrete operator algebra.

There is also a more direct way to identify $\mathcal{A}_d/\widetilde{I}$ with an algebra of concrete operators. To this end, set

$$\mathcal{F}_I = H^2_d \ominus I.$$

The space $\mathcal{F}_I$ is co-invariant for the tuple M_z, so the compression S^I of M_z to $\mathcal{F}_I$ is a commuting row contraction. Let $\mathcal{A}_I$ be the unital norm-closed algebra generated by S^I. Then $\mathcal{A}_d/\widetilde{I}$ is completely isometrically isomorphic to $\mathcal{A}_I$ via a homomorphism sending $[M_{z_i}]$ to S^I_i for each i. Thus, $\mathcal{A}_I$ is the universal operator algebra generated by a commuting row contraction subject to the relations in I.

In this Master's thesis, which is based upon the article [DRS11] by Davidson, Ramsey and Shalit, the isomorphism problem for the algebras $\mathcal{A}_I$ is studied. More explicitly, the following question is considered:

Question 5. Let $I, J \subset \mathbb{C}[z_1, \dots, z_d]$ be homogeneous ideals. Under which conditions are the algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ isomorphic?

We will mostly be concerned with radical homogeneous ideals. In this case, there is a close connection between the structure of the algebra $\mathcal{A}_I$ and the geometry of the vanishing locus $V(I)$. Of course, there are several reasonable notions of isomorphisms between the algebras $\mathcal{A}_I$, most notably, topological and isometric isomorphisms. In the radical case, we will obtain an answer to the above question in terms of the geometry of the vanishing loci $V(I)$ and $V(J)$ for both these notions. The main theorem concerning isometric isomorphisms is the following result from [DRS11]:

Theorem 6. *Let $I, J \subset \mathbb{C}[z_1, \dots, z_d]$ be radical homogeneous ideals. The algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic if and only if there exists a unitary map U on $\mathbb{C}^d$ which maps $V(J)$ onto $V(I)$.*

In the case of topological isomorphisms, we will show the following theorem:

Theorem 7. *Let $I, J \subset \mathbb{C}[z_1, \dots, z_d]$ be radical homogeneous ideals. The algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic if and only if there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J) \cap \overline{\mathbb{B}_d}$ onto $V(I) \cap \overline{\mathbb{B}_d}$.*

Necessity of the existence of a linear map as in the theorem was proven in [DRS11]. The converse was established by the authors of [DRS11] for the case of tractable varieties, and was conjectured to be true in general. The preceding theorem gives a positive answer to this conjecture.

We will obtain analogous results for the WOT-closures of the algebras $\mathcal{A}_I$. In the non-radical case, we can at least classify the algebras $\mathcal{A}_I$ up to isometric isomorphism.

In more detail, the contents of this Master's thesis are as follows: In the first chapter, we study a rather general class of spaces of holomorphic functions on circular sets in $\mathbb{C}^d$. This class contains in particular the Drury-Arveson space H^2_d and the algebra $\mathcal{A}_d$, which are the main examples we have in mind. We establish Cesàro-convergence of the homogeneous expansions of functions in those spaces (Proposition 1.6), and deduce a Nullstellensatz for homogeneous polynomial ideals (Theorem 1.7). In the case where the space of holomorphic functions is itself an algebra, we prove - under some mild hypotheses - a Nullstellensatz for homogeneous ideals in the algebra (Theorem 1.22).

The purpose of the second chapter is to recall some well-known facts about the Drury-Arveson space H^2_d, and to explain the identification of $\mathcal{A}_d/\widetilde{I}$ and $\mathcal{A}_I$ (Theorem 2.17), which was alluded to earlier. We show that in the radical case, the algebra $\mathcal{A}_I$ is in a natural way an algebra of continuous functions on $V(I) \cap \overline{\mathbb{B}_d}$. In the final section, we examine the effect of a unitary change of variables on the Drury-Arveson space and on the algebra $\mathcal{A}_d$. This will already prove one implication of Theorem 6.

The main results of the third chapter are Theorem 3.36 and Theorem 3.28, which establish necessity of the existence of the linear maps in Theorem 6 and Theorem 7. To this end, we show that the maximal ideal space of the algebra $\mathcal{A}_I$ can be identified with $V(I) \cap \overline{\mathbb{B}_d}$, and prove that algebra isomorphisms between $\mathcal{A}_I$ and $\mathcal{A}_J$ induce biholomorphic maps between $V(J) \cap \mathbb{B}_d$ and $V(I) \cap \mathbb{B}_d$. Those isomorphisms with the property that the induced biholomorphic map fixes the origin will play an important role, and they will be called vacuum-preserving. Examining biholomorphic maps on homogeneous varieties, we will see in Proposition 3.27 that two isomorphic algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are isomorphic via a vacuum-preserving isomorphism, and that such a vacuum-preserving isomorphism is induced by a linear map on $\mathbb{C}^d$.

In the fourth chapter, we show that the necessary condition for $\mathcal{A}_I$ and $\mathcal{A}_J$ being topologically isomorphic, established in the third chapter, is also sufficient. We begin by reducing the problem to the case where the varieties are unions of subspaces, a fact which was established in [DRS11]. The remaining parts of this chapter are new. We observe that in order to treat the case of unions of subspaces, it suffices to show that finite algebraic sums of full Fock spaces over subspaces of $\mathbb{C}^d$ are closed. We achieve this with the help of the notion of the Friedrichs angle. The proof consists of two main steps, namely the reduction to subspaces with trivial joint intersection (Lemma 4.36), and the solution of the this case (Theorem 4.46)

The fifth chapter deals with the isomorphism problem for the algebras $\mathcal{A}_I$, where $I \subset \mathbb{C}[z_1, \dots, z_d]$ is a homogeneous, but not necessarily radical ideal. Theorem 5.9 is from [DRS11] and classifies the algebras $\mathcal{A}_I$ up to isometric isomorphism. For the case of topological isomorphisms, we only obtain partial results. We show in Theorem 5.13 that if two algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are isomorphic, then they are isomorphic via an isomorphism which is induced by an invertible linear map on $\mathbb{C}^d$. In a very particular case, this allows us to find a necessary and sufficient condition for $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic in terms of the ideals I and J (Theorem 5.17).

In the sixth chapter, we consider the WOT-closures $\mathcal{L}_I$ of the algebras $\mathcal{A}_I$. We establish results which are analogous to those from the second chapter for the algebras $\mathcal{A}_I$, namely that $\mathcal{L}_I$ can be identified with a quotient of the multiplier algebra of the Drury-Arveson space, and that in the radical case, $\mathcal{L}_I$ is naturally a function algebra. As in the norm-closed case, we continue by studying the maximal ideal space of $\mathcal{L}_I$, and prove that algebra isomorphisms induce certain biholomorphisms. The main theorem is Theorem 6.13 which asserts that in the radical case, the isomorphism classes of the algebras $\mathcal{L}_I$ are the same as those of the algebras $\mathcal{A}_I$, so that the results from previous chapters give an answer to the isomorphism problem for the algebras $\mathcal{L}_I$.

As mentioned before, this thesis is based upon the paper [DRS11]. For the most part, we closely follow the exposition given there. New are Chapter 4 without the first section and the last section of Chapter 5. The content of the former is submitted for peer review, and is also available under [Har12].

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1. Spaces of holomorphic functions on circular sets

1.1. The basic setup

Let f be a holomorphic function on the open unit ball in $\mathbb{C}^d$. Then f admits a locally uniformly convergent expansion $f = \sum_{n=0}^{\infty} f_n$, where each f_n is a homogeneous polynomial of degree n. Indeed, this follows by regrouping the terms in the Taylor expansion of f around the origin. The purpose of this chapter is to establish a global homogeneous expansion for certain spaces of holomorphic functions. Moreover, we will deduce a Nullstellensatz for homogeneous ideals. In subsequent chapters, we will only need these results for the Drury-Arveson space (see Section 2.1) and algebras of multipliers on this space. Already the authors of [DRS11] observed that the homogeneous expansion and the Nullstellensatz remain valid in a broader class of spaces. We will adapt the techniques from [DRS11, Section 6] to the following even more general situation.

Throughout this chapter, let $\Omega \subset \mathbb{C}^d$ be an open connected set containing the origin which is *circular* in the sense that $e^{it}z \in \Omega$ for all $t \in \mathbb{R}$ and $z \in \Omega$. Suppose that B is a locally convex Hausdorff space of analytic functions on Ω which satisfies the following properties:

(a) B is quasi-complete,

(b) for all $z \in \Omega$, the evaluation functional $B \to \mathbb{C}, f \mapsto f(z)$, is continuous,

(c) for all $f \in B$ and $t \in \mathbb{R}$, the function f_t defined by $f_t(z) = f(e^{it}z)$ for $z \in \Omega$ is contained in B, and the map $\mathbb{R} \to B, t \mapsto f_t$, is continuous.

For the sake of brevity, we will call such a space *admissible*.

A comment on the assumptions seems to be in order. While (a) and (b) are quite natural, and are satisfied by most spaces of interest, assumption (c) is more restrictive in comparison. We will show that every admissible space admits a dense subspace consisting of polynomials (see Proposition 1.6), so that condition (c) is, for example, not satisfied by $H^\infty(\mathbb{D})$, equipped with the sup norm.

We begin by exhibiting some examples of admissible spaces. The following simple observation is often useful for establishing condition (c).

Lemma 1.1. *Let X be a locally convex Hausdorff space of analytic functions on Ω with the property that $f_t \in X$ for all $f \in X$ and $t \in \mathbb{R}$. Suppose that X admits a family of seminorms p generating the topology on X which satisfy $p(f_t) = p(f)$ for all $f \in X$ and $t \in \mathbb{R}$. Then the map*

$$\mathbb{R} \times X \to X, \quad (t, f) \mapsto f_t$$

is continuous if and only if for all $f \in X$, the map

$$\mathbb{R} \to X, \quad t \mapsto f_t$$

is continuous at 0.

Proof. The non-trivial implication follows from the estimate

$$p(f_t - g_s) = p(f_{t-s} - g) \leq p(f_{t-s} - f) + p(f - g)$$

for $f, g \in X$ and $s, t \in \mathbb{R}$. □

We write $\mathcal{O}(\Omega)$ for the space of all holomorphic functions on Ω, equipped with the topology of locally uniform convergence. Recall that $\mathcal{O}(\Omega)$ is a Fréchet space, and in particular complete.

Lemma 1.2. *The space $\mathcal{O}(\Omega)$ is admissible.*

Proof. The only requirement of admissible spaces that needs to be proved is the last one. To this end, let K be a compact subset of Ω. Then

$$\widehat{K} = \{e^{it} z : \, z \in K, t \in [0, 2\pi]\}$$

is again a compact subset of Ω, and it contains K. It is easy to check that for $f \in \mathcal{O}(\Omega)$ and $t \in \mathbb{R}$, we have $||f_t||_{\widehat{K}} = ||f||_{\widehat{K}}$. Thus $\mathcal{O}(\Omega)$ admits a family of seminorms as in the hypothesis of Lemma 1.1. Since for each $f \in \mathcal{O}(\Omega)$, we have

$$\lim_{t \to 0} ||f_t - f||_{\widehat{K}} = 0$$

by uniform continuity of f on $\widehat{K}$, the assertion follows from Lemma 1.1. □

If we know a priori that the polynomials are dense in a Banach space of analytic functions on Ω, then it is often easy to show that it is admissible. The following lemma covers for example the Hardy spaces $H^p(\mathbb{B}_d)$ and the Bergman spaces $A^p(\mathbb{B}_d)$ for $1 \leq p < \infty$, and also the ball algebra, that is, the algebra of all continuous functions on the closed unit ball $\overline{\mathbb{B}_d}$ which are holomorphic on the open unit ball $\mathbb{B}_d$, endowed with the sup norm.

Lemma 1.3. *Let E be a Banach space of analytic functions on Ω with continuous point evaluations such that*

(a) *the polynomials form a dense subset of E,*

(b) *for all $f \in E$ and $t \in \mathbb{R}$, the function f_t is contained in E and $||f_t|| = ||f||$.*

Then E is admissible.

Proof. We infer again from Lemma 1.1 that it suffices to show that the set

$$A = \{f \in E : \lim_{t \to 0} ||f_t - f|| = 0\}$$

equals E. It is straightforward to prove that A is norm-closed in E. Indeed, if $f \in \overline{A}, \varepsilon > 0$ and $g \in A$ with $||f - g|| < \frac{\varepsilon}{4}$, choose $\delta > 0$ such that $||g_t - g|| < \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ satisfying $|t| < \delta$. Then for all $t \in \mathbb{R}$ with $|t| < \delta$, we have

$$||f_t - f|| \le ||f_t - g_t|| + ||g_t - g|| + ||g - f|| = 2||f - g|| + ||g_t - g|| < \varepsilon,$$

and hence $f \in A$. Moreover, A is obviously a subspace of E. To finish the proof, we show that the monomials are contained in A. If z^α is monomial, then

$$(z^\alpha)_t = e^{|\alpha| it} z^\alpha,$$

thus

$$||(z^\alpha)_t - z^\alpha|| = |e^{|\alpha| it} - 1| \, ||z^\alpha|| \xrightarrow{t \to 0} 0,$$

so $z^\alpha \in A$, as asserted. □

As mentioned at the beginning of the section, we will mostly be concerned with spaces that arise in the context of reproducing kernel Hilbert spaces. For the general theory of reproducing kernel Hilbert spaces, we refer to Appendix A and the references therein. A natural condition for a Hilbert function space on Ω is that the kernel K reflects the symmetry of Ω, that is, that $K(z, w) = K(e^{it} z, e^{it} w)$ for all $z, w \in \Omega$ and $t \in \mathbb{R}$. In what follows, such a space will be called *circular*. Lemma A.5 shows that H is circular if and only if $f \mapsto f_t$ defines a unitary operator on H for each $t \in \mathbb{R}$.

Lemma 1.4. *Let H be a circular reproducing kernel Hilbert space of analytic functions on Ω without common zeros. Then the following assertions hold:*

(a) *H is admissible.*

(b) *The multiplier algebra, equipped with the strong or the weak operator topology, is admissible.*

(c) *If the polynomials are multipliers on H, then the norm-closure of the polynomials in* $\mathrm{Mult}(H)$*, equipped with the norm topology, is admissible.*

Proof. (a) The first two requirements on admissible spaces are clear. As for the third, we use again Lemma 1.1 to reduce the problem to showing that $t \mapsto f_t$ is continuous at 0 for each $f \in H$. In fact, since $||f_t|| = ||f||$ for all $t \in \mathbb{R}$, it suffices to show that the map is weakly continuous at 0. So let $(t_n)_n$ be a sequence of real numbers that converges to 0. Since $\{K(\cdot, w); w \in \Omega\}$ is total in H and the sequence (f_{t_n}) is bounded, it even suffices to show that

$$\langle f_{t_n}, K(\cdot, w)\rangle \xrightarrow{n\to\infty} \langle f, K(\cdot, w)\rangle$$

for all $w \in \Omega$. But this follows at once from the continuity of f and the identities

$$\langle f_{t_n}, K(\cdot, w)\rangle = f(e^{it_n} w) \quad \text{and} \quad \langle f, K(\cdot, w)\rangle = f(w).$$

(b) The observation that $\mathrm{Mult}(H)$ is WOT-closed, and hence SOT-closed (see Lemma A.10), and that point evaluations are WOT-continuous, and hence SOT-continuous (see Lemma A.9 (c)), establishes the first two requirements on admissible spaces. To prove the third, let U_t denote the unitary operator on H defined by $U_t f = f_t$, where $t \in \mathbb{R}$. If φ is a multiplier on H and $t \in \mathbb{R}$, then

$$(U_t M_\varphi U_{-t})f = (\varphi \cdot f_{-t})_t = \varphi_t \cdot f$$

for all $f \in H$, so that $\varphi_t \in \mathrm{Mult}(H)$ with $M_{\varphi_t} = U_t M_\varphi U_{-t}$. Part (a) shows that $t \mapsto U_t$ is SOT-continuous, and since multiplication is SOT-continuous on bounded sets, we deduce that the map

$$\mathbb{R} \to \mathrm{Mult}(H), \quad t \mapsto M_{\varphi_t} = U_t M_\varphi U_{-t}$$

is SOT-continuous, and, a fortiori, WOT-continuous.

(c) The proof of (b) shows that $||M_{\varphi_t}|| = ||M_\varphi||$ for $\varphi \in \mathrm{Mult}(H)$ and $t \in \mathbb{R}$, so that the assertion is an immediate consequence of Lemma 1.3. □

Note that the example of $H^\infty(\mathbb{D})$, which is the multiplier algebra of the Hardy space $H^2(\mathbb{D})$, shows that the whole multiplier algebra of a circular reproducing kernel Hilbert space, endowed with the norm topology, need not be admissible.

1.2. Homogeneous expansion and a Nullstellensatz

In order to obtain a homogeneous expansion of functions in admissible spaces, we require some results concerning vector valued integration. To set the stage, let K be a compact Hausdorff space, let μ be a regular Borel measure on K and let X be a locally convex Hausdorff space. Moreover, suppose that $f : K \to X$ is a continuous function. If there exists an element $y \in X$ such that

$$x'(x) = \int_K (x' \circ f)\, d\mu \quad \text{for all } x' \in X', \tag{1.1}$$

then we call x the (weak) integral of f over K and write $x = \int_K f\, d\mu$. Here, X' denotes the space of all continuous linear functionals on X. Clearly, the Hahn-Banach theorem implies that the integral, if existent, is uniquely determined by (1.1). See also [Rud91, Definition 3.26].

Remark 1.5. Let K, X, and f be as above.

(1) Suppose that $\int_K f\, d\mu$ exists, and let p be a continuous seminorm on X. Using equation (1.1) together with an obvious application of the Hahn-Banach theorem, we see that

$$p\Big(\int_K f\, d\mu\Big) \leq \int_K (p \circ f)\, d\mu.$$

In particular, if V is a closed absolutely convex neighborhood of 0 and if f takes values in V, then $\int_K f\, d\mu \in \mu(K)\, V$.

(2) It is well known that $\int_K f\, d\mu$ always exists if X is a Banach space. In fact, the weak integral is equal to the integral obtained by the familiar construction using Riemann sums.

(3) For a large class of locally convex Hausdorff spaces, the existence of $\int_K f\, d\mu$ follows from [Rud91, Theorem 2.27]. There, the requirement on X is that the closed convex hull of a compact subset of X be compact. Rudin shows in [Rud91, Theorem 3.20 (c)] that all Fréchet spaces meet this condition. It turns out that this remains true in all quasi-complete locally convex Hausdorff spaces [Bou87, IV, p.37, Theorem 3].

(4) Although the results in this section are formulated for a rather general class of function spaces, we will ultimately be only concerned with Banach spaces of functions and with multiplier algebras of reproducing kernel Hilbert spaces, endowed with the weak or the strong operator topology. The existence of the integral in these cases follows from the preceding remarks. However, it seems worthwhile to give a direct argument for the latter case. That is, we wish to show that the integral exists if X is a WOT-closed subspace of $\mathcal{L}(H)$ for some Hilbert space H, equipped with the weak or the strong operator topology.

Notice that, by the Hahn-Banach theorem, it suffices to show that the integral exists in $\mathcal{L}(H)$. Moreover, it is well-known that the SOT-continuous linear functionals on $\mathcal{L}(H)$ coincide with the WOT-continuous ones, and are given by

$$\mathcal{L}(H) \to \mathbb{C}, \quad T \mapsto \sum_{i=1}^{r} \langle T x_i, y_i \rangle \tag{1.2}$$

for some vectors $x_1, \dots, x_r, y_1, \dots, y_r \in H$. Since being SOT-continuous is a stronger condition than being WOT-continuous for a function $f : K \to X$, it therefore suffices to consider the case where X is endowed with the weak operator topology.

One way to establish the existence of the integral is to argue that WOT-compact sets are always norm-bounded by the uniform boundedness principle, and since the unit ball of $\mathcal{L}(H)$ is WOT-compact and convex, the WOT-closed convex hull of WOT-compact sets is WOT-compact. Thus, we can apply [Rud91, Theorem 2.27].

A more elementary argument goes as follows: Define a sesquilinear form

$$(x, y) \mapsto \int_K \langle f(t)x, y \rangle \, d\mu(t)$$

on H. Using the uniform boundedness principle, we see that the sesquilinear form is continuous, so by the Lax-Milgram theorem, there exists an operator $T \in \mathcal{L}(H)$ such that

$$\langle T x, y \rangle = \int_K \langle f(t)x, y \rangle \, d\mu(t)$$

for all $x, y \in H$. Since WOT-continuous linear functionals are of the form (1.2), we conclude that $T = \int_K f \, d\mu$.

Having justified the use of vector-valued integration in admissible spaces, we can establish the desired homogeneous expansion of functions in these spaces. To shorten notation, let us write $\mathbb{C}[z]$ instead of $\mathbb{C}[z_1, \dots, z_d]$ if d is understood.

Proposition 1.6. *Let B be an admissible space of analytic functions on Ω. Let $f \in B$, and let $f = \sum_{n=0}^{\infty} f_n$ be the homogeneous expansion of f in a neighborhood of the origin.*

(a) *For all $n \in \mathbb{N}$, we have*

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f_t \, e^{-int} \, dt \in B. \tag{1.3}$$

For $n < 0$, the integral is zero.

(b) *The series* $\sum_{n=0}^{\infty} f_n$ *is Cesàro-convergent to* f *in the topology of* B. *In particular,* f *lies in the closed linear span of its homogeneous components.*

(c) B *contains* $B \cap \mathbb{C}[z]$ *as a dense subspace.*

Proof. (a) For $f \in B$ and $n \in \mathbb{Z}$, let $P_n(f)$ denote the right-hand side of equation (1.3). By the preceding remark, the integral makes sense. So $P_n(f) \in B$ is holomorphic, and since evaluation at a point in Ω is continuous, we have

$$P_n(f)(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}z)e^{-itn}\, dt$$

for all $z \in \Omega$, where the integral is just an ordinary complex-valued integral. Since the homogeneous expansion of f converges uniformly in a small ball $B_r(0) \subset \Omega$, we deduce that

$$P_n(f)(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} f_k(z)e^{i(k-n)t} dt = f_n(z)$$

for all $z \in B_r(0)$. The identity theorem shows that $P_n(f) = f_n$ on Ω. This observation finishes the proof of (a).

To show part (b), let

$$D_k(t) = \sum_{j=-k}^{k} e^{-ijt} \quad \text{and} \quad K_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt}$$

denote the Dirichlet and Fejér kernel, respectively. By part (a), we have

$$\frac{1}{2\pi} \int_0^{2\pi} f_t D_k(t)\, dt = \sum_{j=0}^{k} f_j$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} f_t K_n(t)\, dt = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{k} f_j.$$

Thus it suffices to show that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f_t K_n(t)\, dt = f_0 = f.$$

We adjust the proof for the normed case in [Kat68, Lemma I 2.2] to the locally convex case. First, we recall three properties of the Fejér kernel:

1. $K_n(t) \geq 0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$.
2. $\frac{1}{2\pi} \int_0^{2\pi} K_n(t)\, dt = 1$.
3. For all $0 < \delta < \pi$,

$$\lim_{n \to \infty} \int\limits_{\delta}^{2\pi - \delta} K_n(t)\, dt = 0.$$

The proof of these well known identities can be found in [Kat68, Lemma I 2.5]. Now, let p be a continuous semi-norm on B and let $\varepsilon > 0$ be arbitrary. By continuity of the map $t \mapsto f_t$, we find a $\delta > 0$ such that $p(f_t - f_0) < \frac{\varepsilon}{2}$ for all t with $|t| \leq \delta$. By the third property, there is an $N \in \mathbb{N}$ such that

$$\sup_{t \in [0, 2\pi]} p(f_t - f_0) \frac{1}{2\pi} \int\limits_{\delta}^{2\pi - \delta} K_n(t)\, dt < \frac{\varepsilon}{2}$$

for all $n \geq N$. Using this estimate and the second property, we obtain for all $n \geq N$

$$\begin{aligned} p\left(\frac{1}{2\pi} \int\limits_0^{2\pi} f_t K_n(t)\, dt - f_0 \right) &= p\left(\frac{1}{2\pi} \int\limits_0^{2\pi} K_n(t)(f_t - f_0)\, dt \right) \\ &\leq \frac{1}{2\pi} \int\limits_{-\delta}^{\delta} K_n(t) p(f_t - f_0)\, dt + \frac{1}{2\pi} \int\limits_{\delta}^{2\pi - \delta} K_n(t) p(f_t - f_0)\, dt \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where we have used the estimate from the first part of the preceding remark. This completes the proof of part (b), and (c) is an immediate consequence of (b). □

With the above proposition in hand, we can now deduce a Nullstellensatz for admissible spaces B of analytic function on Ω. For $J \subset B$, we write

$$V_\Omega(J) = \{z \in \Omega; f(z) = 0 \text{ for all } f \in J\}.$$

Conversely, for a set $X \subset \Omega$, we define

$$I_B(X) = \{f \in B; f(z) = 0 \text{ for all } z \in X\}.$$

Theorem 1.7. *Let B be an admissible space of functions containing the polynomials, and let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. Then*

$$I_B(V_\Omega(I)) = \overline{\sqrt{I}}.$$

Proof. One inclusion is elementary. If $f \in \sqrt{I}$, say $f^N \in I$ for some $N \in \mathbb{N}$, then f^N, and hence also f, vanish on $V_\Omega(I)$, so $f \in I_B(V_\Omega(I))$. Since evaluation at a point in Ω is continuous, $I_B(V_\Omega(I))$ is closed, from which we infer that $I_B(V_\Omega(I)) \supset \overline{\sqrt{I}}$.

Conversely, let $f \in I_B(V_\Omega(I))$, and let $f = \sum_{n=0}^\infty f_n$ be its homogeneous expansion in a neighborhood of 0. We will show that each f_n vanishes on $V(I)$. Since I is homogeneous,

$$\mathbb{C}V(I) = V(I).$$

In particular, given $z \in V_\Omega(I) = V(I) \cap \Omega$, we have $e^{it}z \in V_\Omega(I)$ for all $t \in \mathbb{R}$. It follows from Proposition 1.6 and continuity of the point evaluations that

$$f_n(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}z)e^{-int}\,dt = 0.$$

We conclude that f_n vanishes on $V(I)$, so $f_n \in \sqrt{I}$ by Hilbert's Nullstellensatz (see, for example, [Eis95, Theorem 1.6.]). Another application of Proposition 1.6 shows that f lies in the closed linear span of its homogeneous components, hence $f \in \overline{\sqrt{I}}$. □

1.3. Properties of the homogeneous expansion

Let B be an admissible space of analytic functions on a circular set Ω. According to Proposition 1.6, the homogeneous expansion of a function f in B is Cesàro-convergent to f in the topology of B. In this section, we will take a closer look at this homogeneous expansion, and the projections P_n on B sending a function to its homogeneous component of degree n.

Remark 1.8. By definition of P_n, we have

$$P_n(f) = \sum_{|\alpha|=n} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^{|\alpha|}}(0) z^\alpha$$

for all $f \in B$ and $n \in \mathbb{N}$. Since $\mathbb{C}[z]_n$, the space of homogeneous polynomials of degree n, is finite-dimensional, it follows that the projections P_n are continuous if and only if all linear functionals

$$B \to \mathbb{C}, \quad f \mapsto \frac{\partial^\alpha f}{\partial z^{|\alpha|}}(0)$$

are continuous.

The following lemma shows that the projections P_n are automatically continuous in many cases. The second part is useful especially in combination with Lemma 1.1.

Lemma 1.9. *Let B be an admissible space of functions.*

(a) *If B is barrelled or bornological, then convergence in B implies locally uniform convergence on Ω, and all projections P_n are continuous.*

(b) *If $(t, f) \mapsto f_t$ is jointly continuous as a map from $\mathbb{R} \times B$ into B, then all projection P_n are continuous.*

Proof. (a) Since B is quasi-complete, it is barreled in both cases. By continuity of the functions in B, the point evaluations over a compact set in Ω are point-wise bounded, hence they are equicontinuous. It follows that convergence in B implies locally uniform convergence, so the assertion follows from the preceding remark.

(b) By linearity of P_n, it is sufficient to proof continuity at 0. Given a closed absolutely convex neighborhood U of 0 in B, a simple compactness argument shows that there is a neighborhood V of 0 in B such that $g_t \in U$ for all $g \in V$ and $t \in [0, 2\pi]$. Hence the first part of Remark 1.5 shows that

$$P_n(g) = \frac{1}{2\pi} \int_0^{2\pi} g_t e^{-int}\, dt \in U$$

for all $g \in V$, so that P_n is continuous at 0. □

If H is a circular reproducing kernel Hilbert space, then the preceding lemma shows that the projections P_n are continuous. In fact, a stronger statement holds.

Lemma 1.10. *Let H be a circular reproducing kernel Hilbert space. Then the projections P_n sending a function to its homogeneous component of degree n are orthogonal projections onto mutually orthogonal spaces, and the series $\sum_{n=0}^{\infty} P_n$ converges to the identity in the strong operator topology.*

Proof. Equation (1.3) in Proposition 1.6 shows that the projections P_n are contractive, so they are orthogonal projections. It is clear that $P_n P_m = 0$ for $n \neq m$, hence the ranges of the projections are mutually orthogonal. It follows that $\sum_{n=0}^{\infty} P_n$ converges in the strong operator topology to an operator which acts as the identity on $\mathbb{C}[z] \cap H$. Since the latter is a dense subspace of H (see part (c) of Proposition 1.6), the assertion follows. □

Also for the multiplier algebra of a circular reproducing kernel Hilbert space, we obtain continuity of the projections.

Lemma 1.11. *Let H be a circular reproducing kernel Hilbert space containing the constant function 1. On $\mathrm{Mult}(H)$, endowed with the strong or the weak operator topology, the projections P_n are continuous.*

Proof. On H, all linear functionals

$$\lambda_\alpha : H \to \mathbb{C}, \quad f \mapsto \frac{\partial^\alpha f}{\partial z^{|\alpha|}}(0)$$

are continuous (this follows for example from part (a) in Lemma 1.9). It follows that

$$\mathrm{Mult}(H) \to \mathbb{C}, \quad \varphi \mapsto \frac{\partial^\alpha \varphi}{\partial z^{|\alpha|}}(0) = \lambda_\alpha(M_\varphi 1)$$

is WOT-continuous, and hence also SOT-continuous. So an application of Remark 1.8 finishes the proof. □

If the projections P_n are continuous, then we can go back and forth between homogeneous ideals of polynomials and their closures.

Lemma 1.12. *Let B be an admissible space of functions containing the polynomials, and suppose that the projections P_n are continuous. Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. If $f \in \overline{I} \subset B$ with homogeneous decomposition $f = \sum_{n=0}^\infty f_n$, then $f_n \in I$ for all $n \in \mathbb{N}$. In particular, $\overline{I} \cap \mathbb{C}[z] = I$.*

Proof. By homogeneity of I, each P_n maps I into $I \cap \mathbb{C}[z]_n$. Since the P_n are continuous, and since $I \cap \mathbb{C}[z]_n$ is finite-dimensional and hence closed in B, each P_n maps $\overline{I}$ into $I \cap \mathbb{C}[z]_n$, which shows the first assertion, and the second one obviously follows from this observation. □

In the setting of circular reproducing kernel Hilbert spaces, we can again say a bit more.

Lemma 1.13. *Let H be a circular reproducing kernel Hilbert space, and let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. If P_n denotes the projection sending a function in H to its homogeneous component of degree n, then $P_n P_{\overline{I}} = P_{\overline{I}} P_n$ holds for all $n \in \mathbb{N}$.*

Proof. Lemma 1.12 shows that $\overline{I}$ is invariant under each P_n. According to Lemma 1.10, each P_n is an orthogonal projection, so that $\overline{I}$ is reducing for P_n, which proves the assertion. □

We have seen in Proposition 1.6 that the homogeneous expansion of a function f in B is Cesàro-convergent to f in the topology of B. In general, the series need not converge itself. Nevertheless, it is possible to establish convergence in some cases. For example, Lemma 1.10 shows that the series is always convergent in circular reproducing kernel Hilbert spaces. The following simple lemma is sometimes useful to deduce convergence of a related series. It asserts that the coefficients of a Cesàro-convergent series, although possibly unbounded, cannot grow to rapidly.

Lemma 1.14. *Let E be a topological vector space and $(a_n)_n$ be a sequence in E such that $\sum_{n=0}^{\infty} a_n$ is Cesàro-convergent. Then*

$$\frac{a_n}{n} \xrightarrow{n\to\infty} 0.$$

Proof. Write

$$b_k = \sum_{j=0}^{k} a_j \quad \text{and } c_n = \frac{1}{n+1}\sum_{k=0}^{n} b_k.$$

Then $b_n = (n+1)c_n - nc_{n-1}$ and $a_n = b_n - b_{n-1}$. Since $(c_n)_n$ is convergent, we have

$$\frac{b_n}{n} = c_n - c_{n-1} + \frac{c_n}{n} \xrightarrow{n\to\infty} 0,$$

hence $\lim_{n\to\infty} \frac{a_n}{n} = 0$ as well. □

As a first application, we show that Proposition 1.6 implies the following complex analytic fact.

Lemma 1.15. *Let $\Omega \subset \mathbb{C}^d$ be a circular connected open set containing the origin. Let $f \in \mathcal{O}(\Omega)$, and let $f = \sum_{n=0}^{\infty} f_n$ be the homogeneous expansion of f in a neighborhood of the origin. Then the series $\sum_{n=0}^{\infty} f_n$ converges locally uniformly on Ω to f.*

Proof. Recall from Lemma 1.2 that $\mathcal{O}(\Omega)$ is admissible, so that Proposition 1.6 shows that the homogeneous expansion of f is Cesàro-convergent to f in the topology of $\mathcal{O}(\Omega)$. To show that the series is itself convergent, let $z_0 \in \Omega$ and $r > 0$ such that $K = \overline{B_{2r}(z_0)} \subset \Omega$. Then there is a $\lambda > 1$ with

$$\lambda\overline{B_r(z_0)} \subset \overline{B_{2r}(z_0)}.$$

Now Lemma 1.14 yields constant $M > 0$ such that

$$\sup_{z\in K} |p_n(z)| \leq Mn \quad \text{for all } n \geq 1.$$

Thus for $z \in \overline{B_r(z_0)}$, we have

$$|f_n(z)| = \frac{1}{\lambda^n}|f_n(\lambda z)| \leq \frac{1}{\lambda^n} nM.$$

Therefore, by the Weierstraß M-test, $\sum_{n=0}^{\infty} f_n$ converges uniformly on $\overline{B_r(z_0)}$. We conclude that the series $\sum_{n=0}^{\infty} f_n$ converges locally uniformly on Ω, and since ordinary convergence implies Cesàro convergence to the same limit, its limit is necessarily f. □

1.4. Algebras of analytic functions

In this section, we study the situation, when an admissible space B of analytic functions is in addition an algebra. The goal is to deduce a Nullstellensatz for ideals in B under suitable conditions. We will consider the following situation: B is an admissible algebra of functions on a circular set Ω which contains the polynomials and satisfies the following two properties:

(a) Multiplication in B is separately continuous.

(b) The projections P_n sending a function to its homogeneous component of degree n are continuous.

We will call such algebras *strongly admissible.* Note that the space $\mathcal{O}(\Omega)$ and all admissible Banach algebras are strongly admissible. This follows from part (a) of Lemma 1.9.

Condition (a) ensures, roughly speaking, that the multiplicative structure of B is compatible with the topology on B in some sense, and seems quite natural. Since we do not require joint continuity of multiplication, also multiplier algebras of circular reproducing kernel Hilbert spaces, endowed with the strong or the weak operator topology, are covered, see the lemma below. Condition (b) makes it possible to recover homogeneous ideals of polynomials from their closures (see Lemma 1.12), and will thus allow us to go back and forth between homogeneous ideals of polynomials and closed homogeneous ideals in B.

Lemma 1.16. *Let H be a circular reproducing kernel Hilbert space of analytic functions on Ω. Then $\mathrm{Mult}(H)$, equipped with the strong or the weak operator topology, is strongly admissible.*

Proof. We know that $\mathrm{Mult}(H)$ is admissible in both topologies, and it is well known that multiplication is separately continuous in both topologies. Continuity of the projections follows from Lemma 1.11. □

We will frequently need the following simple fact, whose proof is a straightforward application of separate continuity of multiplication.

Lemma 1.17. *Let B be a strongly admissible algebra of functions, and let X and Y be subsets of B. Then $\overline{X} \cdot \overline{Y} \subset \overline{X \cdot Y}$.* □

The following lemma is a preliminary statement relating ideals in $\mathbb{C}[z]$ to ideals in B. We say that an ideal $J \subset B$ is homogeneous, if for every $f \in J$, all homogeneous components $P_n(f)$ of f are again in J.

Lemma 1.18. *Let B be a strongly admissible algebra of functions on Ω. Then the maps*

$$\begin{aligned} \{\textit{homogeneous ideals in } \mathbb{C}[z]\} &\leftrightarrow \{\textit{closed homogeneous ideals in } B\} \\ I &\mapsto \overline{I} \\ J \cap \mathbb{C}[z] &\leftarrow\!\shortmid J. \end{aligned}$$

are bijections which are inverse to each other.

Proof. Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. Then $\mathbb{C}[z] \cdot I \subset I$, and since $\mathbb{C}[z]$ is a dense subset of B by Proposition 1.6, we conclude with the help of Lemma 1.17 that $B \cdot \overline{I} \subset \overline{I}$. Hence $\overline{I}$ is an ideal in B. Lemma 1.12 shows that $\overline{I}$ is also homogeneous, and that $\overline{I} \cap \mathbb{C}[z] = I$.

Conversely, let $J \subset B$ be a closed homogeneous ideal in B. Then it is obvious that $J \cap \mathbb{C}[z]$ is a homogeneous ideal in $\mathbb{C}[z]$, and as an application of Proposition 1.6, we see that $J = \overline{J \cap \mathbb{C}[z]}$. □

We will see that the correspondence between homogeneous ideals of polynomials and closed homogeneous ideals in B is compatible with forming radicals. We begin by showing that radicals of homogeneous ideals are again homogeneous.

Lemma 1.19. *In a strongly admissible algebra of functions, radicals of homogeneous ideals are homogeneous.*

Proof. Let B be a strongly admissible algebra, and let $J \subset B$ be a homogeneous ideal. To prove that $\sqrt{J}$ is homogeneous, let $f \in \sqrt{J}$, say $f^N \in J$ for some natural number N. If

$$f = \sum_{k=n}^{\infty} f_k$$

is the homogeneous decomposition of f in a neighborhood of the origin, where $f_n \neq 0$, we have to show that each f_k belongs to $\sqrt{J}$. For $j \in \mathbb{N}$, let R_j denote the closed homogeneous ideal $\overline{\bigoplus_{k=j}^{\infty} \mathbb{C}[z]_k}$. As an application of Proposition 1.6, we see that we can write $f = f_n + r$, where $r \in R_{n+1}$. Lemma 1.17 shows that $R_i R_j \subset R_{i+j}$ for $i, j \in \mathbb{N}$, so that $f^N = f_n^N + \widetilde{r}$ for some $\widetilde{r} \in R_{nN+1}$. Since J is homogeneous, we deduce that $f_n^N \in J$, and hence $f_n \in \sqrt{J}$. The homogeneity of $\sqrt{J}$ now follows recursively by considering $f - f_n$. □

Showing that radicals of closed homogeneous ideals are again closed is less straightforward. In the proof, we will make use of the Noetherian property of the polynomial ring $\mathbb{C}[z]$.

Lemma 1.20. *In a strongly admissible algebra of functions, radicals of closed homogeneous ideals are closed.*

Proof. Let B be a strongly admissible algebra and let $J \subset B$ be a homogeneous ideal. Suppose that $f \in \overline{\sqrt{J}}$. We need to show that $f^N \in J$ for some natural number N. The main issue is the following: If we approximate f by elements in $\sqrt{J}$, then a suitable power of each approximating element lies in J. However, this process might involve arbitrarily large exponents, so that we cannot immediately conclude that a power of f is contained in J. We will circumvent this problem by passing to $\sqrt{J} \cap \mathbb{C}[z]$, and using the Noetherian property of $\mathbb{C}[z]$.

First, we note that since $\sqrt{J}$ is homogeneous by Lemma 1.19, an application of Proposition 1.6 shows that

$$\overline{\sqrt{J}} = \overline{\sqrt{J} \cap \mathbb{C}[z]}.$$

Moreover, it is elementary that $\sqrt{J} \cap \mathbb{C}[z] = \sqrt{J \cap \mathbb{C}[z]}$. Since $\mathbb{C}[z]$ is Noetherian, the ideal $I = J \cap \mathbb{C}[z]$ contains a power of its radical [AM69, Proposition 7.14.], say $\sqrt{I}^N \subset I$ for some $N \in \mathbb{N}$. Since $f \in \overline{\sqrt{I}}$, an obvious inductive application of Lemma 1.17 thus shows that

$$f^N \in \overline{\sqrt{I}^N} \subset \overline{I} = J,$$

as desired. $\square$

It is well known that radicals of homogeneous ideals in $\mathbb{C}[z]$ are again radical. In fact, the proof is similar to the one given for ideals in B. Hence both sets in Lemma 1.18 are closed with respect to forming radicals. This allows for a refinement of Lemma 1.18, from which the desired Nullstellensatz for ideals in B will easily follow.

Proposition 1.21. *Let B be a strongly admissible algebra of functions on Ω. Then the maps*

$$\begin{aligned} \{\textit{homogeneous ideals in } \mathbb{C}[z]\} &\leftrightarrow \{\textit{closed homogeneous ideals in } B\} \\ I &\mapsto \overline{I} \\ J \cap \mathbb{C}[z] &\leftarrow\!\shortmid J. \end{aligned}$$

are bijections which are inverse to other. These bijections are compatible with forming radicals. In particular, radical ideals in B correspond to radical ideals in $\mathbb{C}[z]$.

Proof. In view of Lemma 1.18 and the above discussion, it suffices to show that the correspondence is compatible with forming radicals. It is clear that

$$\sqrt{J} \cap \mathbb{C}[z] = \sqrt{J \cap \mathbb{C}[z]}$$

for all closed homogeneous ideals $J \subset B$. Conversely, let $I \subset \mathbb{C}[z]$ be a homogeneous ideal, and set $J = \overline{I} \subset B$. Then J and $\sqrt{J}$ are closed homogeneous ideals in B, the latter by Lemma 1.19 and Lemma 1.20. Hence Lemma 1.18 shows that

$$\overline{\sqrt{I}} = \overline{\sqrt{J \cap \mathbb{C}[z]}} = \overline{\sqrt{J} \cap \mathbb{C}[z]} = \sqrt{J} = \sqrt{\overline{I}},$$

which finishes the proof. $\square$

We are now in the position to deduce a Nullstellensatz for closed homogeneous ideals in strongly admissible algebras of functions.

Theorem 1.22. *Let B be a strongly admissible algebra of functions on Ω and let $J \subset B$ be a closed homogeneous ideal. Then*

$$I_B(V_\Omega(J)) = \sqrt{J}.$$

Proof. Let $I = J \cap \mathbb{C}[z]$. Then I is a homogeneous ideal in $\mathbb{C}[z]$ satisfying $J = \overline{I}$ and $\sqrt{J} = \overline{\sqrt{I}}$ by Proposition 1.21. Since the point evaluations are continuous, we have $V_\Omega(J) = V_\Omega(I)$. Hence we infer from the Nullstellensatz (Theorem 1.7) that

$$I_B(V_\Omega(J)) = I_B(V_\Omega(I)) = \overline{\sqrt{I}} = \sqrt{J}. \qquad \square$$

2. Universal operator algebras

2.1. The Drury-Arveson space

In this section, we recall the definition and some properties of the Drury-Arveson space. This space was discovered by Drury [Dru78] and Arveson [Arv98], who put it into the context of reproducing kernel Hilbert spaces and symmetric Fock space. It turns out that for many questions in multivariate operator theory, the Drury-Arveson space, and not the Hardy space on the unit ball, is the "correct" generalization of the Hardy space on the unit disk.

Definition 2.1. The Drury-Arveson space H^2_d is defined as the reproducing kernel Hilbert space on the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ with reproducing kernel

$$K(z,w) = \frac{1}{1 - \langle z, w \rangle}.$$

The expansion

$$K(z,w) = \sum_{n=0}^{\infty} \langle z, w \rangle^n$$

for $z, w \in \mathbb{B}_d$ shows that K is positive definite in the sense of Definition A.2. so that there is indeed a Hilbert function space with reproducing kernel K. Note that if $d = 1$, then H^2_d is the classical Hardy space on the unit disk. It is obvious that K is invariant under multiplication by complex scalars of modulus one, thus H^2_d is a circular reproducing kernel Hilbert space in the sense of the discussion preceding Lemma 1.4. A concrete description of the functions belonging to H^2_d is given by the following lemma.

Lemma 2.2. *For the Drury-Arveson space H^2_d, the following assertions are true:*

(a) *H^2_d is a space of holomorphic functions on $\mathbb{B}_d$, and it contains the polynomials as a dense subspace.*

(b) *If $f = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}_d)$, then $f \in H^2_d$ if and only if*

$$||f||^2 = \sum_{\alpha \in \mathbb{N}^d} \frac{|a_\alpha|^2}{\gamma_\alpha} < \infty,$$

where $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$.

(c) *For $\alpha, \beta \in \mathbb{N}^d$, we have $\gamma_\alpha \langle z^\alpha, z^\beta \rangle = \delta_{\alpha\beta}$. In particular, different monomials and hence homogeneous polynomials of different degree are orthogonal in H^2_d.*

Proof. See Section 1 and Lemma 3.8 in [Arv98]. □

As mentioned before, H^2_d is in some sense the right generalization of the Hardy space on the unit disk for the purposes of multivariate operator theory. The role of the unilateral shift is played by the operator tuple introduced in the following lemma. Recall that a d-tuple $T = (T_1, \dots, T_d)$ of bounded linear operators on a Hilbert space H is called a row contraction if the row operator

$$H^d \to H, \quad (x_i)_{i=1}^d \mapsto \sum_{i=1}^d T_i x_i$$

is a contraction. Equivalently, $\sum_{i=1}^d T_i T_i^* \leq 1$.

Lemma 2.3. *The coordinate functions z_i are multipliers on H^2_d, and the commuting operator tuple $M_z = (M_{z_1}, \dots, M_{z_d})$ is a row contraction. More precisely, $\sum_{i=1}^d M_{z_i} M_{z_i}^* = 1 - P_0$, where P_0 denotes the orthogonal projection from H^2_d onto the constant functions.*

Proof. See Section 2 in [Arv98]. □

The tuple M_z is usually called the d-shift. Note that if $d = 1$, the operator M_z is just the unilateral shift. In the following, we will denote the unital non-selfadjoint norm-closed algebra generated by M_z by $\mathcal{A}_d$. Identifying a polynomial p with $p(M_z) = M_p$, where M_p is the multiplication operator on H^2_d induced by p, we can regard $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_d]$ as a subalgebra of $\mathcal{A}_d$. Since the multiplier norm on H^2_d dominates the sup norm on the open unit ball by Lemma A.9 (b), we can naturally think of elements in $\mathcal{A}_d$ as continuous functions on the closed unit ball. Recall that $\mathcal{A}_1$ is the disk algebra $A(\mathbb{D})$. The following generalization of the celebrated von Neumann inequality is due to Drury [Dru78] and Arveson [Arv98].

Theorem 2.4. *Let $T = (T_1, \dots, T_n)$ be a commuting row contraction on a Hilbert space H. Then the algebra homomorphism*

$$\mathbb{C}[z] \to \mathcal{L}(H), \quad p \mapsto p(T_1, \dots, T_d)$$

extends to a completely contractive representation of $\mathcal{A}_d$. In particular, if $p \in \mathbb{C}[z]$, then

$$||p(T_1, \dots, T_d)|| \leq ||p(M_z)|| = ||p||_{\mathrm{Mult}(H^2_d)}.$$

Proof. See [Arv98, Theorem 8.1], and also [EP02, Corollary 8,5]. □

Recall that the multiplier algebra of the Hardy space $H^2(\mathbb{D})$ can be isometrically identified with $H^\infty(\mathbb{D})$, so that for $d = 1$, the last inequality really is von Neumann's inequality. For $d \geq 2$, however, the multiplier norm on H^2_d is in general strictly larger then the sup norm on $\overline{\mathbb{B}_d}$ (see [Arv98, Theorem 3.3]), which is the multiplier norm on the Hardy space on the unit ball. In particular, the natural analogue of von Neumann's inequality for commuting row contractions $T = (T_1, \ldots, T_d)$, namely

$$||p(T_1, \ldots, T_d)|| \leq ||p||_{\overline{\mathbb{B}_d}}$$

for all polynomials p, does not hold when $d \geq 2$. The failure of this inequality for general d was already observed by Varopoulos [Var74].

The Drury-Arveson space shares another property with the Hardy space on the unit disk, which the Hardy space on the unit balls fails to fulfill: H^2_d is a complete Nevanlinna-Pick space (see, for example, [AM02, Theorem 7.28]). In fact, H^2_∞, which is defined as the reproducing kernel Hilbert space on the unit ball in ℓ^2 with kernel $K(z, w) = (1 - \langle z, w\rangle)^{-1}$, satisfies a certain universal property among complete Nevanlinna-Pick spaces (see, for example, [AM02, Theorem 8.2]).

Closely related to the fact that H^2_d is a complete Nevanlinna-Pick space is the presence of the following commutant lifting theorem. The connection between commutant lifting and Nevanlinna-Pick interpolation goes back to work of Sarason [Sar67].

Theorem 2.5. *Let $\mathcal{E}$ be a finite dimensional Hilbert space, and let $X \subset H^2_d \otimes \mathcal{E}$ be a closed subspace which is invariant under each $M_\varphi^* \otimes 1_{\mathcal{E}}$ for $\varphi \in \mathrm{Mult}(H^2_d)$. Suppose that $T \in \mathcal{L}(X)$ is a contraction with*

$$T P_X (M_\varphi \otimes 1)\big|_X = P_X (M_\varphi \otimes 1) T$$

for all $\varphi \in \mathrm{Mult}(H^2_d)$. Then there exists a multiplier $\Phi \in \mathrm{Mult}(H^2_d \otimes \mathcal{E})$ such that $||\Phi|| \leq 1$ and

$$T = P_X M_\Phi\big|_X.$$

Proof. See [BTV01, Theorem 5.1], and also [AT02, Theorem 2] for a simplified proof in the scalar case. A proof using a non-commutative commutant lifting theorem of Popescu is given in [DP98b, Proposition 4.4]. □

Arveson [Arv98] observed that the Drury-Arveson space can be identified with the symmetric Fock space over $\mathbb{C}^d$. This description of H^2_d is sometimes more convenient than the definition as a space of holomorphic functions on the unit ball. To indicate how this can be done, we need some preliminaries.

Let $E = \mathbb{C}^d$, and let n be a natural number. We denote the symmetric group on n letters by S_n. For a permutation $\sigma \in S_n$, let U_σ be the unique unitary operator on $E^{\otimes n}$ satisfying

$$U_\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$$

for all $v_1, \ldots, v_n \in E$.

Definition 2.6. The symmetric n-fold tensor power of E is defined by

$$E^n = \{\omega \in E^{\otimes n} : U_\sigma \omega = \omega \text{ for all } \sigma \in S_n\}.$$

The Hilbert space

$$\mathcal{F}_s(E) = \bigoplus_{n=0}^{\infty} E^n$$

is called the *symmetric Fock space* over E, and

$$\mathcal{F}(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

is called the *full Fock space* over E.

The orthogonal projection from $E^{\otimes n}$ onto the symmetric tensor power E^n can be easily expressed as an average of the unitary operators U_σ.

Lemma 2.7. *Let E be a finite dimensional Hilbert space, and let $n \in \mathbb{N}$. The orthogonal projection from $E^{\otimes n}$ onto E^n is given by*

$$Q_n \omega = \frac{1}{n!} \sum_{\sigma \in S_n} U_\sigma \omega$$

for $\omega \in E^{\otimes n}$.

Proof. Clearly Q_n acts as the identity on E^n, and since $U_\sigma U_\tau = U_{\tau \circ \sigma}$ for $\sigma, \tau \in S_n$, the image of Q_n is contained in E^n. From the identity $U_\sigma^* = U_{\sigma^{-1}}$ for $\sigma \in S_n$, we infer that Q_n is self-adjoint, which finishes the proof. $\square$

We can now describe the identification of H^2_d with $\mathcal{F}_s(\mathbb{C}^d)$. A moment's reflection about the respective degree one parts reveals that the natural choice is to make the isomorphism from H^2_d onto $\mathcal{F}_s(\mathbb{C}^d)$ anti-linear instead of linear. Indeed, homogeneous polynomials of degree one act naturally as linear forms on $\mathbb{C}^d$, so that the degree one part of $\mathbb{C}[z]$ should be identified with the dual space of $\mathbb{C}^d$.

We will write $Q = \bigoplus_{n=0}^{\infty} Q_n$ for the orthogonal projection from $\mathcal{F}(\mathbb{C}^d)$ onto $\mathcal{F}_s(\mathbb{C}^d)$. Moreover, for a d-tuple $e = (e_1, \ldots, e_d)$ of vectors in $\mathbb{C}^d$ and a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we set $e^\alpha = Q(e^{\otimes \alpha}) = Q(e_1^{\alpha_1} \otimes \ldots \otimes e_d^{\alpha_d})$.

Proposition 2.8. *There is a unique anti-unitary operator $J : H^2_d \to \mathcal{F}_s(\mathbb{C}^d)$ with*

$$J(\langle \cdot, \lambda \rangle^n) = \lambda^{\otimes n} \tag{2.1}$$

for all $\lambda \in \mathbb{C}^d$ and $n \in \mathbb{N}$. The operator J has the following properties:

(a) *If $e = (e_1, \ldots, e_d)$ is the usual basis of $\mathbb{C}^d$, then $J(z^\alpha) = e^\alpha$ for all $\alpha \in \mathbb{N}^d$.*

(b) *For homogeneous polynomials p and q, we have $J(p \cdot q) = Q(J(p) \otimes J(q))$.*

(c) *For a linear map A on $\mathbb{C}^d$ and a homogeneous polynomial p of degree n, we have $J(p \circ A^*) = A^{\otimes n}(J(p))$.*

Proof. Uniqueness follows from the fact that the set of kernel functions

$$K(\cdot, \lambda) = \sum_{n=0}^{\infty} \langle \cdot, \lambda \rangle^n,$$

where $\lambda \in \mathbb{B}_d$, is total in H^2_d (see Lemma A.4 (c)). To establish existence of J, set $E = \mathbb{C}^d$. A simple combinatorial argument shows that for $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| = |\beta| = n$, we have

$$\begin{aligned}\langle e^\alpha, e^\beta \rangle_{E^{\otimes n}} &= \sum_{\sigma \in S_n} \frac{1}{n!} \langle U_\sigma \big(e_1^{\alpha_1} \otimes \ldots \otimes e_d^{\otimes \alpha_d}\big), e_1^{\beta_1} \otimes \ldots \otimes e_d^{\alpha_d} \rangle_{E^{\otimes n}} \\ &= \delta_{\alpha\beta} \frac{\alpha!}{n!} = \langle z^\alpha, z^\beta \rangle_{H^2_d},\end{aligned}$$

from which we conclude that there exists a unique anti-unitary operator J from H^2_d onto $\mathcal{F}_s(\mathbb{C}^d)$ which satisfies property (a).

Next, we show that the just defined operator J satisfies (b). By linearity, it is enough to prove the assertion if p and q are monomials, say $p = z^\alpha$ and $q = z^\beta$ with $|\alpha| = n$ and $|\beta| = m$. Using the identity $QU_\sigma = Q$ for all $\sigma \in S_{n+m}$, we obtain

$$J(z^{\alpha+\beta}) = Q\, e^{\otimes(\alpha+\beta)} = Q(e^{\otimes\alpha} \otimes e^{\otimes\beta}) = Q(e^\alpha \otimes e^\beta) = Q(J(z^\alpha) \otimes J(z^\beta)),$$

as desired.

Having established this fact, we can now show that (2.1) holds. So let $\lambda \in \mathbb{C}^d$, and note that with the usual convention $\lambda^{\otimes 0} = 1 \in \mathbb{C}$, the case $n = 0$ is trivial. By construction of J, (2.1) is valid for $n = 1$. Thus part (b) and an obvious inductive argument show that (2.1) holds for all $n \in \mathbb{N}$.

Part (c) now follows by linearity from the identity

$$J(\langle \cdot, \lambda \rangle^n \circ A^*) = J(\langle \cdot, A\lambda \rangle^n) = (A\lambda)^{\otimes n} = A^{\otimes n} J(\langle \cdot, \lambda \rangle^n)$$

for $\lambda \in \mathbb{C}^d$ and $n \in \mathbb{N}$. □

As a first application, we show that the multiplier norm coincides with the norm of H^2_d for homogeneous polynomials (compare [SS09, Lemma 9.5]).

Lemma 2.9. *For any homogeneous polynomial p, we have $||p||_{H^2_d} = ||p||_{\mathrm{Mult}(H^2_d)}$.*

Proof. Let us write $||\cdot||$ for the norm in H^2_d. The inequality $||p|| \le ||p||_{\mathrm{Mult}(H^2_d)}$ is trivial because $||1|| = 1$. To establish the reverse inequality, we first note that the multiplication operator M_p maps $\mathbb{C}[z]_m$, the space of homogeneous polynomials of degree m, into $\mathbb{C}[z]_{n+m}$ for all natural numbers m. Since homogeneous polynomials of different degrees are orthogonal in H^2_d, it suffices to show that $||p\cdot q|| \le ||p||\,||q||$ holds for all homogeneous polynomials $q \in \mathbb{C}[z]_m$ and all $m \in \mathbb{N}$. But this follows at once from part (b) of Proposition 2.8. Indeed,

$$||p\cdot q|| = ||Q(J(p)\otimes J(q))||_{E^{n+m}} \le ||J(p)\otimes J(q)||_{E^n\otimes E^m} = ||p||\,||q||. \qquad \square$$

Example 2.10. Of course, the estimate $||p\cdot q|| \le ||p||\,||q||$ need not be true for non-homogeneous polynomials p and q, so that in general, $||p||_{\mathrm{Mult}(H^2_d)} > ||p||$. To give a concrete example, let $d = 1$ and $p = q = 1 + z$. Then

$$||p\cdot q||^2 = ||1 + 2z + z^2||^2 = 6 > 4 = ||p||^2\,||q||^2.$$

Remark 2.11. For homogeneous polynomials of degree 1, the statement of the preceding lemma is also a direct consequence of the fact that M_z is a row contraction (see Lemma 2.3). To show the non-trivial inequality, let $p = \sum_{i=1}^d a_i z_i$ with $a_i \in \mathbb{C}$, and note that for $q \in \mathbb{C}[z]$, we have

$$||p\cdot q||^2 = \Big|\Big|\sum_{i=1}^d z_i\cdot(a_i q)\Big|\Big|^2 \le \sum_{i=1}^d |a_i|^2\,||q||^2 = ||p||^2\,||q||^2.$$

2.2. Universal operator algebras for row contractions

According to Arveson's von Neumann inequality (Theorem 2.4), the unital non-selfadjoint norm-closed algebra $\mathcal{A}_d$ generated by the d-shift $S = M_z$ on the Drury-Arveson space H^2_d is the universal operator algebra generated by a commuting row contraction in the following sense: For any commuting row contraction $T = (T_1,\dots,T_d)$ on a Hilbert space H, there is a unique unital completely contractive algebra homomorphism

$$\Phi : \mathcal{A}_d \to \mathcal{L}(H) \quad \text{with} \quad \Phi(S_i) = T_i \quad \text{for } i = 1,\dots,d.$$

Suppose now that $I \subset \mathbb{C}[z_1,\dots,z_d]$ is a homogeneous ideal. Theorem 2.4 immediately implies the existence of a universal object for commuting row contractions subject to the relations in I. Note that if $N \subset M \subset \mathcal{L}(H)$ are closed subspaces for some Hilbert space H, then the quotient M/N can be endowed with a family of matrix norms by identifying $M_n(M/N)$ with $M_n(M)/M_n(N)$.

Corollary 2.12. *Let $I \subset \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous ideal. Suppose that $T = (T_1, \dots, T_d)$ is a commuting row contraction on a Hilbert space H satisfying $p(T) = 0$ for all $p \in I$. Then there is a unique unital completely contractive algebra homomorphism*

$$\Phi : \mathcal{A}_d/\widetilde{I} \to \mathcal{L}(H) \quad \textit{with} \quad \Phi([S_i]) = T_i \quad \textit{for } i = 1, \dots, d,$$

where $\widetilde{I}$ denotes the closure of I in $\mathcal{A}_d$. □

From a certain point of view, this result is not completely satisfactory, since the algebra $\mathcal{A}_d/\widetilde{I}$ is not an algebra of operators on a Hilbert space. In particular, the equivalence class of the tuple $S = M_z$, which is the universal row-contractive solution of the equations in I in the above sense, is not an operator tuple. The algebra $\mathcal{A}_d/\widetilde{I}$ is, however, an abstract operator algebra in the sense of Blecher, Ruan and Sinclair (see, for example, [ER00, Chapter 17]), hence it is completely isometrically isomorphic to a concrete operator algebra.

There is also a direct way of identifying $\mathcal{A}_d/\widetilde{I}$ with a concrete operator algebra. To this end, let $\mathcal{F}_I = H^2_d \ominus I$. Note that $\mathcal{F}_I$ is co-invariant for M_z, so that the compressions

$$S_i^I = P_{\mathcal{F}_I} M_{z_i}\big|_{\mathcal{F}_I}$$

form a commuting row contraction $S^I = (S_1^I, \dots, S_d^I) \in \mathcal{L}(\mathcal{F}_I)^n$. If $p \in I$, then M_p maps H^2_d into $\overline{I}$, so that

$$p(S^I) = P_{\mathcal{F}_I} M_p\big|_{\mathcal{F}_I} = 0,$$

that is, the d-tuple S^I satisfies the relations in I. Let $\mathcal{A}_I$ denote the unital non-selfadjoint norm-closed algebra generated by S^I. The goal of this section is to show that there is a completely isometric algebra isomorphism

$$\Phi : \mathcal{A}_d/\widetilde{I} \to \mathcal{A}_I \quad \text{with} \quad \Phi([S_i]) = S_i \quad \text{for } i = 1, \dots, d.$$

In view of Corollary 2.12, this means that $\mathcal{A}_I$ is the universal operator algebra generated by a commuting row contraction subject to the relations in I.

Let us begin by observing that $\mathcal{A}_d/\widetilde{I}$ need not be isomorphic to $\mathcal{A}_I$ for non-homogeneous ideals $I \subset \mathbb{C}[z]$.

Example 2.13. Let $d = 1$ and $I = \langle z - 1 \rangle \subset \mathbb{C}[z]$. It is easy to check that $H^2_1 \ominus I = \{0\}$, thus $\mathcal{A}_I = \{0\}$. On the other hand, $\mathcal{A}_1$ is the disk algebra $A(\mathbb{D})$, and

$$\delta_1 : \mathcal{A}_1 \to \mathbb{C}, \quad f \mapsto f(1)$$

is a non-trivial continuous linear functional on $\mathcal{A}_1$. Hence $\widetilde{I} \subset \ker(\delta_1) \neq \mathcal{A}_1$, that is, $\mathcal{A}_1/\widetilde{I} \neq \{0\}$.

To show that $\mathcal{A}_d/\widetilde{I}$ and $\mathcal{A}_I$ are isomorphic if I is homogeneous, we will use the commutant lifting theorem for H^2_d (Theorem 2.5), and the following lemma for approximating multipliers of H^2_d by elements in $\mathcal{A}_d$. If $f : \mathbb{B}_d \to X$ is a map, where X is a set, and if $0 < r < 1$, we define

$$f_{(r)} : \mathbb{B}_d \to X, \quad z \mapsto f(rz).$$

For the notion of a vector-valued multiplier, we refer to Section A.3 in the appendix.

Lemma 2.14. *Let $0 < r < 1$ and let $\mathcal{E}$ be a finite dimensional Hilbert space.*

(a) *Let $\varphi \in \operatorname{Mult}(H^2_d)$. Then $\varphi_{(r)} \in \mathcal{A}_d$, and if $\varphi = \sum_{n=0}^\infty \varphi_n$ is the homogeneous expansion of φ, then the series*

$$\varphi_{(r)} = \sum_{n=0}^\infty r^n \varphi_n$$

converges in the norm of $\mathcal{A}_d$.

(b) *If $\Phi \in \operatorname{Mult}(H^2_d \otimes \mathcal{E})$ is a vector-valued multiplier with $||\Phi||_M \leq 1$, then $\Phi_{(r)} \in \operatorname{Mult}(H^2_d \otimes \mathcal{E})$ with $||\Phi_{(r)}||_M \leq 1$.*

Proof. (a) By Lemma 1.4 (b) and Proposition 1.6, the series $\sum_{n=0}^\infty \varphi_n$ is Cesàro-convergent to φ in the strong operator topology. Consequently, Lemma 1.14 shows that the sequence $(\frac{\varphi_n}{n})_n$ converges to zero in the strong operator topology. Therefore, by the uniform boundedness principle, there is a constant $M > 0$ such that

$$||\varphi_n||_M \leq Mn \quad \text{for all } n \geq 1.$$

We conclude that the series $\sum_{n=0}^\infty r^n \varphi_n$ converges absolutely in the Banach algebra $\operatorname{Mult}(H^2_d)$ to a function $\psi \in \mathcal{A}_d$. By evaluating ψ at points in $\mathbb{B}_d$, we see that $\psi = \varphi_{(r)}$.

(b) We will use the characterization of contractive multipliers from Lemma A.12, and the Schur product theorem (Theorem A.13). So let $K(z,w) = (1 - \langle z, w\rangle)^{-1}$ be the reproducing kernel of H^2_d. Since Φ is a contractive multiplier, the map

$$\mathbb{B}_d \times \mathbb{B}_d \to \mathcal{L}(\mathcal{E}), \quad (z,w) \mapsto K(rz,rw)(1 - \Phi(rz)\Phi(rw)^*)$$

is positive definite. Moreover,

$$(z,w) \mapsto \frac{K(z,w)}{K(rz,rw)} = \frac{1 - r^2\langle z, w\rangle}{1 - \langle z, w\rangle} = 1 + (1 - r^2)\langle z, w\rangle \frac{1}{1 - \langle z, w\rangle}$$

defines a positive definite map from $\mathbb{B}_d \times \mathbb{B}_d$ to $\mathbb{C}$, for example by the Schur product theorem. Consequently, another application of the (vector-valued) Schur product theorem shows that

$$\begin{aligned}\mathbb{B} \times \mathbb{B} \to \mathcal{L}(\mathcal{E}), (z,w) \mapsto & K(z,w)(1 - \Phi(rz)\Phi(rw)^*) \\ &= \frac{K(z,w)}{K(rz,rw)} K(rz,rw)(1 - \Phi(rz)\Phi(rw)^*)\end{aligned}$$

is also positive definite, so that $\Phi_{(r)}$ is a contractive multiplier on $H^2_d \otimes \mathcal{E}$. □

Remark 2.15. Part (a) of the preceding lemma is a special case of the following much more general fact: Any function $\varphi : \mathbb{B}_d \to \mathbb{C}$ which admits a holomorphic extension to a neighborhood of the closed unit ball is an element of $\mathcal{A}_d$, and if $\varphi = \sum_{n=0}^\infty \varphi_n$ is the homogeneous expansion of φ, which is uniformly convergent in a neighborhood of the closed unit ball, then the series $\varphi = \sum_{n=0}^\infty \varphi_n$ is also convergent in the multiplier norm of H^2_d.

This assertion can be shown using the analytic functional calculus for commuting tuples of operators (see [EP96, Theorem 2.5.7]). An outline of the proof goes as follows: The Taylor spectrum of M_z is the closed unit ball (see [GRS05, Proposition 2.6]), so that any function φ as above defines a bounded linear operator $\varphi(M_z)$ on H^2_d. The continuity properties of the analytic functional calculus imply that $\varphi(M_z) = \sum_{n=0}^\infty \varphi_n(M_z) = \sum_{n=0}^\infty M_{\varphi_n}$ with convergence in $\mathcal{L}(H^2_d)$. We conclude that $\varphi(M_z)$ is a multiplication operator on H^2_d, and by evaluating $\varphi(M_z)1$ at points in $\mathbb{B}_d$, we see that $\varphi(M_z) = M_\varphi$.

In the discussion following Corollary 2.12, we have used that if I is a homogeneous ideal, then $\mathcal{F}_I$ is co-invariant under each M_{z_i}, and that multiplication by elements in I maps H^2_d into $\overline{I}$. The following lemma contains more general results along these lines.

Lemma 2.16. *Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal, and let $\overline{I}^{SOT}$ denote the SOT-closure of I in* $\operatorname{Mult}(H^2_d)$.

(a) $\overline{I}^{SOT} \cdot H^2_d \subset \overline{I}$ *and* $\operatorname{Mult}(H^2_d) \cdot \overline{I} \subset \overline{I}$.

(b) *If $\varphi \in \operatorname{Mult}(H^2_d)$, then $P_{\mathcal{F}_I} M_\varphi \big|_{\mathcal{F}_I} = 0$ if and only if $\varphi \in \overline{I}^{SOT}$. In this case, $\varphi_{(r)} \in \widetilde{I}$ for $0 < r < 1$.*

(c) *If $p \in \mathbb{C}[z]$ is a polynomial, then $p(S^I) = 0$ if and only if $p \in I$.*

Proof. (a) If $\varphi \in I$, then $M_\varphi(\mathbb{C}[z]) \subset I$, hence $M_\varphi(H^2_d) \subset \overline{I}$. The first assertion thus follows by approximating a multiplier in $\overline{I}^{SOT}$ by elements in I. The second assertion is proved similarly: If $\varphi \in \mathbb{C}[z]$, then $M_\varphi(I) \subset I$, hence $M_\varphi(\overline{I}) \subset \overline{I}$. Since $\mathbb{C}[z]$ is

SOT-dense in $\mathrm{Mult}(H^2_d)$ (see Lemma 1.4 (b) and Proposition 1.6), we conclude that $\overline{I}$ is invariant for all multipliers.

(b) The "if"-part follows from the first assertion in (a). Conversely, given a multiplier φ with $P_{F_I} M_\varphi\big|_{\mathcal{F}_I} = 0$, the operator M_φ maps $\mathcal{F}_I$ into $\overline{I}$. Combined with the second assertion in (b), we infer that M_φ maps H^2_d into $\overline{I}$. In particular, $\varphi = M_\varphi 1 \in \overline{I}$. Thus, if $\varphi = \sum_{n=0}^\infty \varphi_n$ is the homogeneous expansion of φ, we have $\varphi_n \in I$ for all $n \in \mathbb{N}$ by Lemma 1.12. So the first assertion follows from the fact that the homogeneous expansion of φ is Cesàro-convergent to φ in the strong operator topology (see Lemma 1.4 (b) and Proposition 1.6). Moreover, Lemma 2.14 (a) shows that

$$\varphi_{(r)} = \sum_{n=0}^\infty r^n \varphi_n \in \widetilde{I}$$

for $0 < r < 1$.

Part (c) is an immediate consequence of part (b) and its proof, since for polynomials, the homogeneous expansion is just a finite sum. $\square$

We are now in the position to prove that the completely contractive homomorphism from $\mathcal{A}_d/\widetilde{I}$ into $\mathcal{A}_I$ given by Corollary 2.12 is a completely isometric isomorphism.

Theorem 2.17. *Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. There is a unique unital completely isometric isomorphism*

$$\mathcal{A}_d/\widetilde{I} \to \mathcal{A}_I$$

mapping $[M_{z_i}]$ to S_i^I for $i = 1, \ldots, d$. It is given by

$$\mathcal{A}_d/\widetilde{I} \to \mathcal{A}_I, \quad [M_\varphi] \mapsto P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}.$$

Proof. Uniqueness is clear. To prove existence, note that

$$\mathcal{A}_d \to \mathcal{A}_I, \quad M_\varphi \mapsto P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$$

defines a unital completely contractive homomorphism, and that its kernel contains I. Thus it induces a unital completely contractive homomorphism from $\mathcal{A}_d/\widetilde{I}$ into $\mathcal{A}_I$ (alternatively, this follows from Corollary 2.12).

We have to show that this homomorphism is in fact completely isometric and surjective, or equivalently, that the map

$$A = \{p(S^I) : p \in \mathbb{C}[z]\} \to \mathcal{A}_d/\widetilde{I}, \quad p(S^I) = P_{\mathcal{F}_I} p(M_z)\big|_{\mathcal{F}_I} \mapsto [p(M_z)]$$

is well defined and extends to a complete contraction on $\mathcal{A}_I$. Well-definedness follows from Lemma 2.16 (c). To show that it is completely contractive, let $N \in \mathbb{N}$ and let

$$T = (p_{ij}(S^I))_{i,j=1}^N \in M_N(A)$$

with $||T|| \leq 1$. By Lemma 2.16 (a), $\mathcal{F}_I$ is co-invariant for all multipliers. Since $p_{ij}(S^I) = P_{\mathcal{F}_I} M_{p_{ij}}\big|_{\mathcal{F}_I}$, each $p_{ij}(S^I)$ commutes with all compressed multipliers, that is

$$p_{ij}(S^I) P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I} = P_{\mathcal{F}_I} M_\varphi p_{ij}(S^I)$$

for all $i, j = 1, \ldots, N$ and all $\varphi \in \mathrm{Mult}(H^2_d)$. Identifying $M_N(A)$ with a subset of $\mathcal{L}(\mathcal{F}_I \otimes \mathbb{C}^N)$ (see the discussion following Definition A.11), Theorem 2.5 yields a multiplier $\Phi \in \mathrm{Mult}(H^2_d \otimes \mathbb{C}^N)$ with $||\Phi|| \leq 1$ and

$$T = P_{\mathcal{F}_I \otimes \mathbb{C}^N} M_\Phi\big|_{\mathcal{F}_I \otimes \mathbb{C}^N}.$$

Write $\Phi = (\Phi_{ij})_{i,j=1}^N$ with $\Phi_{ij} \in \mathrm{Mult}(H^2_d)$. Then

$$P_{\mathcal{F}_I} M_{p_{ij}}\big|_{\mathcal{F}_I} = P_{\mathcal{F}_I} M_{\Phi_{ij}}\big|_{\mathcal{F}_I} \quad \text{for all } i, j = 1, \ldots, N.$$

According to Lemma 2.14 (a), all $\Phi_{ij_{(r)}}$ belong to $\mathcal{A}_d$ for $0 < r < 1$, and Lemma 2.16 (b) shows that each $\Phi_{ij_{(r)}}$ equals $p_{ij_{(r)}}$ modulo $\widetilde{I}$ for $0 < r < 1$. Consequently, we obtain the estimate

$$\begin{aligned} ||([p_{ij}(M_z)])_{i,j=1}^N|| &= \lim_{r \uparrow 1} ||([p_{ij}(rM_z)])_{i,j=1}^N|| = \lim_{r \uparrow 1} ||([p_{ij_{(r)}}])_{i,j=1}^N|| \\ &= \lim_{r \uparrow 1} ||([\Phi_{ij_{(r)}}])_{i,j=1}^N|| \leq \sup_{0<r<1} ||\Phi_{(r)}|| \leq 1, \end{aligned}$$

where the last inequality follows from Lemma 2.14 (b). This observation finishes the proof. □

2.3. The radical case

Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. In the preceding chapter, we have identified the algebra $\mathcal{A}_d/\widetilde{I}$ with $\mathcal{A}_I$, which is an algebra of operators on $\mathcal{F}_I = H^2_d \ominus I$. We will show that if the ideal I is radical, then $\mathcal{F}_I$ can be regarded as a space of functions on the intersection of the vanishing locus of I with the open unit ball. This fact will allow us to identify $\mathcal{A}_I$ with a function algebra on this set.

To fix notation, write $V(I)$ for the vanishing locus of I. Moreover, $Z^0(I)$ (respectively $Z(I)$) will denote the intersection of $V(I)$ with the open (respectively closed) unit ball in $\mathbb{C}^d$. With this notation, $\mathcal{F}_I$ is isomorphic to the restricted Hilbert function space $H^2_d\big|_{Z^0(I)}$ for radical homogeneous ideals I (see Lemma A.7 for the notion of a restriction of a reproducing kernel Hilbert space).

Lemma 2.18. *Let $I \subset \mathbb{C}[z]$ be a radical homogeneous ideal. Then*

$$U : \mathcal{F}_I \to H^2_d\big|_{Z^0(I)}, \quad f \mapsto f\big|_{Z^0(I)}$$

is a unitary, whose inverse is given by $U^{-1}(f\big|_{Z^0(I)}) = P_{\mathcal{F}_I} f$ for $f \in H^2_d$.

Proof. Let $f \in \mathcal{F}_I$. It is immediate from

$$||f\big|_{Z^0(I)}|| = \inf\{||g|| : g \in H^2_d \text{ with } g\big|_{Z^0(I)} = f\big|_{Z^0(I)}\}$$

(see Lemma A.7) that the linear map U is contractive. On the other hand, if $g \in H^2_d$ with $g\big|_{Z^0(I)} = f\big|_{Z^0(I)}$, then the Nullstellensatz (Theorem 1.7 in combination with Lemma 1.4 (a)) shows that $g - f \in \overline{I}$. Hence

$$||f|| = ||P_{\mathcal{F}_I} g|| \leq ||g||,$$

so that U is isometric. To establish surjectivity and the additional claim, let $f \in H^2_d$. Then $P_{\mathcal{F}_I} f - f \in \overline{I}$, and thus $(P_{\mathcal{F}_I} f)\big|_{Z^0(I)} = f\big|_{Z^0(I)}$. □

On the level of multipliers, the preceding lemma implies the desired identification of $\mathcal{A}_I$ with a function algebra.

Corollary 2.19. *Suppose that $I \subset \mathbb{C}[z]$ is a radical homogeneous ideal and let $U : \mathcal{F}_I \to H^2_d\big|_{Z^0(I)}$ be the unitary map given by restriction from Lemma 2.18. Then*

$$\mathcal{A}_I \to \overline{\{p\big|_{Z^0(I)} : p \in \mathbb{C}[z]\}}^{\mathrm{Mult}(H^2_d|_{Z^0(I)})}, \quad T \mapsto UTU^*$$

is a unital completely isometric isomorphism, which maps $P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$ to $\varphi\big|_{Z^0(I)}$ for each $\varphi \in \mathcal{A}_d$. In particular, it sends S^I_i to $z_i\big|_{Z^0(I)}$ for $i = 1, \ldots, d$. Moreover, every function in the range of the above isomorphism extends uniquely to a continuous function on $Z(I)$.

Proof. The unital completely isometric isomorphism

$$\Phi : \mathcal{L}(\mathcal{F}_I) \to \mathcal{L}(H^2_d\big|_{Z^0(I)}), \quad T \mapsto UTU^*,$$

satisfies

$$\Phi(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I})(f\big|_{Z^0(I)}) = UP_{\mathcal{F}_I} M_\varphi U^*(f\big|_{Z^0(I)}) = (P_{\mathcal{F}_I} M_\varphi P_{\mathcal{F}_I} f)\big|_{Z^0(I)} = (\varphi f)\big|_{Z^0(I)}$$

for $\varphi \in \mathcal{A}_d$ and $f \in H^2_d$, where we have used that $\mathcal{F}_I$ is co-invariant under M_φ (see Lemma 2.16 (a)). Since $\mathcal{A}_I$ is the norm-closed algebra generated by the operators $S^I_i = P_{\mathcal{F}_I} M_{z_i}\big|_{\mathcal{F}_I}$, we conclude that Φ maps $\mathcal{A}_I$ onto the norm-closure of the polynomials in $\mathrm{Mult}(H^2_d\big|_{Z^0(I)})$. The additional assertion follows from the fact that the multiplier norm on the restriction of H^2_d to $Z^0(I)$ dominates the sup norm on $Z^0(I)$ by Lemma A.9 (b). □

Because of the above corollary, we can think of $\mathcal{A}_I$ as an algebra of continuous functions on $Z(I)$ for radical homogeneous ideals I.

Remark 2.20. In the radical case, the use of Theorem 2.5 in the proof of Theorem 2.17 can be replaced by a more direct application of the complete Nevanlinna-Pick property of H^2_d. Let us briefly sketch the idea. Suppose that $I \subset \mathbb{C}[z]$ is a radical homogeneous ideal. Then Corollary 2.19 shows that the elements of $\mathcal{A}_I$ can be regarded as multipliers on the restricted Hilbert function space $H^2_d\big|_{Z^0(I)}$. Thus elements of $M_N(\mathcal{A}_I)$ can be viewed as vector-valued multipliers of $H^2_d\big|_{Z^0(I)}$. Since H^2_d is a complete Nevanlinna-Pick space, these multipliers can be extended to multipliers of H^2_d without increasing their norm. Under the identification explained in Corollary 2.19, this extension of multipliers corresponds to dilating an element of $M_N(\mathcal{A}_I)$ to an operator in $\mathrm{Mult}(H^2_d \otimes \mathbb{C}^N)$, which is precisely what was done in the proof of Theorem 2.17.

The results for the radical case can be summarized as follows: The universal operator algebra generated by a commuting row contraction satisfying the relations in a radical homogeneous ideal I has three interpretations:

1. The quotient algebra $\mathcal{A}_d/\widetilde{I}$.

2. The concrete operator algebra $\mathcal{A}_I$.

3. The function algebra obtained by taking the norm-closure of the polynomials in $\mathrm{Mult}(H^2_d\big|_{Z^0(I)})$.

We will see that depending on the context, each description has its benefits.

2.4. Unitary changes of variables

We conclude this chapter with a short section about maps on the Drury-Arveson space which are induced by unitary maps on $\mathbb{C}^d$. It turns out that the group of unitaries on $\mathbb{C}^d$ acts by composition on H^2_d and on $\mathcal{A}_d$, the norm-closure of the polynomials in $\mathrm{Mult}(H^2_d)$. This is due to the fact that the kernel of H^2_d is invariant under unitary transformations.

Lemma 2.21. *Let U be a unitary on $\mathbb{C}^d$. Then*

$$C_U : H^2_d \to H^2_d, \quad f \mapsto f \circ U$$

is a unitary on H^2_d. If $\varphi \in \mathrm{Mult}(H^2_d)$ is a multiplier, then $\varphi \circ U$ is again a multiplier, and $C_U M_\varphi C_U^* = M_{\varphi \circ U}$. Moreover, if $\varphi \in \mathcal{A}_d$, then $\varphi \circ U \in \mathcal{A}_d$ as well.

Proof. Note that U maps $\mathbb{B}_d$ bijectively onto itself, and if $K(z,w) = (1 - \langle z, w\rangle)^{-1}$ is the reproducing kernel of H^2_d, then $K(Uz, Uw) = K(z,w)$ holds for all $z, w \in \mathbb{B}$. Hence Lemma A.5 shows that C_U is a unitary on H^2_d.

Let $\varphi \in \mathrm{Mult}(H^2_d)$. Then for all $f \in H^2_d$,

$$(C_U M_\varphi C_U^*)(f) = (\varphi \cdot (f \circ U^*)) \circ U = (\varphi \circ U) \cdot f,$$

thus $\varphi \circ U$ is again a multiplier with $M_{\varphi \circ U} = C_U M_\varphi C_U^*$. Therefore, the map

$$\mathrm{Mult}(H^2_d) \to \mathrm{Mult}(H^2_d), \quad \varphi \mapsto \varphi \circ U,$$

is a (completely) isometric isomorphism, which clearly maps the algebra of polynomials onto itself. So by definition of $\mathcal{A}_d$, it also preserves $\mathcal{A}_d$. □

Suppose now that I and J are homogeneous ideals in $\mathbb{C}[z]$. If U is a unitary on $\mathbb{C}^d$ such that the operator C_U maps I onto J, then the preceding lemma can be used to obtain a completely isometric isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$.

Lemma 2.22. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals, and let U be a unitary on $\mathbb{C}^d$ such that $J = \{p \circ U : p \in I\}$. Let C_U be the unitary operator on H^2_d given by composition with U from Lemma 2.21. Then C_U restricts to a unitary operator $C_U^I : \mathcal{F}_I \to \mathcal{F}_J$. The map*

$$\Phi_U^I : \mathcal{A}_I \to \mathcal{A}_J, \quad T \mapsto C_U^I T {C_U^I}^*$$

is a unital completely isometric algebra isomorphism that satisfies

$$\Phi_U^I\big(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}\big) = P_{\mathcal{F}_J} M_{\varphi \circ U}\big|_{\mathcal{F}_J}$$

for all $\varphi \in \mathcal{A}_d$. If I and J are radical, then regarding $\mathcal{A}_I$ and $\mathcal{A}_J$ as algebras of functions on $Z(I)$ and $Z(J)$, respectively (see Corollary 2.19), then Φ_U^I is given by

$$\mathcal{A}_I \to \mathcal{A}_J, \quad \varphi \mapsto \varphi \circ U.$$

Proof. The assumption on U implies that C_U maps $\overline{I}$ onto $\overline{J}$, hence it maps $\mathcal{F}_I$ onto $\mathcal{F}_J$. Now, let $\varphi \in \mathcal{A}_d$. Then $\varphi \circ U \in \mathcal{A}_d$ by Lemma 2.21, and for all $f \in \mathcal{F}_J$, we have

$$\begin{aligned} C_U^I\big(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}\big){C_U^I}^* f &= C_U^I P_{\mathcal{F}_I}(\varphi \cdot (f \circ U^*)) = (P_{\mathcal{F}_J} C_U(\varphi \cdot (f \circ U^*)) \\ &= P_{\mathcal{F}_J}((\varphi \cdot (f \circ U^*)) \circ U) = P_{\mathcal{F}_J} M_{\varphi \circ U} f. \end{aligned}$$

Hence

$$C_U^I\big(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}\big){C_U^I}^* = P_{\mathcal{F}_J} M_{\varphi \circ U}\big|_{\mathcal{F}_J} \in \mathcal{A}_J.$$

Since $\mathcal{A}_I$ is the norm-closed algebra generated by the $S_i^I = P_{\mathcal{F}_I} M_{z_i}\big|_{\mathcal{F}_I}$ (and likewise for $\mathcal{A}_J$), we conclude that Φ_U^I indeed maps $\mathcal{A}_I$ onto $\mathcal{A}_J$. The statement about the radical case immediately follows from the identification explained in Corollary 2.19. □

On the one hand, of course, the above lemma already gives a sufficient condition for two algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ being (completely) isometrically isomorphic. On the other hand, also the case $I = J$ will be important. It can be used to obtain automorphisms of $\mathcal{A}_I$, which will turn out helpful for determining necessary conditions for two algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic (see Proposition 3.27).

Remark 2.23. If the homogeneous ideals I and J in the preceding lemma are radical, then the hypothesis of the lemma can be stated in terms of the vanishing loci $V(I)$ and $V(J)$. More generally, let A be any invertible linear map on $\mathbb{C}^d$. Then

$$AV(J) = V(\{p \circ A^{-1} : p \in J\}).$$

Since J is radical, the ideal $\{p \circ A^{-1} : p \in J\}$ is radical as well. Clearly, we have $I = \{p \circ A^{-1} : p \in J\}$ if and only if $J = \{p \circ A : p \in I\}$. So we deduce from Hilbert's Nullstellensatz that $J = \{p \circ A : p \in I\}$ if and only if $AV(J) = V(I)$.

3. Necessary conditions for isomorphisms between the algebras $\mathcal{A}_I$

3.1. The maximal ideal space of $\mathcal{A}_I$

The aim of this chapter is to exhibit necessary conditions for two algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ being (topologically or isometrically) isomorphic, where I and J are radical homogeneous ideals in $\mathbb{C}[z_1, \ldots, z_d]$. This was done in [DRS11], and we follow the exposition given there. The basic strategy is to use the fact that if $\Phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism of unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$, then Φ induces a homeomorphism between the maximal ideal spaces $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, which is given by

$$\Phi^* : \Delta(\mathcal{B}) \to \Delta(\mathcal{A}), \quad \rho \mapsto \rho \circ \Phi.$$

Here and in the sequel, the maximal ideals spaces are equipped with their respective Gelfand topologies.

First, let us observe that $\mathcal{A}_I$ is a unital commuatative Banach algebra for every proper homogeneous ideal $I \subset \mathbb{C}[z]$.

Remark 3.1. Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. We claim that $\mathcal{A}_I = \{0\}$ if and only if $I = \mathbb{C}[z]$. For a proof, note that $\mathcal{A}_I \subset \mathcal{L}(\mathcal{F}_I)$ always contains the identity on $\mathcal{F}_I$, so that $\mathcal{A}_I = \{0\}$ if and only if $\mathcal{F}_I = \{0\}$. If $I = \mathbb{C}[z]$, then $\mathcal{F}_I = H^2_d \ominus I = \{0\}$ by density of the polynomials in H^2_d, which shows one direction. Conversely, if I is a proper homogeneous ideal, then $1 \in \mathcal{F}_I$, thus $\mathcal{A}_I$ is non-trivial as well.

Because of this observation, we will only consider proper homogeneous ideals $I \subset \mathbb{C}[z]$. To determine the maximal ideal space of the commutative Banach algebra $\mathcal{A}_I$, we need some preliminaries.

Lemma 3.2. *Let H be a Hilbert space, and let $\mathcal{A} \subset \mathcal{L}(H)$ be a unital norm-closed subalgebra. Suppose that ρ is a multiplicative linear functional on $\mathcal{A}$. If $(T_1, \ldots, T_d) \in \mathcal{A}^d$ is a row contraction, then*

$$(\rho(T_1), \ldots, \rho(T_d)) \in \overline{\mathbb{B}_d}.$$

Proof. It is well known that multiplicative linear functionals on Banach algebras are automatically contractive, and that contractive linear functionals are completely contractive (see, for example, [Pau02, Proposition 3.8]). Hence $\rho : \mathcal{A} \to \mathbb{C}$ is a complete contraction. Since T is a row contraction, the operator given by

$$\begin{pmatrix} T_1 & \dots & T_d \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathcal{L}(H^d)$$

is a contraction, so

$$\begin{pmatrix} \rho(T_1) & \dots & \rho(T_d) \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^d)$$

is contractive as well, from which it follows that $\rho(T) \in \overline{\mathbb{B}_d}$. □

Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal. If I is radical, then the algebra $\mathcal{A}_I$ can be regarded as an algebra of continuous functions on $Z(I)$ by Corollary 2.19. Under this identification, every $\lambda \in Z(I)$ gives rise to a non-trivial multiplicative linear functional

$$\delta_\lambda^I : \mathcal{A}_I \to \mathbb{C}, \quad \varphi \mapsto \varphi(\lambda).$$

If I is understood, we will simply write δ_λ.

Although we are primarily concerned with the radical case in this chapter, it seems worthwile to include the non-radical case at this point as well. If I is homogeneous, but not necessarily radical, then

$$R_I : \mathcal{A}_I \to \mathcal{A}_{\sqrt{I}}, \quad T \mapsto P_{\mathcal{F}_{\sqrt{I}}} T\big|_{\mathcal{F}_{\sqrt{I}}}$$

is a completely contractive algebra homomorphism. Note that identifying $\mathcal{A}_I$ and $\mathcal{A}_{\sqrt{I}}$ with a quotient of $\mathcal{A}_d$ according to Theorem 2.17, the map R_I is just the natural quotient map

$$\mathcal{A}_d/\widetilde{I} \to \mathcal{A}_d/\widetilde{\sqrt{I}}.$$

Since $Z(I) = Z(\sqrt{I})$, every $\lambda \in Z(I)$ gives rise to a multiplicative linear functional

$$\delta_\lambda^I = \delta_\lambda^{\sqrt{I}} \circ R_I : \mathcal{A}_I \to \mathbb{C}.$$

Again, we will sometimes simply write δ_λ. The following lemma summarizes some elementary properties of these characters.

Lemma 3.3. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal, and let $\lambda \in Z(I)$.*

(a) *For $\varphi \in \mathcal{A}_d$, we have $\delta_\lambda(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) = \varphi(\lambda)$.*

(b) *If $J \subset I$ is another homogeneous ideal, then*

$$\delta_\lambda^I(P_{\mathcal{F}_I} T\big|_{\mathcal{F}_I}) = \delta_\lambda^J(T)$$

holds for all $T \in \mathcal{A}_J$.

(c) *If $\lambda \in Z^0(I)$, then $\delta_\lambda(T) = \langle T1, K(\cdot, \lambda) \rangle$ for all $T \in \mathcal{A}_I$.*

Proof. Part (a) is immediate from the definition of δ_λ and the identification explained in Corollary 2.19. To show (b), note that part (a) implies that the statement holds if T is of the form $P_{\mathcal{F}_J} M_\varphi\big|_{\mathcal{F}_J}$ for some $\varphi \in \mathcal{A}_d$. It is clear that the set of all these elements is dense in $\mathcal{A}_J$ (in fact, Theorem 2.17 shows that it is all of $\mathcal{A}_J$), so that (b) holds for all $T \in \mathcal{A}_J$.

For the proof of (c), observe that the assumptions on I imply that $1 \in \mathcal{F}_I$. Using the defining property of the kernel function, we obtain for $\varphi \in \mathcal{A}_d$, the identity

$$\langle P_{\mathcal{F}_I} M_\varphi 1, K(\cdot, \lambda) \rangle = \langle M_\varphi 1, K(\cdot, \lambda) \rangle = \varphi(\lambda),$$

since $K(\cdot, \lambda) \in \mathcal{F}_I$. Thus, the same reasoning as above establishes the assertion. □

We can now show that every multiplicative linear functional on $\mathcal{A}_I$ arises in the way described above.

Proposition 3.4. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal. The map*

$$Z(I) \to \Delta(\mathcal{A}_I), \quad \lambda \mapsto \delta_\lambda$$

is a homeomorphism, where $\Delta(\mathcal{A}_I)$ is endowed with its Gelfand topology. Its inverse is given by

$$\Delta(\mathcal{A}_I) \to Z(I), \quad \rho \mapsto (\rho(S_1^I), \dots, \rho(S_d^I)).$$

Proof. By the above discussion, every $\lambda \in Z(I)$ gives rise to an element δ_λ of $\Delta(\mathcal{A}_I)$, and Lemma 3.3 (a) shows that $(\delta_\lambda(S_1^I), \dots, \delta_\lambda(S_d^I)) = \lambda$, which implies injectivity of the map $\lambda \mapsto \delta_\lambda$. Conversely, let $\rho \in \Delta(\mathcal{A}_I)$, and set

$$\lambda = (\rho(S_1^I), \dots, \rho(S_d^I)) \in \mathbb{C}^d.$$

Then $\lambda \in \overline{\mathbb{B}_d}$ according to Lemma 3.2, and given $p \in I$, Lemma 2.16 (c) shows that

$$p(\lambda) = \rho(p(S^I)) = 0,$$

so that $\lambda \in Z(I)$. By definition of λ, the multiplicative linear functionals ρ and δ_λ coincide on each S_i^I. Because $\mathcal{A}_I$ is the unital norm-closed algebra generated by these elements, we conclude that $\rho = \delta_\lambda$.

To finish the proof, note that continuity of the second map is trivial, whereas continuity of the first map follows from the fact that the elements of $\mathcal{A}_{\sqrt{I}}$ are continuous functions on $Z(I)$ under the identification explained in Corollary 2.19. $\square$

The following corollary indicates why the radical case is easier in general.

Corollary 3.5. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal. Then $R_I : \mathcal{A}_I \to \mathcal{A}_{\sqrt{I}}$ is the Gelfand transform modulo the identifications explained in Corollary 2.19 and Proposition 3.4. In particular, $\mathcal{A}_I$ is semi-simple if and only if I is radical.*

Proof. It is clear from Proposition 3.4 and the definition of δ_λ that R_I is the Gelfand transform. If I is radical, then R_I is just the identity, thus $\mathcal{A}_I$ is semi-simple. Conversely, if $\mathcal{A}_I$ is semi-simple and $p \in \sqrt{I}$, then

$$R_I(p(S^I)) = p(S^{\sqrt{I}}) = 0,$$

hence $p(S^I) = 0$. From Lemma 2.16 (c), we infer that $p \in I$, so that I is radical. $\square$

In general, the kernel of the Gelfand transform on a unital commutative Banach algebra $\mathcal{A}$ consists precisely of the quasi-nilpotent elements of $\mathcal{A}$. In the case of the algebras $\mathcal{A}_I$, this result can be strengthened due to the Noetherian property of $\mathbb{C}[z]$.

Lemma 3.6. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal. Then there is a natural number N such that*

$$\ker(R^I) = \bigcap_{\lambda \in Z^0(I)} \ker(\delta_\lambda) = \{T \in \mathcal{A}_I : T^N = 0\}.$$

Proof. The first equality follows from Corollary 3.5 and the fact that the elements of $\mathcal{A}_{\sqrt{I}}$, regarded as functions on $Z(I)$, are continuous. Moreover, it is trivial that nilpotent elements are contained in the kernel of every multiplicative linear functional.

To prove the remaining inclusion, note that by the Noetherian property of $\mathbb{C}[z]$, there is a natural number N such that $J^N \subset I$, where $J = \sqrt{I}$ (see, for example, [AM69, Proposition 7.14.]). Identifying $\mathcal{A}_I$ with $\mathcal{A}_d/\widetilde{I}$ and $\mathcal{A}_J$ with $\mathcal{A}_d/\widetilde{J}$ according to Theorem 2.17, we have

$$\ker(R^I) = \widetilde{J}/\widetilde{I}.$$

The continuity of multiplication in $\mathcal{A}_d$ shows that

$$\big(\widetilde{J}\big)^N \subset \widetilde{J^N} \subset \widetilde{I},$$

which finishes the proof. $\square$

We now come to the study of algebra homomorphisms between algebras of the type $\mathcal{A}_I$. To this end, let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a unital algebra homomorphism. Then Φ induces a continuous map

$$\Phi^* : \Delta(\mathcal{A}_J) \to \Delta(\mathcal{A}_I), \quad \rho \mapsto \rho \circ \Phi.$$

Identifying $\Delta(\mathcal{A}_J)$ with $Z(J)$ and $\Delta(\mathcal{A}_I)$ with $Z(I)$ according to Proposition 3.4, Φ^* can be regarded as a continuous map from $Z(J) \to Z(I)$. The structure of the algebras $\mathcal{A}_I$ allows us to say more about Φ^*.

Proposition 3.7. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a unital algebra homomorphism. Then there is a continuous map $F : \overline{\mathbb{B}_d} \to \mathbb{C}^d$ such that $F\big|_{\mathbb{B}_d}$ is holomorphic, the restriction of F to $Z(J)$ acts as*

$$F\big|_{Z(J)} = \Phi^*,$$

and such that the components of F are in $\mathcal{A}_d$.

Proof. For $i = 1, \ldots, d$, Theorem 2.17 yields a multiplier $\varphi_i \in \mathcal{A}_d$ such that

$$P_{\mathcal{F}_J} M_{\varphi_i}\big|_{\mathcal{F}_J} = \Phi(S_i^I).$$

Define

$$F : \overline{\mathbb{B}_d} \to \mathbb{C}^d, \quad z \mapsto (\varphi_1(z), \ldots, \varphi_d(z)).$$

Then F is continuous and $F\big|_{\mathbb{B}_d}$ is holomorphic. Using Lemma 3.3 (a), we obtain for all $\lambda \in Z(J)$ and $i = 1, \ldots, d$ the identity

$$\Phi^*(\delta_\lambda)(S_i^I) = \delta_\lambda\Big(P_{\mathcal{F}_J} M_{\varphi_i}\big|_{\mathcal{F}_J}\Big) = \varphi_i(\lambda).$$

Proposition 3.4 now implies that $F(\lambda) \in Z(I)$, and that $\Phi^*(\delta_\lambda) = \delta_{F(\lambda)}$, as asserted. ∎

The above proposition implies that Φ^* restricts to a holomorphic map on the analytic set $Z^0(I) \subset \mathbb{B}_d$ in the sense that for any point $\lambda \in Z^0(I)$, there is an open neighborhood U of λ in $\mathbb{C}^d$ and a holomorphic map F on U which coincides with Φ^* on U.

Example 3.8. Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals, and let U be a unitary on $\mathbb{C}^d$ such that $J = \{p \circ U : p \in I\}$. In particular, this means that $UZ(J) \subset Z(I)$. By Lemma 2.22, U induces an algebra isomorphism $\Phi = \Phi_U^I : \mathcal{A}_I \to \mathcal{A}_J$ which maps $P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$ to $P_{\mathcal{F}_J} M_{\varphi \circ U}\big|_{\mathcal{F}_J}$ for each $\varphi \in \mathcal{A}_d$. Thus, given $\lambda \in Z(J)$, we have

$$\Phi^*(\delta_\lambda)(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) = \delta_\lambda(P_{\mathcal{F}_J} M_{\varphi \circ U}\big|_{\mathcal{F}_J}) = (\varphi \circ U)(\lambda) = \delta_{(U\lambda)}(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I})$$

for all $\varphi \in \mathcal{A}_d$, where we have used Lemma 3.3 (a). We conclude that the map $\Phi^* : Z(J) \to Z(I)$ is just the restriction of U to $Z(J)$.

If I and J are radical homogeneous ideals, we can view $\mathcal{A}_I$ and $\mathcal{A}_J$ as algebras of continuous functions on $Z(I)$ and $Z(J)$, respectively (see Corollary 2.19). Under this identification, Φ acts as a composition operator.

Lemma 3.9. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals, and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a unital algebra homomorphism. Regarding $\mathcal{A}_I$ and $\mathcal{A}_J$ as algebras of functions on $Z(I)$ and $Z(J)$, respectively, Φ is given by composition with Φ^*, that is,*

$$\Phi(\varphi) = \varphi \circ \Phi^*$$

for $\varphi \in \mathcal{A}_I$.

Proof. To avoid confusion, let us write F for Φ^*, regarded as a map from $Z(J)$ to $Z(I)$. Let $\varphi \in \mathcal{A}_I$, viewed as an algebra of functions on $Z(I)$. Then for all $\lambda \in Z(J)$, we have

$$\Phi(\varphi)(\lambda) = \delta_\lambda(\Phi(\varphi)) = \Phi^*(\delta_\lambda)(\varphi) = \delta_{F(\lambda)}(\varphi) = (\varphi \circ F)(\lambda). \qquad \square$$

Remark 3.10. The preceding lemma can be seen as a special instance of the following general fact: If $\Phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of unital commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$, and if $R_{\mathcal{A}} : \mathcal{A} \to C(\Delta(\mathcal{A}))$ and $R_{\mathcal{B}} : \mathcal{B} \to C(\Delta(\mathcal{B}))$ are the respective Gelfand transforms, then the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Phi"] \arrow[d, "R_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "R_{\mathcal{B}}"] \\ C(\Delta(\mathcal{A})) \arrow[r, "f \mapsto f \circ \Phi^*"'] & C(\Delta(\mathcal{B})) \end{tikzcd}$$

commutes, where $\Phi^* : \Delta(\mathcal{B}) \to \Delta(\mathcal{A})$ is the induced map. In the setting of the lemma, the algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are semi-simple (see Corollary 3.5), so that modulo the identifications we have used in this section, Φ is given by composition with Φ^*.

The closed graph theorem and the last lemma easily imply an automatic continuity result for homomorphisms between $\mathcal{A}_I$ and $\mathcal{A}_J$ in the radical case. In fact, this result is just a special case of the theorem that every homomorphism of a commutative Banach algebra into a semi-simple commutative Banach algebra is continuous [Rud91, Theorem 11.10].

Corollary 3.11. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals. Every unital algebra homomorphism $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is continuous.* $\square$

3.2. Automorphisms of $\mathcal{A}_d$

As a by-product, the results established in the preceding section allow us to determine the group of algebra automorphisms of $\mathcal{A}_d$. We therefore diverge from our initial path of finding conditions for two algebras of the type $\mathcal{A}_I$ being isomorphic, and examine the case $I = \{0\}$ more carefully. To this end, it is convenient to think about $\mathcal{A}_d$ as an algebra of holomorphic functions on the open unit ball $\mathbb{B}_d$, and we will do so throughout this chapter. It turns out that there is a close relation between automorphisms of $\mathcal{A}_d$ and biholomorphic self-maps of the unit ball $\mathbb{B}_d$.

To fix notation, let

$$\operatorname{Aut}(\mathbb{B}_d) = \{\psi : \mathbb{B}_d \to \mathbb{B}_d : \psi \text{ is biholomorphic}\}$$

be the set of all automorphisms of $\mathbb{B}_d$, which is a group under composition. Similarly, we write $\operatorname{Aut}(\mathcal{A}_d)$ for the group of all algebra automorphisms of $\mathcal{A}_d$.

As an application of Proposition 3.7, we see that every algebra automorphism of $\mathcal{A}_d$ is a composition operator with an automorphism of $\mathbb{B}_d$.

Lemma 3.12. *For any $\Phi \in \operatorname{Aut}(\mathcal{A}_d)$, there exists a biholomorphic map $\psi \in \operatorname{Aut}(\mathbb{B}_d)$ such that*

$$\Phi(\varphi) = \varphi \circ \psi \quad \textit{for all} \quad \varphi \in \mathcal{A}_d.$$

Proof. Proposition 3.7, applied with $I = J = \{0\}$, asserts that there are continuous maps F and $\widetilde{F}$ from $\overline{\mathbb{B}_d}$ to $\mathbb{C}^d$ that are holomorphic on $\mathbb{B}_d$ such that

$$\Phi^* = F \quad \text{and} \quad (\Phi^{-1})^* = \widetilde{F}.$$

Consequently, F and $\widetilde{F}$ take values in $\overline{\mathbb{B}_d}$, and they are inverse to each other. It follows from the maximum modulus principle, or from a well-known theorem about injective holomorphic maps [Ran86, Theorem 1.2.14], that F and $\widetilde{F}$ map $\mathbb{B}_d$ into $\mathbb{B}_d$, hence F restricts to an automorphism ψ of $\mathbb{B}_d$. Lemma 3.9 finally shows that

$$\Phi(\varphi) = \varphi \circ \psi \quad \text{for all} \quad \varphi \in \mathcal{A}_d,$$

as asserted. □

Remark 3.13. The proof of the preceding lemma shows that the automorphism ψ extends to a homeomorphism of the closed unit ball. This is nothing special. In fact, it is well known that all automorphisms of the open unit ball extend to homeomorphisms of the closed unit ball (see, for example, [Rud08, Section 2.2]). Note that if we think of $\mathcal{A}_d$ as an algebra of functions on $\overline{\mathbb{B}_d}$, then the identity $\Phi(\varphi) = \varphi \circ \psi$ holds on all of $\overline{\mathbb{B}_d}$.

The goal of this section is to show that conversely, every automorphism of $\mathbb{B}_d$ induces an algebra automorphism of $\mathcal{A}_d$ by composition. In fact, we will see that this algebra automorphism is unitarily implemented. The most simple case when the automorphism is just a unitary on $\mathbb{C}^d$ has been treated with in Lemma 2.21. For the general case, we recall some properties of automorphisms of $\mathbb{B}_d$.

For $a \in \mathbb{B}_d$, write P_a for the orthogonal projection of $\mathbb{C}^d$ onto the subspace spanned by a, that is $P_0 = 0$ and

$$P_a(z) = \frac{\langle z, a \rangle}{\langle a, a \rangle} z$$

for $z \in \mathbb{C}^d$ if $a \neq 0$. Let $Q_a = 1 - P_a$ and let $s_a = (1 - |a|^2)^{1/2}$. Finally, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle}. \tag{3.1}$$

Then φ_a is an automorphism of $\mathbb{B}_d$ which maps 0 to a, and it is an involution [Rud08, Theorem 2.2.2]. If ψ is an arbitrary automorphism of $\mathbb{B}_d$ and $a = \psi^{-1}(0)$, then there is a unique unitary operator U on $\mathbb{C}^d$ such that

$$\psi = U \circ \varphi_a.$$

Moreover, the identity

$$1 - \langle \psi(z), \psi(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}$$

holds for all $z, w \in \mathbb{B}_d$ [Rud08, Theorem 2.2.5]. If $K(z, w) = \frac{1}{1 - \langle z, w \rangle}$ denotes the reproducing kernel of H^2_d, we can rewrite the last identity as

$$K(\psi(z), \psi(w)) = \frac{1}{s_a^2} \frac{K(z, w)}{K(z, a)\overline{K(w, a)}}. \tag{3.2}$$

We can use this fact to show that automorphisms of the unit ball induce automorphisms of the multiplier algebra on H^2_d by composition. In [DRS11], this result was proven using a construction of Voiculescu concerning a non-commutative analogue of the multiplier algebra on H^2_d. The essential ingredient of the proof presented here is Lemma A.5 from the theory of reproducing kernel Hilbert spaces.

Lemma 3.14. *Let ψ be an automorphism of $\mathbb{B}_d$ and set $a = \psi^{-1}(0)$. Then*

$$V_\psi : H^2_d \to H^2_d, \quad f \mapsto s_a K(\cdot, a)(f \circ \psi)$$

is a unitary operator. If $\varphi \in \mathrm{Mult}(H^2_d)$ is a multiplier, then $\varphi \circ \psi$ is again a multiplier and

$$V_\psi M_\varphi V_\psi^* = M_{\varphi \circ \psi}.$$

Proof. First note that by Lemma 2.21, it suffices to consider the case $\psi = \varphi_a$. Indeed, if $\psi = U \circ \varphi_a$ where U is a unitary, then $V_\psi = V_{\varphi_a} \circ V_U$. The fact that V_{φ_a} is a unitary immediately follows from equation (3.2) and Lemma A.5. Note that for $z \in \mathbb{B}_d$,

$$(K(\cdot, a) \circ \varphi_a)(z) = K(\varphi_a(z), \varphi_a(0)) = \frac{1}{s_a^2 K(\cdot, a)}(z)$$

by equation (3.2). Using this identity and the fact that φ_a is an involution, we obtain

$$V_{\varphi_a}^2 f = s_a K(\cdot, a)((s_a K(\cdot, a)(f \circ \varphi_a)) \circ \varphi_a) = s_a^2 K(\cdot, a) \frac{1}{s_a^2 K(\cdot, a)} f = f$$

for all $f \in H_d^2$, thus the unitary V_{φ_a} is an involution as well, and hence selfadjoint. To finish the proof, let $\varphi \in \operatorname{Mult}(H_d^2)$. Then for all $f \in H_d^2$,

$$\begin{aligned}
(V_{\varphi_a} M_\varphi V_{\varphi_a}^*) f &= s_a K(\cdot, a)((\varphi s_a K(\cdot, a)(f \circ \varphi_a)) \circ \varphi_a) \\
&= s_a^2 K(\cdot, a)(\varphi \circ \varphi_a) \frac{1}{s_a^2 K(\cdot, a)} f \\
&= (\varphi \circ \varphi_a) f,
\end{aligned}$$

so that $\varphi \circ \varphi_a \in \operatorname{Mult}(H_d^2)$ and $V_{\varphi_a} M_\varphi V_{\varphi_a}^* = M_{\varphi \circ A}$. □

To show that composition with an automorphism of $\mathbb{B}_d$ induces an automorphism of $\mathcal{A}_d$, we need the following simple observation. It could also be deduced from Remark 2.15.

Lemma 3.15. *Let $\psi \in \operatorname{Aut}(\mathbb{B}_d)$. Then the components of ψ are in $\mathcal{A}_d$.*

Proof. Writing $\psi = U \circ \varphi_a$ where $a = \psi^{-1}(0)$ and U is a unitary, we see that is sufficient to show that the components of φ_a are in $\mathcal{A}_d$. By equation (3.1), it clearly suffices to show that

$$K(\cdot, a) = \frac{1}{1 - \langle \cdot, a \rangle} \in \mathcal{A}_d.$$

To this end, we observe that Lemma 2.9 yields

$$||\langle \cdot, a \rangle||_M^2 = ||\langle \cdot, a \rangle||_{H_d^2}^2 = \sum_{i=1}^d |a_i|^2 < 1,$$

hence the series $\sum_{n=0}^\infty \langle \cdot, a \rangle^n$ converges absolutely in the Banach algebra $\mathcal{A}_d$, and the limit is necessarily $K(\cdot, a)$. □

Combining these results, we obtain the following characterization of automorphisms of $\mathcal{A}_d$.

Theorem 3.16. *Let $\mathcal{A}_d$ be the norm-closure of the polynomials in* $\mathrm{Mult}(H^2_d)$.

(a) *Any $\psi \in \mathrm{Aut}(\mathbb{B}_d)$ gives rise to a completely isometric automorphism*

$$C_\psi : \mathcal{A}_d \to \mathcal{A}_d, \quad \varphi \mapsto \varphi \circ \psi$$

of $\mathcal{A}_d$. In fact, C_ψ is unitarily implemented.

(b) *The map*

$$\mathrm{Aut}(\mathbb{B}_d) \to \mathrm{Aut}(\mathcal{A}_d), \quad \psi \mapsto C_\psi$$

is bijective and satisfies $C_{\psi_1} C_{\psi_2} = C_{\psi_2 \circ \psi_1}$ for $\psi_1, \psi_2 \in \mathrm{Aut}(\mathbb{B}_d)$. Its inverse is given by

$$\mathrm{Aut}(\mathcal{A}_d) \to \mathrm{Aut}(\mathbb{B}_d), \quad \Phi \mapsto (\Phi(z_1), \dots, \Phi(z_d)).$$

(c) *Every algebra automorphism of $\mathcal{A}_d$ is completely isometric.*

Proof. (a) Taking Lemma 3.14 into account, it remains to show that the isometric isomorphism

$$\widetilde{C}_\psi : \mathrm{Mult}(H^2_d) \to \mathrm{Mult}(H^2_d), \quad \varphi \mapsto \varphi \circ \psi$$

maps $\mathcal{A}_d$ into $\mathcal{A}_d$. As an application of Lemma 3.15, we see that $\widetilde{C}_\psi(z_i) \in \mathcal{A}_d$ for $i = 1, \dots, d$. Thus, the result follows from the fact that $\mathcal{A}_d$ is the norm closed algebra generated by the coordinate functions z_i.

(b) By part (a), any $\psi \in \mathrm{Aut}(\mathbb{B}_d)$ induces an automorphism $C_\psi \in \mathrm{Aut}(\mathcal{A}_d)$, and clearly, $C_\psi(z_i) = \psi_i$, where ψ_i denotes the i-th component function on ψ. Conversely, any $\Phi \in \mathrm{Aut}(\mathcal{A}_d)$ is of the form C_ψ for some $\psi \in \mathrm{Aut}(\mathbb{B}_d)$ by Lemma 3.12. Since the relation for $C_{\psi_1 \circ \psi_2}$ is obvious, we have established (b).

(c) This is an immediate consequence of (a) and (b). □

3.3. Holomorphic maps on homogeneous varieties

Let I and J be homogeneous ideals in $\mathbb{C}[z]$. According to Proposition 3.7, a unital algebra homomorphism from $\mathcal{A}_I$ to $\mathcal{A}_J$ always induces a map Φ^* from $Z(J)$ to $Z(I)$ which is holomorphic on $Z^0(J)$ in the sense that for all $\lambda \in Z^0(I)$, there is an open neighborhood U of λ in $\mathbb{C}^d$ and a holomorphic map F on U which coincides with F on U. In fact, if I and J are radical, the algebra homomorphism is uniquely determined by this induced map (see Lemma 3.9). To get a better understanding of these algebra homomorphisms, it is therefore helpful to examine holomorphic maps on sets of the form $Z^0(I)$.

The general scheme will be to use the homogeneity of I to embed the unit disk into $Z^0(I)$ via $t \mapsto t\frac{z}{||z||}$, where z is a non-zero point in $Z^0(I)$, and apply results from classical complex analysis. We begin with a variant of the maximum modulus principle.

Lemma 3.17. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal and let $F : Z^0(I) \to \overline{\mathbb{B}_d}$ be a holomorphic map. If there is a point $z \in Z^0(I)$ such that $F(z) \in \partial\mathbb{B}_d$, then F is constant.*

Proof. Clearly, we may assume that $Z^0(I) \supsetneq \{0\}$. Since I is homogeneous, we can choose a point $w_0 \in Z^0(I) \setminus \{0\}$ and a real number $t_0 \in [0,1)$ such that $z = t_0 w_0$. Then $tw_0 \in Z^0(I)$ for all $t \in \overline{\mathbb{D}}$. By assumption,

$$f : \overline{\mathbb{D}} \to \mathbb{C}, \quad t \mapsto \langle F(tw_0), F(z)\rangle$$

is continuous and holomorphic on $\mathbb{D}$. Moreover, f satisfies $|f(t)| \leq 1$ for all $t \in \mathbb{D}$ and $f(t_0) = 1$. By the maximum modulus principle, f is the constant function 1. But this can only happen if $F(tw_0) = F(z)$ for all $t \in \overline{\mathbb{D}}$. In particular, $F(0) = F(z) \in \partial\mathbb{B}_d$. Now, if $w \in Z^0(I)$ is arbitrary, the same argument, applied to the function

$$\overline{\mathbb{D}} \to \mathbb{C}, \quad t \mapsto \langle F(tw), F(0)\rangle,$$

shows that $F(w) = F(0)$. $\square$

An immediate application to isomorphisms between algebras of the type $\mathcal{A}_I$ is the following result.

Lemma 3.18. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be an algebra isomorphism. Then Φ^*, regarded as a map from $Z(J)$ to $Z(I)$, maps $Z^0(J)$ biholomorphically onto $Z^0(I)$.*

Proof. This is a direct consequence of Proposition 3.7 and the maximum modulus principle (Lemma 3.17). $\square$

The following lemma is a generalization of the classical Schwarz lemma to homogeneous varieties.

Lemma 3.19. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal and let $F : Z^0(I) \to \mathbb{B}_d$ be holomorphic with $F(0) = 0$.*

(a) $||F(z)|| \leq ||z||$ *and* $||\frac{d}{dt}F(tz)\big|_{t=0}|| \leq ||z||$ *for all* $z \in Z^0(I)$.

(b) *Suppose that there exists a point* $z \in Z^0(I) \setminus \{0\}$ *with* $||F(z)|| = ||z||$ *or* $||\frac{d}{dt}F(tz)\big|_{t=0}|| = ||z||$. *Then there is a* $z_0 \in \partial\mathbb{B}_d$ *such that*

$$F\Big(t\frac{z}{||z||}\Big) = tz_0 \quad \textit{for all } t \in \mathbb{D}. \tag{3.3}$$

In particular, the map F *sends the disk* $\mathbb{C}z \cap \mathbb{B}_d$ *biholomorphically onto the disk* $\mathbb{C}F(z) \cap \mathbb{B}_d$.

Proof. All assertions are trivial if $Z^0(I) = \{0\}$. Otherwise, let $z \in Z^0(I) \setminus \{0\}$. For $w \in \overline{\mathbb{B}_d}$, we define a holomorphic map

$$f_w : \mathbb{D} \to \mathbb{C}, \quad t \mapsto \Big\langle F\Big(t\frac{z}{||z||}\Big), w\Big\rangle.$$

The assumptions on F imply that $f_w(0) = 0$ and that f_w maps $\mathbb{D}$ into $\mathbb{D}$. Hence, an application of the classical Schwarz lemma shows that $|f_w(t)| \le |t|$ for all $t \in \mathbb{D}$. If $F(z) \neq 0$, we set $w = \frac{F(z)}{||F(z)||}$ and $t = ||z||$ to deduce the inequality $||F(z)|| \le ||z||$, which is evident if $F(z) = 0$. This proves half of (a).

If $||F(z)|| = ||z||$ for some $z \in Z^0(I) \setminus \{0\}$, we see that $f_w(||z||) = ||z||$, where $w = \frac{F(z)}{||z||}$. In this case, the Schwarz lemma shows that this is only possible if f_w is the identity, that is,

$$\Big\langle F\Big(t\frac{z}{||z||}\Big), w\Big\rangle = t \quad \text{for all } t \in \mathbb{D}.$$

Since $||F(t\frac{z}{||z||})|| \le |t|$ by the first half of (a), we conclude that equation (3.3) holds with $z_0 = w = \frac{F(z)}{||z||}$, which establishes the first half of (b).

To finish the proof, let $z \in Z^0(I) \setminus \{0\}$, and define $w_0 = \frac{d}{dt}F(t\frac{z}{||z||})\big|_{t=0}$. If $w_0 \neq 0$, set $w = \frac{w_0}{||w_0||}$ in the definition of f_w to deduce that

$$||w_0|| = \frac{\langle w_0, w_0\rangle}{||w_0||} = f_w'(0) \le 1,$$

once again by the Schwarz lemma. Consequently, $||\frac{d}{dt}F(tz)\big|_{t=0}|| \le ||z||$. If equality holds, then $f_w'(0) = 1$, so the Schwarz lemma tells us that f_w is the identity, and the same argument as above shows that equation (3.3) holds with $z_0 = w$. □

In one-dimensional complex analysis, the Schwarz lemma can be used to show that every automorphism of the unit disk which fixes the origin is linear. For analytic sets of the form $Z^0(I)$, we now obtain a similar result, which is a variant of Cartan's uniqueness theorem (see, for example, [Rud08, Theorem 2.1.3]).

Corollary 3.20. *Let* $I, J \subsetneq \mathbb{C}[z]$ *be homogeneous ideals and let* $F : Z^0(I) \to Z^0(J)$ *be a biholomorphic map with* $F(0) = 0$. *Then there is a linear map* A *on* $\mathbb{C}^d$ *such that* $F = A\big|_{Z^0(I)}$.

Proof. We may assume without loss of generality that $Z^0(I) \supsetneq \{0\}$. Let A be the derivative of G at 0, where G is a holomorphic function on a neighborhood U of the origin which coincides with F on $U \cap Z^0(I)$. Then

$$\frac{d}{dt} F(tz)\Big|_{t=0} = Az \quad \text{for all } z \in Z^0(I).$$

The Schwarz lemma for homogeneous varieties (Lemma 3.19), applied to F and its inverse, shows that $||F(z)|| = ||z||$ for all $z \in Z^0(I)$. Let $z \in Z^0(I) \setminus \{0\}$. By part (b) of the same lemma, there is a $z_0 \in \partial\mathbb{B}_d$ such that

$$F\Big(t\frac{z}{||z||}\Big) = tz_0 \quad \text{for all } t \in \mathbb{D}.$$

Note that z_0 necessarily satisfies $||z||z_0 = \frac{d}{dt}F(tz)\big|_{t=0}$, from which we conclude that

$$F(z) = ||z||z_0 = Az.$$

Hence $F = A\big|_{Z^0(I)}$. □

Another useful consequence of the Schwarz lemma for homogeneous varieties is the following result concerning biholomorphisms which do not fix the origin.

Corollary 3.21. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $F : Z^0(I) \to Z^0(J)$ be a biholomorphic map with $F(0) \neq 0$. Let $b = F(0)$ and $a = F^{-1}(0)$. Then F maps the disk $D_1 = \mathbb{C}a \cap \mathbb{B}_d$ biholomorphically onto the disk $D_2 = \mathbb{C}b \cap \mathbb{B}_d$.*

Proof. Let φ_a be an automorphism of $\mathbb{B}_d$ which maps a to 0 and vice versa, and which restricts to an automorphism of D_1 (the automorphism φ_a defined in the discussion preceding Lemma 3.14 will do). Then

$$h : D_1 \to \mathbb{B}_d, \quad h = F \circ \varphi_a,$$

is holomorphic with $h(0) = 0$ and $h(a) = b$. Observe that the disk D_1 is of the form $Z^0(I_0)$ for some homogeneous ideal $I_0 \subsetneq \mathbb{C}[z]$. Thus, the Schwarz lemma (Lemma 3.19) yields $||b|| = ||h(a)|| \leq ||a||$. A similar argument, applied to F^{-1} in place of F, shows that $||a|| \leq ||b||$, hence $||a|| = ||b||$. Using part (b) of the Schwarz lemma, we deduce that h maps D_1 biholomorphically onto D_2. The assertion is now immediate since φ_a maps D_1 biholomorphically onto itself. □

Suppose that $F : Z^0(I) \to Z^0(J)$ is a biholomorphism which does not fix the origin. Then there is way of constructing a biholomorphism from $Z^0(I)$ onto $Z^0(J)$ which does fix the origin with the help of certain "turn" maps. To make this precise, let $t \in \mathbb{R}$, and define $F_t : \mathbb{B}_d \to \mathbb{B}_d, z \mapsto e^{it}z$. It is clear that for any homogeneous ideal $I \subset \mathbb{C}[z]$, we have $F_t(Z^0(I)) = Z^0(I)$ for all $t \in \mathbb{R}$.

Lemma 3.22. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals, and let $F : Z^0(I) \to Z^0(J)$ be a biholomorphic map. Then there are real numbers s and t such that the biholomorphic map*

$$F \circ F_t \circ F^{-1} \circ F_s \circ F : Z^0(I) \to Z^0(J)$$

fixes the origin.

Proof. If F itself fixes the origin, we set $s = t = 0$, and we are done. Thus, we may suppose that $a = F(0)$ and $b = F^{-1}(0)$ are non-zero. We define $D_1 = \mathbb{C}a \cap \mathbb{B}_d$ and $D_2 = \mathbb{C}b \cap \mathbb{B}_d$. Then Corollary 3.21 shows that F maps the disk D_2 biholomorphically onto the disk D_1. We will use an argument from plane conformal geometry to establish the assertion. To this end, let

$$C = \{e^{is}a : s \in \mathbb{R}\} \subset D_1$$

denote the circle in D_1 around the origin with radius $||a||$. Identifying D_1 and D_2 with $\mathbb{D}$, the biholomorphic map $F : D_2 \to D_1$ is an automorphism of the unit disk. Since automorphisms of the unit disk extend to Möbius maps on the Riemann sphere, which map circles to circles or straight lines (see, for example, Section 14.3 in [Rud74]), $C' = F^{-1}(C)$ is again a circle. It easily follows, for example by the maximum modulus principle, that F^{-1} maps the interior of C into the interior of C'. Thus, the point $b = F^{-1}(0)$ lies in the interior of C'. Clearly, $0 = F^{-1}(a) \in C'$, that is, C' is a circle passing through the origin. Now, an elementary geometry argument shows that there is a point in C', say $F^{-1}(c)$ with $c \in C$, and a real number t such that $e^{it}F^{-1}(c) = F^{-1}(0)$. By definition of C, there is an $s \in \mathbb{R}$ such that $c = e^{is}a$. It follows that

$$e^{it}F^{-1}(e^{is}F(0)) = F^{-1}(0).$$

The assertion readily follows from this observation. □

3.4. Algebra isomorphisms

Suppose that $I \subsetneq \mathbb{C}[z]$ is a homogeneous ideal. In Proposition 3.4, we have identified the maximal ideal space $\Delta(\mathcal{A}_I)$ of $\mathcal{A}_I$ with $Z(I)$. Since I is homogeneous, $\Delta(\mathcal{A}_I)$ always contains a distinguished element δ_0, which corresponds to $0 \in Z(I)$. Note that δ_0 is the unique multiplicative linear functional on $\mathcal{A}_I$ mapping 1 to 1 and S_i^I to 0 for each i. Lemma 3.3 (c) shows that it is given by

$$\delta_0(T) = \langle T1, 1\rangle \quad \text{for all } T \in \mathcal{A}_I.$$

When identifying H_d^2 with the symmetric Fock space $\mathcal{F}_s(\mathbb{C}^d)$, the vector in $\mathcal{F}_s(\mathbb{C}^d)$ which corresponds to the constant function $1 \in H_d^2$ is usually called the vacuum vector. Therefore, δ_0 is called the vacuum state.

Now, suppose that $J \subsetneq \mathbb{C}[z]$ is another homogeneous ideal. Among all algebra homomorphisms from $\mathcal{A}_I$ into $\mathcal{A}_J$, those homomorphisms Φ with the property that $\Phi^* : \Delta(\mathcal{A}_J) \to \Delta(\mathcal{A}_I)$ preserves the vacuum state play a special role.

Definition 3.23. Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a unital algebra homomorphism. We call Φ vacuum-preserving if $\Phi^*(\delta_0) = \delta_0$.

Vacuum-preserving isomorphisms are rather easy to understand in the radical case.

Proposition 3.24. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals, and suppose that $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is a vacuum-preserving algebra isomorphism. Then there exists an invertible linear map A on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$, and such that regarding $\mathcal{A}_I$ and $\mathcal{A}_J$ as algebras of functions on $Z(I)$ and $Z(J)$, respectively, Φ is given by*

$$\mathcal{A}_I \to \mathcal{A}_J, \quad \varphi \mapsto \varphi \circ A.$$

Proof. Since Φ is vacuum-preserving, Lemma 3.18 and Corollary 3.20 imply the existence of a linear map A on $\mathbb{C}^d$ such that the continuous map Φ^* coincides with A on $Z^0(J)$, hence also on $Z(J)$. In particular, A maps $Z(J)$ onto $Z(I)$. The same reasoning, applied to the inverse of Φ, yields a linear map B which maps $Z(I)$ onto $Z(J)$, and such that $A \circ B = \mathrm{id}_{Z(I)}$ and $B \circ A = \mathrm{id}_{Z(J)}$. It follows that A is invertible on $\mathrm{span}(Z(J))$, so that it can be chosen to be invertible on $\mathbb{C}^d$. Corollary 3.9 finally shows that Φ is given by composition with A. $\square$

Remark 3.25. In the above proof, we only needed the hypothesis that the ideals I and J are radical in the last sentence. Thus, if I and J are not necessarily radical, then there is still an invertible linear map A on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$, and such that Φ^*, regarded as a map from $Z(J)$ to $Z(I)$, is the restriction of A to $Z(J)$. However, in the non-radical case, this does not completely determine the isomorphism Φ.

To deduce a necessary criterion for two algebras of the type $\mathcal{A}_I$ being isomorphic, we will show that whenever $\mathcal{A}_I$ and $\mathcal{A}_J$ are isomorphic, then they are isomorphic via a vacuum-preserving homomorphism. We will do this with the help of Lemma 3.22. To this end, we need a particular class of automorphisms of $\mathcal{A}_I$.

Remark 3.26. Let t be a real number. Then t gives rise to a unitary U_t on $\mathbb{C}^d$ by multiplication with e^{it}. For any homogeneous ideal $I \subset \mathbb{C}[z]$, we have

$$I = \{p \circ U_t : p \in I\}.$$

Thus, Lemma 2.22 shows that U_t induces an isometric automorphism $\Phi_t^I = \Phi_{U_t}^I$ of $\mathcal{A}_I$. By Example 3.8, the map Φ_t^{I*} is given by multiplication with e^{it}. Note that if $I = \{0\}$, that is, $\mathcal{A}_I = \mathcal{A}_d$, then for every $\varphi \in \mathcal{A}_d$, the function $\Phi_t^I(\varphi)$ is just the function φ_t defined at the beginning of Section 1.1.

The existence of vacuum-preserving isomorphisms is guaranteed by the following result.

Proposition 3.27. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be an algebra isomorphism. Then there exists a vacuum-preserving algebra isomorphism $\Psi : \mathcal{A}_I \to \mathcal{A}_J$. If Φ is a topological (respectively isometric) isomorphism, then Ψ can be chosen to be a topological (respectively isometric) isomorphism as well. More precisely, there are real numbers s and t such that*

$$\Psi = \Phi \circ \Phi_s^I \circ \Phi^{-1} \circ \Phi_t^J \circ \Phi$$

is a vacuum-preserving algebra isomorphism.

Proof. Clearly, it suffices to prove the last assertion. To this end, we set $F = \Phi^*$, and regard it as a map from $Z(J)$ onto $Z(I)$. Lemma 3.18 shows that F maps $Z^0(J)$ biholomorphically onto $Z^0(I)$. Using the fact that the mapping which assigns to a unital commutative Banach algebra its maximal ideal space and to a homomorphism of unital Banach algebras its induced map between the maximal ideal spaces is a contravariant functor, we see that the assertion follows as an application of Lemma 3.22 and Remark 3.26. $\square$

We can now establish a necessary criterion for $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic. Recall from Corollary 3.11 that algebraic and topological isomorphisms are the same in the radical case.

Theorem 3.28. *Let $I, J \subset \mathbb{C}[z]$ be radical homogeneous ideals. If $\mathcal{A}_I$ and $\mathcal{A}_J$ are algebraically isomorphic, then there exists an invertible linear map A on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$.*

Proof. Remark 3.1 deals with the trivial cases where $I = \mathbb{C}[z]$ or $J = \mathbb{C}[z]$. If both I and J are proper ideals, the assertion immediately follows from Proposition 3.27 and Proposition 3.24. $\square$

Remark 3.29. The proof of Theorem 3.28 follows closely the one given in [DRS11]. However, they differ in two aspects. Firstly, the existence of the two disks in the first paragraph of the proof of Lemma 3.22 was established in [DRS11] using the notion of a singular nucleus of a homogeneous variety. Secondly, the proof of Corollary 3.20 was an adaption of the proof of the Cartan uniqueness theorem (see [Rud08, Theorem 2.1.3]). In the approach presented here, the heart of both results is Lemma 3.19, which is essentially an application of the Schwarz lemma from one-dimensional complex analysis.

3.5. Isometric isomorphisms

The goal of this section is to refine Theorem 3.28 in the situation where $\mathcal{A}_I$ and $\mathcal{A}_J$ are not only algebraically, but isometrically, isomorphic. We will show that the map A can be chosen to be a unitary on $\mathbb{C}^d$ in this case. To this end, we need to take a closer look at the isometric structure of $\mathcal{A}_I$.

To start with, let $I \subset \mathbb{C}[z]$ be a homogeneous ideal, and observe that $\mathbb{C}[z]/I$ can be regarded as a dense subspace of both $\mathcal{F}_I$ and $\mathcal{A}_I$. Indeed, both maps

$$\mathbb{C}[z] \to \mathcal{F}_I, \quad p \mapsto P_{\mathcal{F}_I} p$$

and

$$\mathbb{C}[z] \to \mathcal{A}_I, \quad p \mapsto p(S^I) = P_{\mathcal{F}_I} M_p\big|_{\mathcal{F}_I}$$

have kernel I (see Lemma 1.12 and Lemma 2.16 (c)), so they induce embeddings of $\mathbb{C}[z]/I$ into $\mathcal{F}_I$ and $\mathcal{A}_I$, respectively. It is clear that both maps have dense image. Since $\mathbb{C}[z]/I$ carries a natural grading, we have a notion of a graded homomorphism between algebras of the type $\mathcal{A}_I$.

Definition 3.30. Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals, and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a unital algebra homomorphism. We call Φ a *graded homomorphism* if it restricts to a graded homomorphism between $\mathbb{C}[z]/I$ and $\mathbb{C}[z]/J$, viewed as a subspace of $\mathcal{A}_I$ and $\mathcal{A}_J$, respectively.

We emphasize that a graded homomorphism in particular maps "polynomial multipliers" to "polynomial multipliers", that is, an element of the form $p(S^I)$ for some polynomial p is mapped to $q(S^J)$ for some polynomial q.

In the radical case, vacuum-preserving isomorphisms are automatically graded.

Lemma 3.31. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals. Every vacuum-preserving algebra isomorphism $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is graded.*

Proof. Proposition 3.24 shows that there exists an (invertible) linear map A on $\mathbb{C}^d$ such that regarding $\mathcal{A}_I$ and $\mathcal{A}_J$ as function algebras on $Z(I)$ and $Z(J)$, respectively, Φ is given by composition with A. It follows that for $p \in \mathbb{C}[z]$, we have $\Phi(p(S^I)) = (p \circ A)(S^J)$, so that Φ restricts to

$$\mathbb{C}[z]/I \to \mathbb{C}[z]/J, \quad [p] \mapsto [p \circ A],$$

which is obviously graded. □

For $n \in \mathbb{N}$, let $(\mathbb{C}[z]/I)_n$ denote the degree n part of the graded algebra $\mathbb{C}[z]/I$. The embeddings of $\mathbb{C}[z]/I$ into $\mathcal{F}_I$ and $\mathcal{A}_I$ induce norms on $\mathbb{C}[z]/I$, which will be different in general. However, they coincide on each graded part $(\mathbb{C}[z]/I)_n$.

Lemma 3.32. *Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal and let n be a natural number. Suppose that $p \in (\mathbb{C}[z])_n$. Then*

$$||p(S^I)||_{\mathcal{A}_I} = ||P_{\mathcal{F}_I} p||_{\mathcal{F}_I}.$$

Proof. The assertion is trivial if $I = \mathbb{C}[z]$. Otherwise, let $p \in \mathbb{C}[z]_n$. Since $1 \in \mathcal{F}_I$, we have $||P_{\mathcal{F}_I} p|| = ||p(S^I)1|| \leq ||p(S^I)||$. We will deduce the non-trivial equality from Lemma 2.9. First, note that $P_{\mathcal{F}_I} p$ and $P_{\bar{I}} p$ are homogeneous polynomials of degree n by Lemma 1.13. Lemma 1.12 shows that $P_{\bar{I}} p \in I$, hence for any $f \in \mathcal{F}_I$, the function $(P_{\bar{I}} p) \cdot f$ is contained in $\bar{I}$ by Lemma 2.16 (a). We conclude that for $f \in \mathcal{F}_I$, we have

$$\begin{aligned} ||p(S^I)f|| = ||P_{\mathcal{F}_I}(p \cdot f)|| &= ||P_{\mathcal{F}_I}((P_{\mathcal{F}_I} p + P_{\bar{I}} p) \cdot f)|| \\ &= ||P_{\mathcal{F}_I}((P_{\mathcal{F}_I} p) \cdot f)|| \leq ||P_{\mathcal{F}_I} p|| \, ||f||, \end{aligned}$$

where we have used the crucial Lemma 2.9 in the last inequality. This observation finishes the proof. ☐

In this section, we will only need the above lemma for homogeneous polynomials of degree 1. In this case, the use of Lemma 2.9 can be replaced with an application of Remark 2.11.

Clearly, every homomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$ given by composition with a linear map is graded. The converse of this fact is also true, and, more importantly in our situation, if a graded isomorphism restricts to an isometry between the degree one parts of $\mathbb{C}[z]/I$ and $\mathbb{C}[z]/J$, then it is given by composition with a unitary.

Lemma 3.33. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals, and let*

$$\Phi : \mathbb{C}[z]/I \to \mathbb{C}[z]/J$$

be a unital graded homomorphism. Then there is a linear map A on $\mathbb{C}^d$ such that

$$\Phi([p]) = [p \circ A]$$

for all $p \in \mathbb{C}[z]$. Moreover, the following statements hold:

(a) *If Φ is an isomorphism, then A can be chosen to be invertible, and we have $J = \{p \circ A : p \in I\}$ in this case.*

(b) *If Φ is an isomorphism which maps $(\mathbb{C}[z]/I)_1$, viewed as a subspace of $\mathcal{A}_I$, isometrically to $(\mathbb{C}[z]/J)_1$, viewed as a subspace of $\mathcal{A}_J$, then A can be chosen to be unitary.*

Proof. Let

$$\Phi_1 : (\mathbb{C}[z]/I)_1 \to (\mathbb{C}[z]_J)_1$$

be the restriction of Φ to $(\mathbb{C}[z]/I)_1$. Then Φ_1 lifts to a linear map Ψ_1 on $\mathbb{C}[z]_1$, that is, there is a linear map Ψ_1 on $\mathbb{C}^d$ such that the diagram

$$\begin{tikzcd} \mathbb{C}[z]_1 \arrow[r, "\Psi_1"] \arrow[d] & \mathbb{C}[z]_1 \arrow[d] \\ (\mathbb{C}[z]/I)_1 \arrow[r, "\Phi_1"] & (\mathbb{C}[z]/J)_1 \end{tikzcd}$$

commutes. Note that if Φ and hence Φ_1 is invertible, then Ψ_1 can be chosen to be invertible. If Φ_1 is in addition isometric under the identifications $(\mathbb{C}[z]/I)_1 \subset \mathcal{A}_I$ and $(\mathbb{C}[z]/J)_1 \subset \mathcal{A}_J$, then Lemma 3.32 asserts that it is also isometric if we endow $(\mathbb{C}[z]/I)_1$ with the norm of $\mathcal{F}_I \subset H^2_d$, and similarly for $(\mathbb{C}[z]/J)_1$. Thus, the map Ψ_1 can be chosen to be unitary on $\mathbb{C}[z]_1 \subset H^2_d$ in this case.

Now, let A be the unique linear map on $\mathbb{C}^d$ such that

$$\Psi_1(\langle \cdot, \lambda \rangle) = \langle \cdot, A^* \lambda \rangle \quad \text{for all } \lambda \in \mathbb{C}^d.$$

Then A is invertible (respectively unitary) if Ψ_1 is, and $\Psi_1(p) = p \circ A$ holds for all $p \in \mathbb{C}[z]_1$ (these facts can be deduced, for example, from Proposition 2.8).

We conclude that

$$\Phi([p]) = [p \circ A]$$

holds for all $p \in \mathbb{C}[z]_1$, and since Φ is a unital algebra homomorphism, it holds for all $p \in \mathbb{C}[z]$. In particular, if Φ is bijective, then $p \circ A \in J$ if and only if $p \in I$. □

The preceding lemma enables us to prove a refinement of Proposition 3.24 for isometric isomorphisms.

Proposition 3.34. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals, and suppose that $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is a vacuum-preserving isometric isomorphism. Then there exists a unitary map U on $\mathbb{C}^d$ which maps $V(J)$ onto $V(I)$, and such that Φ is the isomorphism Φ^I_U from Lemma 2.22. In particular, if we regard $\mathcal{A}_I$ and $\mathcal{A}_J$ as algebras of functions on $Z(I)$ and $Z(J)$, respectively, Φ is given by*

$$\mathcal{A}_I \to \mathcal{A}_J, \quad \varphi \mapsto \varphi \circ U.$$

Proof. By Lemma 3.31, the isomorphism Φ is graded, so that Lemma 3.33 yields a unitary U on $\mathbb{C}^d$ such that $J = \{p \circ U : p \in I\}$, and such that

$$\Phi(P_{\mathcal{F}_I} M_p \big|_{\mathcal{F}_I}) = P_{\mathcal{F}_J} M_{p \circ U} \big|_{\mathcal{F}_J}$$

holds for all polynomials p. It follows that $UV(J) = V(I)$ (compare Remark 2.23) and that Φ coincides with the isomorphism Φ^I_U from Lemma 2.22. □

In particular, we see that vacuum-preserving isometric isomorphisms lift to automorphisms of $\mathcal{A}_d$.

Corollary 3.35. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals, and suppose that $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is a vacuum-preserving preserving isometric isomorphism. Then Φ is unitarily implemented, and it lifts to an automorphism $\widehat{\Phi}$ of $\mathcal{A}_d$ in the sense that the diagram*

$$\begin{tikzcd} \mathcal{A}_d \arrow[r, "\hat{\Phi}"] \arrow[d] & \mathcal{A}_d \arrow[d] \\ \mathcal{A}_I \arrow[r, "\Phi"] & \mathcal{A}_J \end{tikzcd}$$

commutes, where the vertical arrows are the natural quotient maps.

Proof. By Proposition 3.34, there is a unitary map U on $\mathbb{C}^d$ such that $\Phi = \Phi_U^I$. Note that Φ_U^I is unitarily implemented. Moreover, if Φ_U denotes the automorphism of $\mathcal{A}_d$ given by composition with U (see Lemma 2.21), then we have

$$P_{\mathcal{F}_J} \Phi_U(M_\varphi)\big|_{\mathcal{F}_J} = \Phi_U^I(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I})$$

for all $\varphi \in \mathcal{A}_d$, so that Φ_U is the desired lifting of Φ. □

Just as in the previous section, we can now deduce the desired necessary criterion for $\mathcal{A}_I$ and $\mathcal{A}_J$ being isometrically isomorphic, where I and J are radical homogeneous ideals. Combined with the results established in Section 2.4, this allows for a classification of the algebras $\mathcal{A}_I$ up to isometric isomorphism in terms of the geometry of the vanishing loci of the ideals I. If H and K are Hilbert spaces and if $\mathcal{A} \subset \mathcal{L}(H)$ and $\mathcal{B} \subset \mathcal{L}(K)$ are norm-closed subalgebras, we say that $\mathcal{A}$ and $\mathcal{B}$ are unitarily equivalent if there exists a unitary $U : H \to K$ such that

$$\mathcal{A} = \{U^* T U : T \in \mathcal{B}\}.$$

Clearly, the operator algebras $\mathcal{A}$ and $\mathcal{B}$ are completely isometrically isomorphic in this case.

Theorem 3.36. *Let $I, J \subset \mathbb{C}[z]$ be radical homogeneous ideals. Then the following assertions are equivalent:*

(i) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic.*

(ii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are unitarily equivalent.*

(iii) *There is a unitary U on $\mathbb{C}^d$ such that $UV(J) = V(I)$.*

Proof. We merely have to collect what we have already shown. According to Remark 3.1, we may assume that I and J are proper ideals. To establish (iii) $\Rightarrow$ (ii), suppose that $UV(J) = V(I)$. Then $J = \{p \circ U : p \in I\}$ (see Remark 2.23), so that Lemma 2.22 shows that $\mathcal{A}_I$ and $\mathcal{A}_J$ are unitarily equivalent.

The implication (ii) $\Rightarrow$ (i) is trivial, and (i) $\Rightarrow$ (iii) follows from Proposition 3.27 and Proposition 3.34. □

Remark 3.37. So far, we have only considered the case where I and J are (radical) homogeneous ideals of complex polynomials in the same number of variables. However, the more general setting where $I \subset \mathbb{C}[z_1, \ldots, z_d]$ and $J \subset \mathbb{C}[z_1, \ldots, z_{d'}]$ are homogeneous ideals such that possibly $d \neq d'$ can be reduced to the case $d = d'$.

To this end, we may assume that $d' \geq d$. We define I' to be the homogeneous ideal in $\mathbb{C}[z_1, \ldots, z_{d'}]$ generated by I and the monomials $z_{d+1}, \ldots, z_{d'}$. Identifying $\mathbb{C}^d$ with $\mathbb{C}^d \oplus \{0\} \subset \mathbb{C}^{d'}$ in the obvious way, $V(I)$ equals $V(I')$. If I_d denotes the ideal in $\mathbb{C}[z_1, \ldots, z_{d'}]$ generated by $z_{d+1}, \ldots, z_{d'}$, then I' decomposes as a direct sum of vector spaces

$$I' = I \oplus I_d,$$

and I_d is orthogonal to $\mathbb{C}[z_1, \ldots, z_d]$ in $H^2_{d'}$. This observation implies that the natural algebra homomorphism

$$\mathbb{C}[z_1, \ldots, z_d]/I \to \mathbb{C}[z_1, \ldots, z_{d'}]/I'$$

is an isomorphism which extends to a unitary operator U from $\mathcal{F}_I = H^2_d \ominus I$ onto $\mathcal{F}_{I'} = H^2_{d'} \ominus I'$. Note that, in particular, I' is radical if and only if I is. Moreover, it is easy to check that $T \mapsto U^*TU$ defines a completely isometric isomorphism from $\mathcal{A}_{I'}$ onto $\mathcal{A}_I$ which maps $S_i^{I'}$ to S_i^I for $i = 1, \ldots, d$. Hence there is no loss of generality in assuming that $d = d'$.

4. Sufficient conditions for isomorphisms between the algebras $\mathcal{A}_I$

4.1. Linear maps on homogeneous varieties

Suppose that I and J are radical homogeneous ideals in $\mathbb{C}[z]$. In the preceding chapter, we have deduced a necessary condition for $\mathcal{A}_I$ and $\mathcal{A}_J$ being topologically isomorphic, namely that there exists an invertible linear map A on $\mathbb{C}^d$ mapping $Z(J)$ onto $Z(I)$ (see Theorem 3.28). We have seen that if $\mathcal{A}_I$ and $\mathcal{A}_J$ are even isometrically isomorphic, then the map A can be chosen to be unitary. The converse of the latter result, that is, that a unitary on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$, and thus maps $V(J)$ onto $V(I)$, induces an isometric isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$, was almost immediate (see Lemma 2.22).

However, the question, whether the necessary condition for $\mathcal{A}_I$ and $\mathcal{A}_J$ being topologically isomorphic given by Theorem 3.28 is also sufficient, is more difficult. Explicitly, we ask:

Question 4.1. Let $I, J \subset \mathbb{C}[z]$ be radical homogeneous ideals. Suppose that there is an invertible linear map on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$. Does it follow that $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic?

In [DRS11], a positive answer to this question was given for the case of tractable varieties, and it was conjectured that the answer is affirmative in general. The aim of this chapter is to prove this conjecture.

To this end, we first prove a result from [DRS11] concerning invertible linear maps mapping $Z(J)$ onto $Z(I)$, which reduces the above question to the case where the ideals are vanishing ideals of unions of subspaces. We begin by observing that such linear maps must preserve the norm on $Z(J)$.

Lemma 4.2. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals and let A be a linear map on $\mathbb{C}^d$ that takes $Z(I)$ bijectively onto $Z(J)$. Then $||Av|| = ||v||$ for all $v \in Z(I)$.*

Proof. Let $v \in Z(I) \setminus \{0\}$. Since I is homogeneous, $\frac{v}{||v||} \in Z(I)$, so by assumption,

$$\left|\left|A \frac{v}{||v||}\right|\right| \leq 1,$$

that is, $||Av|| \leq ||v||$. To establish the reverse inequality, note that $Av \neq 0$, so there is a $w \in Z(I)$ such that $Aw = \frac{Av}{||Av||}$, that is, $Av = A(||Av||w)$. We conclude that $v = ||Av||w$, and hence $||v|| \leq ||Av||$, as desired. □

This result can be significantly strengthened if $V(I)$ is irreducible. We will show that in this case, the linear map A is automatically isometric on the linear span of $V(I)$. To this end, we need some results from algebraic geometry.

By a *variety* in $\mathbb{C}^d$, we mean the vanishing locus of an ideal of polynomials in d variables. Note that we do not require that varieties are irreducible. A variety V is called homogeneous if it is invariant under multiplication with complex scalars. Equivalently, the vanishing ideal of V is a homogeneous ideal. If V is a variety and $p \in V$, the tangent space of V at p is defined by

$$T_p(V) = V(d_p(f) : f \in I(V)),$$

where

$$d_p(f) = \sum_{i=1}^{d} \frac{\partial f}{\partial z_i}(p) z_i.$$

If V is irreducible, the set of *singular* points of V (see [Har77, Chapter I, Section 5]) is denoted by $\mathrm{Sing}(V)$. Points which are not singular are called *smooth.*

Over $\mathbb{C}$, a point $p \in V$ is smooth if and only if V is a complex submanifold of $\mathbb{C}^d$ in a neighborhood of p, see for example [Tay02, Proposition 13.3.6]. We only need the easy direction of this fact.

Lemma 4.3. *Let $V \subset \mathbb{C}^d$ be an irreducible variety, and suppose that $p \in V$ is a smooth point. Then there is an open neighborhood U of p in $\mathbb{C}^d$ such that $V \cap U$ is a complex submanifold of $\mathbb{C}^d$. Moreover, the tangent space $T_p(V)$ is the set of all tangent vectors $\gamma'(0)$ at $\gamma(0) = p$ of complex analytic curves $\gamma : \Omega \to V$, where $\Omega \subset \mathbb{C}$ is a neighborhood of the origin.*

Proof. By Theorem 5.7.1 and Corollary 5.7.2. in [Tay02], the variety V is a complex submanifold of dimension r in a neighborhood of p, where $r = \dim T_p(V)$. This establishes the first assertion. The second assertion is well known and easy to show. Indeed, let W denote the set of all tangent vectors $\gamma'(0)$, where γ is as in the statement of the lemma. Then a straightforward application of the chain rule shows that $W \subset T_p(V)$. Conversely, let $F : \Omega \to \mathbb{C}^d$ be a regular holomorphic parametrisation of V in a neighborhood of p with $F(0)$, where $\Omega \subset \mathbb{C}^r$ is an open neighborhood of 0. Then the image of the derivate of F at 0 is an r-dimensional subspace of $\mathbb{C}^r$, which is clearly contained in W. Since $T_p(V)$ is itself an r-dimensional subspace of $\mathbb{C}^d$, this observation finishes the proof. □

The following lemma is the first step in showing that a linear map A which is isometric on an irreducible variety V is isometric on the linear span of V.

Lemma 4.4. *Let $V \subset \mathbb{C}^d$ be a variety and let $p \in V$ be a smooth point. If A is a linear map on $\mathbb{C}^d$ such that $||Av|| = ||v||$ for all $v \in V$, then $||Av|| = ||v||$ holds for all $v \in T_p(V)$.*

Proof. Write $A = (a_{ij})_{i,j=1}^d$. Given a smooth point $p \in V$ and $v \in T_p(V)$, Lemma 4.3 yields a holomorphic curve $\gamma = (\gamma_i)_{i=1}^d : \Omega \to V$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, where Ω is an open neighborhood of the origin. Clearly,

$$||Az||^2 = \sum_{i,j,k=1}^d a_{ij}\overline{a_{ik}} z_j \overline{z_k}$$

for all $z \in \mathbb{C}^d$, so that for $t \in \Omega$, we have

$$\partial\bar{\partial}\,||A\gamma(t)||^2 = \sum_{i,j,k=1}^d a_{ij}\overline{a_{ik}} \gamma_j'(t)\overline{\gamma_k'(t)} = ||A\gamma'(t)||^2$$

by standard properties of the ∂ and $\bar{\partial}$ operator. This reasoning also applies when A is the identity map, hence

$$\partial\bar{\partial}||\gamma(t)||^2 = ||\gamma'(t)||^2$$

for all $t \in \Omega$. Since A is isometric on V, we deduce that $||A\gamma'(t)||^2 = ||\gamma'(t)||^2$ for all $t \in \Omega$. Evaluating this identity at $t = 0$ finishes the proof. □

The intermediate goal is to build a variety from the tangent spaces at smooth points of a given variety. The first fact we need is the following identity theorem for irreducible varieties.

Theorem 4.5. *Let $W \subset \mathbb{C}^d$ be an irreducible variety and let $U \subset \mathbb{C}^d$ be an open set such that $U \cap W \neq \emptyset$. If $p \in \mathbb{C}[z]$ is a polynomial that vanishes on $U \cap W$, then p vanishes on W.*

Proof. See [Ken77, Theorem IV. 2.11] and the second sentence of the proof. □

As a consequence, we obtain a connection between the euclidean and the Zariski topology on $\mathbb{C}^d$. Recall that the Zariski topology is the topology whose closed sets are precisely the varieties in $\mathbb{C}^d$.

Corollary 4.6. *Let $V \subset \mathbb{C}^d$ be a variety and let $V_0 \subset V$ be Zariski-open and Zariski-dense in V. Then V_0 is dense in V in the euclidean topology.*

Proof. Assume, for a contradiction, that there is a $p \in V$ and a euclidean open neighborhood U of p in $\mathbb{C}^d$ such that $U \cap V_0 = \emptyset$. Let W an irreducible component of V containing p. Then $\emptyset \neq U \cap W \subset V \setminus V_0$. Hence Theorem 4.5 shows that $I(V \setminus V_0) \subset I(U \cap W) = I(W)$, thus $W \subset V \setminus V_0$ since V_0 is Zariski-open in V. But because V_0 is Zariski-dense in V, we have $W \cap V_0 \neq \emptyset$, since otherwise, V would be contained in the union of the irreducible components not equal to W. This contradiction finishes the proof. □

It is well known that the projection of an algebraic set need not be algebraic in the affine setting. The following theorem shows that what is missing is rather small.

Theorem 4.7. *Let $V \subset \mathbb{C}^{d+e}$ be a variety and let $\pi : \mathbb{C}^{d+e} \to \mathbb{C}^d$ be the projection onto the first d components. If V_d denotes the Zariski-closure of $\pi(V)$ in $\mathbb{C}^d$, then there is a variety $W \subsetneq V_d$ such that*

$$V_d \setminus W \subset \pi(V).$$

Proof. See [CLO92, Chapter 3, Paragraph 2, Theorem 3] □

Combining the last two results, we obtain the following useful fact.

Corollary 4.8. *Let $V \subset \mathbb{C}^{d+e}$ be a variety and let $\pi : \mathbb{C}^{d+e} \to \mathbb{C}^d$ be the projection onto the first d components. Then the euclidean closure of $\pi(V)$ is a variety in $\mathbb{C}^d$.*

Proof. By decomposing V into its irreducible components, we may assume that V is irreducible. Since π is continuous in the Zariski topology, the Zariski closure V_d of $\pi(V)$ is irreducible as well. Theorem 4.7 yields a proper subvariety W of V_d such that $V_d \setminus W \subset \pi(V)$. In particular, $V_d \setminus W$ is non-empty and Zariski-open in V_d, hence, by irreducibility of V_d, it is Zariski-dense in V_d. Consequently, Corollary 4.6 shows that the euclidean closure of $V_d \setminus W$ is V_d. Since $V_d \setminus W \subset \pi(V) \subset V_d$, this observation finishes the proof. □

We are now in the position to construct the desired variety from tangent spaces at smooth points of a given variety.

Lemma 4.9. *Let $V \subset \mathbb{C}^d$ be an irreducible variety. Then the euclidean closure of*

$$\bigcup_{p \in V \setminus \operatorname{Sing}(V)} T_p(V) \subset \mathbb{C}^d.$$

is a homogeneous variety.

Proof. Let $I = I(V)$, set

$$X_0 = \bigcup_{p \in V \setminus \operatorname{Sing}(V)} \{p\} \times T_p(V) \subset \mathbb{C}^d \times \mathbb{C}^d,$$

and let X be the Zariski closure of X_0. If we denote the coordinates on $\mathbb{C}^d \times \mathbb{C}^d$ by (z, w), we see that

$$\bigcup_{p \in V} \{p\} \times T_p(V) = V(\{(f \otimes 1)(z) : f \in I\} \cup \{d_z f(w) : f \in I\})$$

is a variety. Here, $f \otimes 1$ is the polynomial in $\mathbb{C}[z, w]$ with $(f \otimes 1)(z, w) = f(z)$. In particular, X is contained in this variety, so $X_0 = X \cap (\operatorname{Sing}(V) \times \mathbb{C}^d)^c$. Since $\operatorname{Sing}(V)$

is Zariski-closed, the set X_0 is Zariski open in X. An application of Corollary 4.6 yields that $\overline{X_0} = X$, where $\overline{X_0}$ is the euclidean closure of X_0. Let $\pi : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ be the projection onto the last d components. Then by Corollary 4.8, $\overline{\pi(X)} \subset \mathbb{C}^d$ is a variety. Since π is continuous, we have $\overline{\pi(X_0)} = \overline{\pi(X)}$, hence the set in question is a variety. Since all tangent spaces are linear subspaces of $\mathbb{C}^d$, it is necessarily homogeneous. □

We can now prove the result about isometric linear maps on irreducible homogeneous varieties which was alluded to above.

Proposition 4.10. *Let $V \subset \mathbb{C}^d$ be an irreducible homogeneous variety, and let A be a linear map on $\mathbb{C}^d$ such that $||Az|| = ||z||$ for all $z \in V$. Then*

$$||Az|| = ||z|| \quad \textit{for all} \quad z \in \operatorname{span}(V).$$

Proof. We will construct a sequence of homogeneous irreducible algebraic varieties

$$V = V_0 \subset V_1 \subset V_2 \subset \dots$$

such that $||Az|| = ||z||$ for all $z \in V_i$ and all i, and such that either $\dim V_i < \dim V_{i+1}$ or V_i is a subspace of $\mathbb{C}^d$. Since the dimensions of the varieties V_i are bounded from above by d, this process will eventually yield a subspace V_i and thus prove the proposition.

Assume that $i \in \mathbb{N}$ and that V_i has already been constructed. By Lemma 4.4, we know that $||Az|| = ||z||$ for all $z \in T_p(V_i)$ and all $p \in V_i \setminus \operatorname{Sing}(V_i)$. Now, let W denote the euclidean closure of

$$\bigcup_{p \in V_i \setminus \operatorname{Sing}(V_i)} T_p(V_i) \subset \mathbb{C}^d,$$

which is a homogeneous variety according to Lemma 4.9. By continuity of A, the linear map A is isometric on W. It is an easy consequence of the homogeneity of V_i that $p \in T_p(V_i)$ for each $p \in V_i$, hence $V_i \setminus \operatorname{Sing}(V_i) \subset W$. Because the smooth points of V_i are dense in V_i in the Zariski topology (see [Har77, Theorem 5.3]), it follows that V_i is contained in W. The irreducibility of V_i implies that there is an irreducible component V_{i+1} of W, which is necessarily homogeneous (see, for example, Section 3.5 in [Eis95]), such that $V_i \subset V_{i+1}$. Let $\widetilde{W}$ denote the union of the other irreducible components of W.

We finish the proof by showing that if $\dim V_i = \dim V_{i+1}$, then V_i is a linear subspace of $\mathbb{C}^d$. Since both V_i and V_{i+1} are irreducible, the definition of dimension as the supremum of lengths of chains of irreducible subvarieties (see the Definition after Corollary 1.6 in [Har77]) implies that $V_i = V_{i+1}$ holds in this case. Now

$$\bigcup_{p \in V_i \setminus \operatorname{Sing}(V_i)} T_p(V_i) \subset W = \widetilde{W} \cup V_i.$$

Since each $T_p(V_i)$ is irreducible, it is contained in V_i or in $\widetilde{W}$. If all $T_p(V_i)$ for $p \in V_i \setminus \mathrm{Sing}(V_i)$ were contained in $\widetilde{W}$, then also W would be contained in $\widetilde{W}$, which is not possible, since $V_i = V_{i+1}$ is not contained in $\widetilde{W}$. Consequently, there is a smooth point $p \in V_i$ such that $T_p(V_i) \subset V_i$. Since $\dim T_p(V_i) = \dim V_i$, we have that $V_i = T_p(V_i)$ is a linear subspace. □

The following example shows that the condition that V is irreducible is essential.

Example 4.11. Let V be the union of the coordinate axes in $\mathbb{C}^2$, and consider the linear map

$$A = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then A is isometric on V, but it is clearly not isometric on $\mathbb{C}^2$, which is the linear span of V.

The preceding proposition enables us to give an affirmative answer to Question 4.1 in the case of irreducible varieties. Moreover, we obtain a rigidity result for certain algebras of the type $\mathcal{A}_I$. Note that the following theorem in particular covers irreducible varieties and non-linear hypersurfaces.

Theorem 4.12. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous radical ideals, and suppose that $V(J)$ has an irreducible component W such that $V(J) \subset \operatorname{span} W$. If there is an invertible linear map A on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$, then $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic. In particular, $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic if and only if they are isometrically isomorphic.*

Proof. By Lemma 4.2, the linear map A is isometric on $V(J)$. Since W is necessarily homogeneous (see, for example, Section 3.5 in [Eis95]), we can apply Proposition 4.10 to W to deduce that A is isometric on $\operatorname{span} W$. Consequently, the assumption on $V(J)$ implies that A is isometric on $\operatorname{span} V(J)$, so that there is a unitary on $\mathbb{C}^d$ which coincides with A on $V(J)$. The first assertion thus follows from Theorem 3.36, and the second one by an application of Theorem 3.28. □

4.2. Algebra isomorphisms and sums of Fock spaces

It was observed in [DRS11] that Proposition 4.10 reduces Question 4.1 to the case where the ideals are vanishing ideals of unions of subspaces. We will shortly see why this is true. Moreover, we show that in this case, the problem is related to the question whether a certain algebraic sum of subspaces of the full Fock space is closed.

Let $I \subset \mathbb{C}[z]$ be a radical homogeneous ideal. An application of Lemma 1.13 shows that $\mathcal{F}_I$ is graded in the sense that

$$\mathcal{F}_I = \bigoplus_{n=0}^{\infty} \mathcal{F}_I \cap \mathbb{C}[z]_n,$$

and in particular, $\mathcal{D}_I = \mathcal{F}_I \cap \mathbb{C}[z]$ is a dense subspace of $\mathcal{F}_I$. We begin by exhibiting a convenient generating set for each homogeneous part $\mathcal{F}_I \cap \mathbb{C}[z]_n$. Note that $\mathcal{F}_I$ can be identified with $H^2_d\big|_{Z^0(I)}$ in the radical case according to Lemma 2.18. Since the kernel functions form a total set in any Hilbert function space (see Lemma A.4 (c)) this fact can be used to obtain a total set in $\mathcal{F}_I$, and therefore a generating set for each $\mathcal{F}_I \cap \mathbb{C}[z]_n$. This can also be done directly.

Lemma 4.13. *Let $I \subset \mathbb{C}[z]$ be a radical homogeneous ideal. Then for all natural numbers n,*

$$\mathcal{F}_I \cap \mathbb{C}[z]_n = \operatorname{span}\{\langle \cdot, \lambda\rangle^n : \lambda \in Z^0(I)\} = \operatorname{span}\{\langle \cdot, \lambda\rangle^n : \lambda \in V(I)\}.$$

Moreover, the set of the kernel functions $K(\cdot, \lambda)$, where $\lambda \in Z^0(I)$, is dense in $\mathcal{F}_I$.

Proof. Note that for any $\lambda \in \mathbb{B}_d$, we have

$$K(\cdot, \lambda) = \sum_{n=0}^{\infty} \langle \cdot, \lambda\rangle^n \in H^2_d,$$

where K is the reproducing kernel of H^2_d. Using that homogeneous polynomials of different degree are orthogonal in H^2_d, we obtain for $\lambda \in Z^0(I)$ and $f \in \mathbb{C}[z]_n$ the identity

$$\Big\langle f, \langle \cdot, \lambda\rangle^n \Big\rangle_{H^2_d} = \Big\langle f, K(\cdot, \lambda) \Big\rangle_{H^2_d} = f(\lambda).$$

In particular, if $f \in I \cap \mathbb{C}[z]_n$ and $\lambda \in Z^0(I)$, then

$$\Big\langle f, \langle \cdot, \lambda\rangle^n \Big\rangle_{H^2_d} = 0,$$

hence $\langle \cdot, \lambda\rangle^n \in \mathcal{F}_I$. Conversely, if $g \in \mathcal{F}_I \cap \mathbb{C}[z]_n$ is orthogonal to each $\langle \cdot, \lambda\rangle^n$ for $\lambda \in Z^0(I)$, then g vanishes on $Z^0(I)$. By homogeneity of I and g, we infer that g vanishes on $V(I)$, hence $g \in I$ by Hilbert's Nullstellensatz. Consequently, $g = 0$, from which the first equality follows, while the second is obvious.

To establish the second assertion, note that the defining property of the kernel functions immediately implies that $K(\cdot, \lambda) \in \mathcal{F}_I$ for all $\lambda \in Z^0(I)$. If $f \in \mathcal{F}_I$ is orthogonal to each $K(\cdot, \lambda)$, then f vanishes on $Z^0(I)$. From Theorem 1.7, we deduce that $f \in \overline{I}$, so that $f = 0$. □

Suppose now that $I, J \subset \mathbb{C}[z]$ are radical homogeneous ideals and that A is a linear map on $\mathbb{C}^d$ which maps $V(J)$ into $V(I)$. Since

$$\langle \cdot, \lambda \rangle^n \circ A^* = \langle \cdot, A\lambda \rangle^n$$

for all $\lambda \in \mathbb{C}^d$ and $n \in \mathbb{N}$, we conclude with the help of the preceding lemma that A induces a densely defined linear map

$$\mathcal{F}_J \supset \mathcal{D}_J \to \mathcal{F}_I, \quad f \mapsto f \circ A^*. \tag{4.1}$$

The crucial problem is to determine when this map is bounded. If J is the vanishing ideal of a single subspace $V \subset \mathbb{C}^d$ and A is isometric on V, then the map is in fact isometric.

Lemma 4.14. *Let $V \subset \mathbb{C}^d$ be a subspace and let $J \subset \mathbb{C}[z]$ be its vanishing ideal. If A is a linear map on $\mathbb{C}^d$ which is isometric on V, then*

$$C_{A^*} : \mathcal{F}_J \supset \mathcal{D}_J \to H^2_d, \quad f \mapsto f \circ A^*$$

is an isometry.

Proof. The assumption on A implies that there exists a unitary U on $\mathbb{C}^d$ which coincides with A on V. By Lemma 2.21, the map

$$C_{U^*} : H^2_d \to H^2_d, \quad f \mapsto f \circ U^*,$$

is a unitary on H^2_d, and it coincides with C_{A^*} on polynomials of the form $\langle z, \lambda \rangle^n$ for $\lambda \in Z^0(J)$ and $n \in \mathbb{N}$. Since the linear span of these elements forms a dense subset of $\mathcal{D}_J$ by Lemma 4.13, we conclude that $C_{A^*} = C_{U^*}\big|_{\mathcal{D}_J}$ is an isometry. □

When considering more complicated algebraic sets such as unions of subspaces, one of course wishes to decompose the sets into smaller pieces which are easier to deal with. Algebraically, this corresponds to writing an ideal as an intersection of larger ideals. On the level of the spaces $\mathcal{F}_I$, we get the following result.

Lemma 4.15. *Let $J_1, \ldots, J_r \subset \mathbb{C}[z]$ be homogeneous ideals and let $J = J_1 \cap \ldots \cap J_r$. Then*

$$\overline{J} = \overline{J_1} \cap \ldots \cap \overline{J_r},$$

and

$$\mathcal{F}_J = \overline{\mathcal{F}_{J_1} + \ldots + \mathcal{F}_{J_r}}.$$

Proof. It suffices to prove the first claim, since the second will then follow by taking orthogonal complements. To this end, note that the inclusion $\overline{J} \subset \overline{J_1} \cap \ldots \cap \overline{J_r}$ is trivial. Conversely, given an element $f \in \overline{J_k}$ with homogeneous expansion

$$f = \sum_{n=0}^{\infty} f_n,$$

Lemma 1.12 shows that each f_n is contained in J_k, from which the reverse inclusion readily follows. □

The question under which conditions the sum $\mathcal{F}_{J_1} + \ldots + \mathcal{F}_{J_r}$ in the preceding lemma is itself closed will be of central importance. The reason lies in the following well-known observation.

Lemma 4.16. *Let H be a Hilbert space and let $M_1, \ldots, M_r \subset H$ be closed subspaces. Then the following are equivalent:*

(i) *$M_1 + \ldots + M_r$ is closed.*

(ii) *There is a constant $C \geq 0$ such that for all $x \in M_1 + \ldots + M_r$, there exist $x_i \in M_i$ for $i = 1, \ldots, r$ such that $x = x_1 + \ldots + x_r$ and*

$$||x_1||^2 + \ldots + ||x_r||^2 \leq C||x||^2.$$

Proof. Let

$$\Phi : M_1 \oplus \ldots \oplus M_r \to \overline{M_1 + \ldots + M_r}$$

be the map given by addition. Then Φ is a continuous linear map between Hilbert spaces with dense image. If (i) holds, Φ is in fact surjective, so a standard application of the open mapping theorem shows that (ii) holds. Conversely, suppose that (ii) is satisfied. Then the induced map

$$(M_1 \oplus \ldots \oplus M_r)/(\ker \Phi) \to \overline{M_1 + \ldots + M_r}$$

is bounded below. Hence its image is closed, so Φ is surjective. □

In general, $\mathcal{F}_{J_1} + \mathcal{F}_{J_2}$ need not be closed for two radical homogeneous ideals J_1 and J_2, see Example 4.24 below. But thanks to the reduction to unions of subspaces from [DRS11] alluded to earlier, we only need to consider the case where the J_k are vanishing ideals of subspaces in $\mathbb{C}^d$.

To keep the statements of the following results reasonably short, we make an ad-hoc definition which will only be used in this section.

Definition 4.17. Let $J \subset \mathbb{C}[z_1, \ldots, z_d]$ be a radical homogeneous ideal, and let $V(J) = W_1 \cup \ldots \cup W_r$ be the decomposition of $V(J)$ into irreducible components. Denote the vanishing ideal of $\operatorname{span} W_k$ by $\widehat{J_k}$. We call J *admissible* if the algebraic sum $\mathcal{F}_{\widehat{J_1}} + \ldots + \mathcal{F}_{\widehat{J_r}}$ is closed.

Proposition 4.18. *Let I and J be radical homogeneous ideals in $\mathbb{C}[z]$. Suppose that there is a linear map A on $\mathbb{C}^d$ that maps $Z(J)$ bijectively onto $Z(I)$. If J is admissible, then*

$$\mathcal{F}_J \supset \mathcal{D}_J \to \mathcal{F}_I, \quad f \mapsto f \circ A^*$$

is a bounded map.

Proof. Let $V(J) = W_1 \cup \ldots \cup W_r$ be the irreducible decomposition of $V(J)$, and let $\widehat{J}_k$ be the vanishing ideal of $\operatorname{span}(W_k)$. Define

$$S = \operatorname{span}(W_1) \cup \ldots \cup \operatorname{span}(W_r),$$

and denote the vanishing ideal of S by $\widehat{J}$, so that $\widehat{J} = \widehat{J}_1 \cap \ldots \cap \widehat{J}_r$. Since $\mathcal{D}_J \subset \mathcal{D}_{\widehat{J}}$, it suffices to show that $f \mapsto f \circ A^*$ defines a bounded map on $\mathcal{D}_{\widehat{J}}$. By Lemma 4.15, we have

$$\mathcal{F}_{\widehat{J}} = \overline{\mathcal{F}_{\widehat{J}_1} + \ldots + \mathcal{F}_{\widehat{J}_r}}.$$

By Lemma 4.2 and Proposition 4.10, the linear map A is isometric on S. Consequently, Lemma 4.14 shows that $f \mapsto f \circ A^*$ defines an isometry on each $D_{\widehat{J}_k} \subset \mathcal{F}_{\widehat{J}_k}$. We will use the hypothesis that J is admissible in order to show that $f \mapsto f \circ A^*$ defines a bounded map on $\mathcal{D}_{\widehat{J}}$. To this end, we note that since $\mathcal{F}_{\widehat{J}_1} + \ldots + \mathcal{F}_{\widehat{J}_r}$ is closed, Lemma 4.16 yields a constant $C \geq 0$ such that for any $f \in \mathcal{F}_{\widehat{J}}$, there are $f_k \in \mathcal{F}_{\widehat{J}_k}$ with $f = f_1 + \ldots + f_r$ and

$$||f_1||^2 + \ldots + ||f_r||^2 \leq C||f||^2.$$

If f is a homogeneous polynomial of degree n, we can choose the f_k to be homogeneous polynomials of degree n as well. Consequently, if $f \in \mathcal{D}_{\widehat{J}}$, the f_k can be chosen from $\mathcal{D}_{\widehat{J}_k}$. With such a choice, we obtain for $f \in \mathcal{D}_{\widehat{J}}$ the (crude) estimate

$$\begin{aligned} ||f \circ A^*||^2 &= ||f_1 \circ A^* + \ldots + f_r \circ A^*||^2 \\ &\leq r^2 \max_{1 \leq k \leq r} ||f_k \circ A^*||^2 \\ &= r^2 \max_{1 \leq k \leq r} ||f_k||^2 \leq Cr^2||f||^2, \end{aligned}$$

where we have used that $f \mapsto f \circ A^*$ is an isometry on each $\mathcal{D}_{\widehat{J}_k}$. □

In the setting of the preceding proposition, let $C_{A^*} : \mathcal{F}_J \to \mathcal{F}_I$ be the continuous extension of $f \mapsto f \circ A^*$ onto $\mathcal{F}_J$. Note that the homogeneous expansion of the kernel function shows in combination with (4.1) that

$$C_{A^*} K(\cdot, \lambda) = K(\cdot, A\lambda) \quad \text{for all } \lambda \in Z^0(J),$$

so that C_{A^*} is the map considered in [DRS11, Section 7.3]. Conjugation with the adjoint of C_{A^*} then yields a topological isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$.

Corollary 4.19. *Let I and J be radical homogeneous ideals in $\mathbb{C}[z]$. Suppose that A is an invertible linear map which maps $Z(J)$ onto $Z(I)$. If I and J are admissible, then C_{A^*} and $C_{(A^{-1})^*}$ are inverse to each other, and*

$$\Phi : \mathcal{A}_I \to \mathcal{A}_J, \quad T \mapsto (C_{A^*})^* T (C_{(A^{-1})^*})^*,$$

is a completely bounded isomorphism. Regarding $\mathcal{A}_I$ and $\mathcal{A}_J$ as function algebras on $Z(I)$ and $Z(J)$, respectively, Φ is given by composition with A, that is,

$$\Phi(\varphi) = \varphi \circ A \quad \textit{for all } \varphi \in \mathcal{A}_I.$$

Proof. It is clear that C_{A^*} and $C_{(A^{-1})^*}$ are inverse to each other. Now, let $p \in \mathbb{C}[z]$ be a polynomial. Then for all $f \in \mathcal{F}_J$ and $\lambda \in Z^0(J)$, we have

$$\begin{aligned}\langle (C_{A^*})^*(P_{\mathcal{F}_I} M_p\big|_{\mathcal{F}_I})(C_{(A^{-1})^*})^* f, K(\cdot,\lambda)\rangle &= \langle M_p (C_{(A^{-1})^*})^* f, K(\cdot, A\lambda)\rangle \\ &= p(A\lambda)\langle f, C_{(A^{-1})^*} K(\cdot, A\lambda)\rangle \\ &= ((p\circ A)\cdot f)(\lambda) \\ &= \langle (P_{\mathcal{F}_J} M_{p\circ A}\big|_{F_J}) f, K(\cdot,\lambda)\rangle.\end{aligned}$$

Since the kernel functions $K(\cdot,\lambda)$, where $\lambda \in Z^0(I)$, form a total subset of $\mathcal{F}_I$ by Lemma 4.13, it follows that

$$(C_{A^*})^*(P_{F_I} M_p\big|_{F_I})(C_{(A^{-1})^*})^* = P_{\mathcal{F}_J} M_{p\circ A}\big|_{\mathcal{F}_J}.$$

We conclude that the continuous map Φ indeed sends $\mathcal{A}_I$ into $\mathcal{A}_J$, and, arguing in the other direction, we find that Φ is an isomorphism. It is clear that Φ is completely bounded, and the final statement easily follows from the above result (see Corollary 2.19 for the identification) using a density argument. ☐

To improve the corresponding results from [DRS11], we will show that every radical homogeneous ideal $I \subset \mathbb{C}[z_1,\dots,z_d]$ is automatically admissible. To this end, we will work with the description of the Drury-Arveson space as symmetric Fock space, rather than as a Hilbert function space (see Proposition 2.8). This identification allows us to translate the condition that the ideals I and J be admissible in terms of symmetric Fock space. In fact, working with the full Fock space suffices.

Lemma 4.20. *Let $J \subset \mathbb{C}[z]$ be a radical homogeneous ideal, and suppose that $V(J) = W_1 \cup \ldots \cup W_r$ is the irreducible decomposition of $V(J)$. Let $V_k = \operatorname{span}(W_k)$. If the algebraic sum of the full Fock spaces $\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r)$ is closed, then J is admissible.*

Proof. Let $\widehat{J_k}$ be the vanishing ideal of V_k. Then by Lemma 4.13, the linear span of the elements $\langle \cdot,\lambda\rangle^n$ with $\lambda \in V_k$ and $n \in \mathbb{N}$ is dense in $\mathcal{F}_{\widehat{J_k}}$, whereas $\mathcal{F}_s(V_k)$ is the closed linear span of the symmetric tensors $\lambda^{\otimes n}$ with $\lambda \in V_k$ and $n \in \mathbb{N}$. Hence, the anti-unitary J from Proposition 2.8 maps $\mathcal{F}_{\widehat{J_k}}$ onto $\mathcal{F}_s(V_k)$, so that I is admissible if and only if the algebraic sum

$$S = \mathcal{F}_s(V_1) + \ldots + \mathcal{F}_s(V_r)$$

is closed.

Now, let Q be the orthogonal projection from $\mathcal{F}(\mathbb{C}^d)$ onto $\mathcal{F}_s(\mathbb{C}^d)$. Note that for a subspace $V \subset \mathbb{C}^d$, the orthogonal projection from $(\mathbb{C}^d)^{\otimes n}$ onto $V^{\otimes n}$ is just $P_V^{\otimes n}$. Combined with the description of Q in Lemma 2.7, we deduce that for a subspace $V \subset \mathbb{C}^d$, the projections Q and $P_{\mathcal{F}(V)}$ commute and $QP_{\mathcal{F}(V)} = P_{\mathcal{F}_s(V)}$, from which is easily follows that closedness of $\mathcal{F}(V_1)+\ldots+\mathcal{F}(V_r)$ implies closedness of S. Indeed, if x is in the closure of S, then we can write $x = \widetilde{x}_1 + \ldots + \widetilde{x}_r$ with $\widetilde{x}_k \in \mathcal{F}(V_k)$. Setting $x_k = Q\widetilde{x}_k \in \mathcal{F}_s(V_k)$, we have

$$x = Qx = x_1 + \ldots x_r \in S. \qquad \square$$

4.3. The Friedrichs angle

In order to show that sums of full Fock spaces are closed, we will make use of a classical notion of angle between two closed subspaces of a Hilbert space due to Friedrichs [Fri37] (for the history of this and related quantities, see for example [BS10]).

Definition 4.21. Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces. If $M \not\subset N$ and $N \not\subset M$, the Friedrichs angle between M and N is defined to be the angle in $[0, \frac{\pi}{2}]$ whose cosine is

$$c(M,N) = \sup_{\substack{x \in M \ominus (M\cap N)\\ y \in N \ominus (M \cap N)\\ x \neq 0 \neq y}} \frac{|\langle x, y\rangle|}{||x||\,||y||}.$$

Otherwise, we set $c(M,N) = 0$.

Let us record some immediate consequences of this definition.

Remark 4.22. Let $M, N \subset H$ be closed subspaces of a Hilbert space H.

(a) Clearly, $c(M,N) = c(N,M)$.

(b) By the Cauchy-Schwarz inequality, $0 \leq c(M,N) \leq 1$.

(c) Since the spaces $M \ominus (M \cap N)$ and $N \ominus (M \cap N)$ have trivial intersection, the supremum is in fact taken over a subset of $[0,1)$. Because the unit ball of a finite-dimensional space is compact, the supremum is attained if M and N are finite dimensional, and thus $c(M,N) < 1$ in this case.

(d) $c(M,N) = c(M \ominus (M \cap N), N \ominus (M \cap N))$.

(e) It is elementary to see that $c(M,N)$ is the smallest real number $c \geq 0$ such that

$$|\langle x, y\rangle| \leq c||x||\,||y||$$

holds for all $x \in M$ and $y \in N \ominus (M \cap N)$ (or for all $x \in M \ominus (M \cap N)$ and $y \in N$)).

The Friedrichs angle between two subspaces is closely related to the question whether their algebraic sum is closed. The following lemma is the reason why we consider this quantity.

Lemma 4.23. *Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces. Then the following are equivalent:*

(i) *The algebraic sum $M + N$ is closed.*

(ii) $c(M, N) < 1$.

(iii) *There is a constant $C \geq 0$ such that for all $z \in M + N$, there are (necessarily unique) $x \in M$ and $y \in N \ominus (M \cap N)$ such that $z = x + y$ and such that*

$$||x||^2 + ||y||^2 \leq C||z||^2. \tag{4.2}$$

Moreover, the smallest possible constant C in (4.2) *is given by* $\frac{1}{1-c(M,N)}$.

Proof. The implication (iii) $\Rightarrow$ (i) follows from Lemma 4.16. Conversely, suppose that (i) holds. By Lemma 4.16, there is a constant C_0 such that for all $z \in M + N$, there are $x \in M$ and $y \in N$ such that $z = x + y$ and such that

$$||x||^2 + ||y||^2 \leq C_0||z||^2.$$

Now write $y = \widetilde{y} + \hat{y}$, where $\widetilde{y} \in N \ominus (M \cap N)$ and $\hat{y} \in M \cap N$. Setting $\widetilde{x} = x + \hat{y} \in M$, we have $||\widetilde{x}||^2 \leq 2(||x||^2 + ||\hat{y}||^2)$ and $\widetilde{x} + \widetilde{y} = z$, so

$$||\widetilde{x}||^2 + ||\widetilde{y}||^2 \leq 2(||x||^2 + ||\hat{y}||^2) + ||\widetilde{y}||^2 \leq 2(||x||^2 + ||y||^2) \leq 2C_0||z||^2.$$

Thus (4.2) holds with $C = 2C_0$. Moreover, if $x_1 + y_1 = x_2 + y_2$ for elements $x_i \in M, y_i \in N \ominus (M \cap N)$, $i = 1, 2$, then $x_1 - x_2 = y_2 - y_1 \in (N \ominus (M \cap N)) \cap M$, so $x_1 = x_2$ and $y_1 = y_2$, which shows the uniqueness statement. Hence (iii) holds.

To show that (ii) implies (iii), let $z \in M + N$ and write $z = x + y$ where $x \in M$ and $y \in N \ominus (M \cap N)$. Since

$$\Big| ||z||^2 - (||x||^2 + ||y||^2) \Big| \leq 2|\langle x, y\rangle| \leq 2c(M, N)||x|| \, ||y|| \leq c(M, N)(||x||^2 + ||y||^2),$$

(iii) holds with $C = \frac{1}{1-c(M,N)}$.

Finally, we show (iii) $\Rightarrow$ (ii). To this end, let $x \in M$ and $y \in N \ominus (M \cap N)$ with $||x|| = ||y|| = 1$. Since the decomposition of $x + y$ into elements of M and $N \ominus (M \cap N)$ is unique, we have

$$2 = ||x||^2 + ||y||^2 \leq C||x + y||^2$$

by assumption, thus

$$\frac{2}{C} \leq ||x + y||^2 = 2 + 2\,\mathrm{Re}\langle x, y\rangle,$$

that is, $\mathrm{Re}\langle x, y\rangle \geq \frac{1}{C} - 1$. Note that this is true for all $x \in M$ and $y \in N \ominus (M \cap N)$. Multiplying x by a suitable complex scalar of modulus one, we see that

$$-|\langle x, y\rangle| \geq \frac{1}{C} - 1,$$

which shows $c(M, N) \leq 1 - \frac{1}{C}$. This observation finishes the proof. □

Recently, Badea, Grivaux and Müller [BGM10] have introduced a generalization of the Friedrichs angle to more than two subspaces. Although we want to show closedness of sums of arbitrarily many Fock spaces, an inductive argument using the classical definition for two subspaces seems to be more feasible in our case.

As a first application, we exhibit two radical homogeneous ideals $I, J \subset \mathbb{C}[z]$ such that $\mathcal{F}_I + \mathcal{F}_J$ is not closed. When the ideals are not necessarily radical, an example for this phenomenon is also given by Shalit's example of a set of polynomials which is not a stable generating set, see [Sha11, Example 2.6].

Example 4.24. Let $I = \langle y^2 + xz\rangle$ and $J = \langle x\rangle$ in $\mathbb{C}[x, y, z]$. We claim that $\mathcal{F}_I + \mathcal{F}_J$ is not closed. It is well known that for two closed subspaces M and N of a Hilbert space H, closedness of $M + N$ is equivalent to closedness of $M^\perp + N^\perp$ (see also Proposition 4.34 below), so that it suffices to show that $\overline{I} + \overline{J}$ is not closed. To this end, we set for $n \geq 2$

$$f_n = z^{n-2}(y^2 + xz) \quad \text{and} \quad g_n = z^{n-1}x.$$

Clearly, $f_n \in I$ and $g_n \in J$ for all n. Using that different monomials in H^2_d are orthogonal, one easily checks that all f_n and g_n are orthogonal to $I \cap J = \langle x^2z + xy^2\rangle$, so they are orthogonal to $\overline{I} \cap \overline{J} = \overline{I \cap J}$ (see Lemma 4.15) as well. Moreover, a straightforward calculations yields

$$||f_n||^2 = \frac{n+1}{n(n-1)} \quad \text{and} \quad \langle f_n, g_n\rangle = ||g_n||^2 = \frac{1}{n}.$$

Consequently,

$$\frac{\langle f_n, g_n\rangle}{||f_n||\,||g_n||} = \sqrt{\frac{n-1}{n+1}} \xrightarrow{n\to\infty} 1,$$

from which we conclude that $c(\overline{I}, \overline{J}) = 1$, so that $\overline{I} + \overline{J}$ is not closed by Lemma 4.23.

The Friedrichs angle can also be expressed in terms of certain orthogonal projections.

Lemma 4.25. *Let H be a Hilbert space and let M and N be closed subspaces of H. Then $c(M,N) = ||P_M P_N - P_{M\cap N}||$ and $c(M,N)^2 = ||P_N P_M P_N - P_{M\cap N}||$.*

Proof. We first consider the case $M \cap N = \{0\}$. Let $x \in M$ and $y \in N$. Then

$$|\langle x, y\rangle| = |\langle x, P_M P_N y\rangle| \leq ||P_M P_N||\,||x||\,||y||,$$

so $c(M,N) \leq ||P_M P_N||$. On the other hand, we have for all $x \in H$ the estimate

$$\begin{aligned} ||P_M P_N x||^2 = |\langle P_M P_N x, P_N x\rangle| &\leq c(M,N)||P_M P_N x||\,||P_N x|| \\ &\leq c(M,N)||P_M P_N x||\,||x||, \end{aligned}$$

so $||P_M P_N|| \leq c(M,N)$, which completes the proof if $M \cap N = \{0\}$.

The general case from this one by using part (d) of Remark 4.22 and observing that by the first case,

$$\begin{aligned} c(M \ominus (M\cap N), N \ominus (M \cap N)) &= ||(P_M - P_{M\cap N})(P_N - P_{M\cap N})|| \\ &= ||P_M P_N - P_{M\cap N}||. \end{aligned}$$

To show the second assertion, we set $T = P_M P_N - P_{M\cap N}$ and note that

$$T^*T = (P_N P_M - P_{M\cap N})(P_M P_N - P_{M\cap N}) = P_N P_M P_N - P_{M\cap N}.$$

Hence, by the first part,

$$c(M,N)^2 = ||T||^2 = ||T^*T|| = ||P_N P_M P_N - P_{M\cap N}||. \qquad \square$$

Let H be a Hilbert space which is graded in the sense that H is the orthogonal direct sum $H = \bigoplus_{n\in\mathbb{N}} H_n$ for some Hilbert spaces H_n. Denote the orthogonal projection from H to H_n by P_n. We say that a closed subspace $M \subset H$ is graded if $P_n P_M = P_M P_n$ for all $n \in \mathbb{N}$. Equivalently,

$$M = \bigoplus_{n=0}^{\infty} M \cap H_n.$$

Note that M is graded if and only if P_M belongs to the commutant of $\{P_n : n \in \mathbb{N}\}$, which is a von Neumann algebra. In particular, if $M, N \subset H$ are graded, then $\overline{M+N}$ and $M \cap N$ are graded as well. The most important examples of graded Hilbert spaces in our case are full Fock spaces and sums thereof.

The angle between two graded subspaces can be easily expressed in terms of the angles between their graded components by the following formula.

Lemma 4.26. *Let $H = \bigoplus_{n=0}^\infty H_n$ be a graded Hilbert space and let $M, N \subset H$ be graded subspaces. Write $M_n = M \cap H_n$ and $N_n = N \cap H_n$ for $n \in \mathbb{N}$. Then*

$$c(M,N) = \sup_{n\in\mathbb{N}} c(M_n, N_n).$$

Proof. The assertion readily follows from Lemma 4.25 and the fact that for any graded subspace $K \subset H$, we have

$$P_K = \bigoplus_{n=0}^{\infty} P_{K\cap H_n}^{H_n},$$

where $P_{K\cap H_n}^{H_n}$ denotes the orthogonal projection from H_n onto $K \cap H_n$. □

If each of the spaces H_n in the preceding lemma is finite dimensional, then $c(M_n, N_n) < 1$ for all $n \in \mathbb{N}$ (see part (c) of Remark 4.22). In particular, $M + N$ is closed if and only if $\limsup_{n\to\infty} c(M_n, N_n) < 1$. That is, closedness of $M + N$ only depends on the asymptotic behaviour of the sequence $(c(M_n, N_n))_n$. Inspired by condition 7 in [BGM10, Theorem 2.3], we will now introduce a variant of the Friedrichs angle which reflects this fact. For a closed subspace M of a Hilbert space H, we denote the equivalence class of P_M in the Calkin algebra by p_M.

Definition 4.27. Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces. The essential Friedrichs angle is defined to be the angle in $[0, \frac{\pi}{2}]$ whose cosine is

$$c_e(M, N) = ||p_M p_N - p_{M\cap N}||.$$

Some of the elementary properties of the Friedrichs angle also hold for its essential variant.

Lemma 4.28. *Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces.*

(a) $c_e(M, N) = c_e(M \ominus (M \cap N), N \ominus (M \cap N))$.

(b) $c_e(M, N)^2 = ||p_N p_M p_N - p_{M\cap N}||$.

(c) $c_e(M, N) = c_e(N, M)$.

Proof. Part (a) follows from the identity

$$(P_M - P_{M\cap N})(P_N - P_{M\cap N}) = P_M P_N - P_{M\cap N},$$

while (b) is again an application of the C^*-identity, see the proof of Lemma 4.25.

For the proof of (c), we may assume that $M \cap N = \{0\}$ by part (a). According to part (b), we have to show that $||p_M p_N p_M|| = ||p_N p_M p_N||$, or, equivalently, that the spectral radius of $p_N p_M p_N$ equals the spectral radius of $p_M p_N p_M$. But this fact readily follows from the identity

$$\sigma(p_N p_M p_N) \cup \{0\} = \sigma(p_N p_N p_M) \cup \{0\} = \sigma(p_N p_M p_M) \cup \{0\} = \sigma(p_M p_N p_M) \cup \{0\},$$

where we have used that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ for elements a, b of a unital C^*-algebra. □

To determine if $M + N$ is closed, the essential Friedrichs angle is just as good as the usual one, that is, the equivalence of (i) and (ii) in Lemma 4.23 also holds with c_e in place of c as well. This follows from [BGM10, Theorem 2.3]. Before we provide a short proof below, we record a simple lemma.

Lemma 4.29. *Let H be a Hilbert space and let $M_1, \dots, M_r \subset H$ be closed subspaces. Define $T = P_{M_1} P_{M_2} \dots P_{M_r}$ and $M = M_1 \cap \dots \cap M_r$.*

(a) $\ker(1 - T^*T) = M$.

(b) *If $\dim H < \infty$, then $||T|| = 1$ if and only if $M \neq \{0\}$.*

Proof. We first claim that a vector $x \in H$ satisfies $||Tx|| = ||x||$ if and only if $x \in M$. We prove the non-trivial implication by induction on r. The case $r = 1$ is clear. So suppose that $r \geq 2$ and that the assertion is true for $r - 1$ subspaces. Let $x \in H$ such that $||Tx|| = ||x||$. Setting $y = P_{M_2} \dots P_{M_r} x$, we have

$$||x|| = ||P_{M_1} y|| \leq ||y|| \leq ||x||,$$

thus $y \in M_1$ and $||P_{M_2} \dots P_{M_r} x|| = ||x||$. By induction hypothesis, $x \in M_2 \cap \dots \cap M_r$, and hence also $x = y \in M_1$, which finishes the proof of the claim.

Both assertions easily follow from this observation. Clearly, M is contained in $\ker(1 - T^*T)$. Conversely, any $x \in \ker(1 - T^*T)$ satisfies $||x||^2 = ||Tx||^2$, so that $x \in M$ by the above remark, which proves (a).

Part (b) is immediate from the claim as well, since $||T||$ is attained if H is finite dimensional. □

Lemma 4.30. *Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces. Then $M + N$ is closed if and only if $c_e(M, N) < 1$.*

Proof. In view of Lemma 4.23, it suffices to show that $c(M, N) < 1$ if $c_e(M, N) < 1$, since $c_e(M, N) \leq c(M, N)$ holds trivially. To this end, we can assume without loss of generality that $M \cap N = \{0\}$ by Remark 4.22 (d) and Lemma 4.28 (a). Then $||P_N P_M P_N||_e < 1$, so $T = 1 - P_N P_M P_N$ is a self-adjoint Fredholm operator. Lemma 4.29 (a) implies that T is injective, from which we conclude that T is invertible. It follows that $1 \notin \sigma(P_N P_M P_N)$, and hence that $c(M, N) = ||P_N P_M P_N|| < 1$. □

For graded subspaces, we obtain a more concrete description of the essential Friedrichs angle, which gives another proof for the preceding lemma in the graded case. In particular, we see that the essential Friedrichs angle indeed only depends on the asymptotic behaviour of the Friedrichs angles between the graded components.

Lemma 4.31. *Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded Hilbert space, where all H_n are finite dimensional, and let $M, N \subset H$ be graded subspaces. Write $M_n = M \cap H_n$ and $N_n = N \cap H_n$ for $n \in \mathbb{N}$. Then*

$$c_e(M, N) = \limsup_{n \to \infty} c(M_n, N_n).$$

Proof. Let $\varepsilon > 0$ be arbitrary. By definition of c_e, there is a compact operator K on H such that

$$||P_M P_N - P_{M\cap N} + K|| \leq c_e(M,N) + \varepsilon.$$

It is easy to see that $\lim_{n\to\infty} ||P_n K P_n|| = 0$. Furthermore,

$$\begin{aligned} c(M_n, N_n) &= ||P_n(P_M P_N - P_{M\cap N})P_n|| \\ &\leq ||P_n(P_M P_N - P_{M\cap N} + K)P_n|| + ||P_n K P_n|| \\ &\leq c_e(M,N) + \varepsilon + ||P_n K P_n||, \end{aligned}$$

so $\limsup_{n\to\infty} c(M_n, N_n) \leq c_e(M,N)$.

Conversely, for any $k \in \mathbb{N}$, the operator

$$K = \bigoplus_{n=0}^{k} P_n(P_M P_N - P_{M\cap N})P_n$$

has finite rank, and

$$P_M P_N - P_{M\cap N} - K = \bigoplus_{n=k+1}^{\infty} P_n(P_M P_N - P_{M\cap N})P_n.$$

Hence

$$c_e(M,N) \leq ||P_M P_N - P_{N\cap N} - K|| = \sup_{n\geq k+1} c(M_n, N_n)$$

for all natural numbers k, which establishes the reverse inequality. $\square$

Remark. If T is an operator on a Hilbert space H, the infimum

$$\inf\{||T + K|| : K \in \mathcal{K}(H)\}$$

is always attained [HK72]. In particular, we can choose an operator K in the first part of the above proof such that $||P_M P_N - P_{M\cap N} + K|| = c_e(M,N)$.

To deduce other identities for the Friedrichs angle and its essential variant, we need two simple consequences of the Gelfand-Naimark theorem.

Lemma 4.32. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a, b \in \mathcal{A}$ be selfadjoint elements.*

(a) *If $ab = 0$, then $||a + b|| = \max(||a||, ||b||)$.*

(b) *Suppose that a and b commute and that $a \leq b$. If f is a continuous and increasing real-valued function on $\sigma(a) \cup \sigma(b)$, then $f(a) \leq f(b)$.*

Proof. In both cases, the unital C^*-algebra generated by a and b is commutative. By the Gelfand-Naimark theorem, we can therefore regard a and b as real-valued functions on a compact Hausdorff space, where both assertions are elementary. $\square$

Lemma 4.33. *Let H be a Hilbert space and let $M_1, M_2, N_1, N_2 \subset H$ be closed subspaces with $M_1 \perp M_2, M_1 \perp N_2, M_2 \perp N_1, N_1 \perp N_2$. Then*

$$c(M_1 \oplus M_2, N_1 \oplus N_2) = \max(c(M_1, N_1), c(M_2, N_2)).$$

The same is true with c_e in place of c.

Proof. The assertion can be shown using the definition of the Friedrichs angle or working with projections. The latter has the advantage of proving the claim for the essential Friedrichs angle at the same time.

First, we note that the assumptions on the subspaces imply that

$$(M_1 \oplus M_2) \cap (N_1 \oplus N_2) = (M_1 \cap N_1) \oplus (M_2 \cap N_2).$$

Indeed, if $m_1 + m_2 = n_1 + n_2$ is an element of the space on the left-hand side, with $m_i \in M_i, n_i \in N_i$ for $i = 1, 2$, then $m_1 - n_1 = n_2 - m_2$, and the orthogonality relations show that this vector is zero. Hence $m_1 \in M_1 \cap N_1$ and $m_2 \in M_2 \cap N_2$, thus proving the non-trivial inclusion. Using the orthogonality relations once again, we conclude that

$$\begin{aligned}
&P_{N_1 \oplus N_2} P_{M_1 \oplus M_2} P_{N_1 \oplus N_2} - P_{(M_1 \oplus M_2) \cap (N_1 \oplus N_2)} \\
&= (P_{N_1} + P_{N_2})(P_{M_1} + P_{M_2})(P_{N_1} + P_{N_2}) - (P_{M_1 \cap N_1} + P_{M_2 \cap N_2}) \\
&= (P_{N_1} P_{M_1} P_{N_1} - P_{M_1 \cap N_1}) + (P_{N_2} P_{M_2} P_{N_2} - P_{M_2 \cap N_2}).
\end{aligned}$$

Since

$$(P_{N_1} P_{M_1} P_{N_1} - P_{M_1 \cap N_1})(P_{N_2} P_{M_2} P_{N_2} - P_{M_2 \cap N_2}) = 0,$$

both assertions follow from Lemma 4.32 (a). □

We finish this section by showing that the (essential) Friedrichs angle is invariant under taking orthogonal complements.

Proposition 4.34. *Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces. Then*

$$c(M, N) = c(M^\perp, N^\perp).$$

The same is true with c_e in place of c. In particular, $M + N$ is closed if and only if $M^\perp + N^\perp$ is closed.

Proof. In a first step, we reduce to the case where M and N are in generic position, that is, where

$$M \cap N, \quad M \cap N^\perp, \quad M^\perp \cap N, \quad M^\perp \cap N^\perp \tag{4.3}$$

are all zero. To this end, note that Lemma 4.33 in particular implies that given closed subspaces M_1, N_1 and E of H such that E is orthogonal to M_1 and N_1, then $c(M_1 \oplus E, N_1 \oplus E) = c(M_1, N_1)$ (and the same statement with c_e in place of c). Consequently, after replacing M by $M \ominus (M \cap N)$ and N by $N \ominus (M \cap N)$, we may assume that $M \cap N = \{0\}$. Now, let $M' = M \ominus (M \cap N^\perp)$ and $N' = N \ominus (N \cap M^\perp)$. Clearly, we still have $M' \cap N' = \{0\}$. Moreover,

$$(N')^\perp = N^\perp \oplus (N \cap M^\perp) \quad \text{and} \quad (M')^\perp = M^\perp \oplus (M \cap N^\perp),$$

from which it easily follows that $M' \cap (N')^\perp = \{0\}$ and $N' \cap (M')^\perp = \{0\}$. Another straightforward application of Lemma 4.33 thus shows that we may suppose that the first three spaces in (4.3) are trivial. To dispose of the fourth space in (4.3), we replace H by $\overline{M + N}$. It is easy to see that $c(M, N)$ and $c_e(M, N)$ do not depend on the ambient Hilbert space, and using once again Lemma 4.33, we deduce that this operation does not change $c(M^\perp, N^\perp)$ and $c_e(M^\perp, N^\perp)$ either. This observation establishes the desired reduction.

In the situation where all spaces in (4.3) are zero, a theorem of Halmos ([Hal69, Theorem 2]) asserts that there exists a Hilbert space K and positive contractions S and C on K, with $S^2 + C^2 = 1$, such that P_M and P_N are unitarily equivalent to

$$P'_M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P'_N = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

respectively. From

$$P'_M P'_N P'_M = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(1 - P'_M)(1 - P'_N)(1 - P'_M) = \begin{pmatrix} 0 & 0 \\ 0 & C^2 \end{pmatrix},$$

we infer that $P'_M P'_N P'_M$ and $(1-P'_M)(1-P'_N)(1-P'_M)$ are unitarily equivalent, which proves the proposition, observing that the additional claim follows from Lemma 4.23. $\square$

4.4. Reduction to subspaces with trivial joint intersection

Let $V_1, \ldots, V_r$ be subspaces of $\mathbb{C}^d$. The purpose of this section is to reduce the problem of showing closedness of the sum of Fock spaces $\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r) \subset \mathcal{F}(\mathbb{C}^d)$ to the case where $V_1 \cap \ldots \cap V_r = \{0\}$. Note that in [DRS11, Lemma 7.12], Davidson, Ramsey and Shalit reduced the problem of showing boundedness of the map $f \mapsto f \circ A^*$ in the setting of unions of subspaces to the case where the joint intersection of the subspaces is trivial. However, in our situation, it does not suffice to consider only subspaces with trivial joint intersection. The issue is that in the inductive proof of closedness of the sum of r Fock spaces, we will use the inductive hypothesis on $r-1$ subspaces which do not necessarily have trivial joint intersection.

We begin by showing that tensoring with another Hilbert space does not make the angle worse.

Lemma 4.35. *Let H be a Hilbert space and let $M, N \subset H$ be closed subspaces. If E is another non-trivial Hilbert space, then*

$$c(M \otimes E, N \otimes E) = c(M, N).$$

Proof. First, note that $(M \cap N) \otimes E = (M \otimes E) \cap (N \otimes E)$. Since $P_{K \otimes E} = P_K \otimes P_E$ for any closed subspace $K \subset H$, we have

$$\begin{aligned} ||P_{M\otimes E}P_{N\otimes E} - P_{(M\otimes E)\cap(N\otimes E)}|| &= ||P_{M\otimes E}P_{N\otimes E} - P_{(M\cap N)\otimes E}|| \\ &= ||(P_M P_N - P_{M\cap N}) \otimes 1_E|| \\ &= ||P_M P_N - P_{M\cap N}||. \end{aligned}$$ ∎

The following result is the desired reduction to subspaces with trivial joint intersection.

Lemma 4.36. *Let $V_1, \ldots, V_r \subset \mathbb{C}^d$ be subspaces and let $V = V_1 \cap \ldots \cap V_r \neq \{0\}$. Suppose that $\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_{r-1})$ and $\mathcal{F}(V_1 \ominus V) + \ldots + \mathcal{F}(V_{r-1} \ominus V)$ are closed. Then $\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r)$ is closed if and only if $\mathcal{F}(V_1 \ominus V) + \ldots + \mathcal{F}(V_r \ominus V)$ is closed.*

Proof. We claim that it suffices to prove the following assertion: If $W_1, \ldots, W_r \subset \mathbb{C}^d$ are subspaces, and if $E \subset \mathbb{C}^d$ is a non-trivial subspace that is orthogonal to each W_i, then

$$\begin{aligned} &c((W_1 \oplus E)^{\otimes n} + \ldots + (W_{r-1} \oplus E)^{\otimes n}, (W_r \oplus E)^{\otimes n}) \\ = \quad &\max_{j=1,\ldots,n} c(W_1^{\otimes j} + \ldots + W_{r-1}^{\otimes j}, W_r^{\otimes j}). \end{aligned} \tag{4.4}$$

Indeed, setting $E = V$ and $W_i = V_i \ominus V$ for each i, we see from Lemma 4.26 and Lemma 4.23 that this assertion will prove the lemma.

In fact, we will show that

$$\begin{aligned} &c\Big(\sum_{i=1}^{r-1} W_i^{\otimes k} \otimes (W_i \oplus E)^{\otimes n}, W_r^{\otimes k} \otimes (W_r \oplus E)^{\otimes n}\Big) \\ = \quad &\max_{j=k,\ldots,k+n} c\Big(\sum_{i=1}^{r-1} W_i^{\otimes j}, W_r^{\otimes j}\Big) \end{aligned} \tag{4.5}$$

holds for all natural numbers k and n. The assertion (4.4) corresponds to the case $k = 0$, with the usual convention $W^{\otimes 0} = \mathbb{C}$ for a subspace $W \subset \mathbb{C}^d$. We proceed by induction on n. If $n = 0$, this is trivial. So suppose that $n \geq 1$ and that the assertion has been proved for $n - 1$. First, we note that

$$\begin{aligned} & W_i^{\otimes k} \otimes (W_i \oplus E)^{\otimes n} \\ = \; & \big(W_i^{\otimes k+1} \otimes (W_i \oplus E)^{\otimes n-1}\big) \oplus \big(W_i^{\otimes k} \otimes E \otimes (W_i \oplus E)^{\otimes n-1}\big), \end{aligned}$$

holds for all i. So defining

$$\begin{aligned} M_1 &= \sum_{i=1}^{r-1} W_i^{\otimes k+1} \otimes (W_i \oplus E)^{\otimes n-1} \quad \text{and} \\ M_2 &= \sum_{i=1}^{r-1} W_i^{\otimes k} \otimes E \otimes (W_i \oplus E)^{\otimes n-1}, \end{aligned}$$

as well as

$$\begin{aligned} N_1 &= W_r^{\otimes k+1} \otimes (W_r \oplus E)^{\otimes n-1} \quad \text{and} \\ N_2 &= W_r^{\otimes k} \otimes E \otimes (W_r \oplus E)^{\otimes n-1}, \end{aligned}$$

we have

$$\begin{aligned} \sum_{i=1}^{r-1} W_i^{\otimes k} \otimes (W_i \oplus E)^{\otimes n} &= M_1 + M_2 \quad \text{and} \\ W_r^{\otimes k} \otimes (W_r \oplus E)^{\otimes n} &= N_1 + N_2. \end{aligned}$$

Since E is orthogonal to each W_i, we see that $M_1 \perp M_2$, $M_1 \perp N_2$, $M_2 \perp N_1$ and $N_1 \perp N_2$. Consequently, Lemma 4.33 applies to show that the left-hand side of (4.5) equals

$$\max(c(M_1, N_1), c(M_2, N_2)).$$

By induction hypothesis,

$$c(M_1, N_1) = \max_{j=k+1,\ldots,k+n} c\Big(\sum_{i=1}^{r-1} W_i^{\otimes j}, W_r^{\otimes j}\Big).$$

Moreover, an application of Lemma 4.35 combined with the inductive hypothesis shows that

$$c(M_2, N_2) = \max_{j=k,\ldots,k+n-1} c\Big(\sum_{i=1}^{r-1} W_i^{\otimes j}, W_r^{\otimes j}\Big),$$

which finishes the proof. $\square$

Example 4.37. With the formula derived in the proof of the preceding lemma, we can already determine the Friedrichs angle between two full Fock spaces. To begin with, suppose that V_1 and V_2 are two subspaces in $\mathbb{C}^d$ such that $V_1 \cap V_2 = \{0\}$. Then Lemma 4.25 yields for all natural numbers n the identity

$$c(V_1^{\otimes n}, V_2^{\otimes n}) = ||P_{V_1}^{\otimes n} P_{V_2}^{\otimes n}|| = ||P_{V_1} P_{V_2}||^n = c(V_1, V_2)^n.$$

Note that $c(V_1, V_2) < 1$ because $\mathbb{C}^d$ is finite dimensional. If $V_1 \cap V_2 \neq \{0\}$, we set $W_i = V_i \ominus (V_1 \cap V_2)$ for $i = 1, 2$. By formula (4.4), we have

$$c(V_1^{\otimes n}, V_2^{\otimes n}) = \max_{j=1,\dots,n} c(W_1^{\otimes j}, W_2^{\otimes j})$$

for all n. Since W_1 and W_2 have trivial intersection,

$$c(W_1^{\otimes j}, W_2^{\otimes j}) = c(W_1, W_2)^j = c(V_1, V_2)^j$$

by what we have just proved, so

$$c(V_1^{\otimes n}, V_2^{\otimes n}) = c(V_1, V_2)$$

for all n. As an application of Lemma 4.26, we see that in any case,

$$c(\mathcal{F}(V_1), \mathcal{F}(V_2)) = c(V_1, V_2),$$

while Lemma 4.31 shows that

$$c_e(\mathcal{F}(V_1), \mathcal{F}(V_2)) = \begin{cases} c(V_1, V_2), & \text{if } V_1 \cap V_2 \neq \{0\}, \\ 0, & \text{if } V_1 \cap V_2 = \{0\}. \end{cases}$$

In particular, we see that sums of *two* Fock spaces are closed.

We conclude this section with a lemma about the case of trivial joint intersection. In view of the definition of the essential Friedrichs angle, it indicates why the reduction to this case will be helpful.

Lemma 4.38. *Let $V_1, \dots, V_r \subset \mathbb{C}^d$ be subspaces with $V_1 \cap \ldots \cap V_r = \{0\}$. Set $M_i = \mathcal{F}(V_i)$ for $i = 1, \dots, r$. Then $P_{M_1} \dots P_{M_r}$ is a compact operator.*

Proof. We note that for each i,

$$P_{M_i} = \bigoplus_{n=0}^{\infty} P_{V_i}^{\otimes n},$$

hence

$$P_{M_1} \dots P_{M_r} = \bigoplus_{n=0}^{\infty} (P_{V_1} \dots P_{V_r})^{\otimes n}.$$

Since $V_1 \cap \ldots \cap V_r = \{0\}$, and since $\mathbb{C}^d$ is finite dimensional, $||P_{V_1} \dots P_{V_r}|| < 1$ by Lemma 4.29 (b). Therefore,

$$||(P_{V_1} \dots P_{V_r})^{\otimes n}|| = ||(P_{V_1} \dots P_{V_r})||^n \xrightarrow{n \to \infty} 0.$$

From this observation, it is easy to see that $P_{M_1} \dots P_{M_r}$ is compact. $\square$

4.5. A closedness result

In this section, we will deduce a closedness result which will form the inductive step in the proof of our general result on the closedness of algebraic sums of r Fock spaces. Because of Lemma 4.36 and Lemma 4.38, we will consider the following situation throughout this section: Let $r \geq 2$, and let $M_1, \ldots, M_r$ be closed subspaces of a Hilbert space H which satisfy the following two conditions:

(a) Any algebraic sum of $r-1$ or fewer subspaces of the M_i is closed, that is, for any subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$ with $k \leq r-1$, the sum

$$M_{i_1} + \ldots + M_{i_k}$$

is closed.

(b) Any product of the P_{M_i} containing each P_{M_i} at least once is compact, that is, for any collection of (not necessarily distinct) indices $i_1, \ldots, i_k$ such that $\{i_1, \ldots, i_k\} = \{1, \ldots, r\}$, the operator

$$P_{M_{i_1}} P_{M_{i_2}} \ldots P_{M_{i_k}}$$

is compact.

Our goal is to show that under these assumptions, the sum $M_1 + \ldots + M_r$ is closed. Note that for $r = 2$, the first condition is empty, while the second is equivalent to demanding that $P_{M_1} P_{M_2}$ be compact.

Recall that for a closed subspace $M \subset H$, we denote the equivalence class of P_M in the Calkin algebra by p_M. Moreover, we define $\mathcal{A}$ to be the unital C^*-subalgebra of the Calkin algebra generated by $p_{M_1}, \ldots, p_{M_r}$. The following proposition is the key step in proving that the sum $M_1 + \ldots + M_r$ is closed. It crucially depends on condition (b).

Proposition 4.39. *For any irreducible representation π of $\mathcal{A}$ on a Hilbert space K, there is an $i \in \{1, \ldots, r\}$ such that $\pi(p_{M_i}) = 0$.*

In particular, there are representations $\pi_1, \ldots, \pi_r$ of $\mathcal{A}$ such that $\pi_i(p_{M_i}) = 0$ for each i, and such that $\pi = \bigoplus_{i=1}^r \pi_i$ is a faithful representation of $\mathcal{A}$.

Proof. We write $p_i = p_{M_i}$. Suppose that $\pi(p_2), \ldots, \pi(p_r)$ are all non-zero. We have to prove that $\pi(p_1) = 0$. First, note that by condition (b),

$$\pi(p_1 a_1 p_2 a_2 \ldots a_{r-1} p_r) = 0 \tag{4.6}$$

holds if each of the a_i is a monomial in the p_j. By linearity and continuity, (4.6) therefore holds for all $a_1, \ldots, a_{r-1} \in \mathcal{A}$.

Since π is irreducible, and since $\pi(p_r) \neq 0$, we have

$$\bigvee_{a_{r-1}\in\mathcal{A}} \pi(a_{r-1}p_r)K = K.$$

Consequently, (4.6) implies that $\pi(p_1a_1p_2a_2\dots a_{r-2}p_{r-1}) = 0$. Iterating this process yields the conclusion $\pi(p_1) = 0$, as desired.

To establish the additional assertion, let π_i be the direct sum of all irreducible GNS representations π_f with $\pi_f(p_i) = 0$, which is understood to be zero if there are no such representations. Then $\pi = \bigoplus_{i=1}^r \pi_i$ contains every irreducible GNS representation of $\mathcal{A}$ as a summand by the first part, and is therefore faithful. □

We will use the preceding proposition to get a good estimate of the essential Friedrichs angle

$$c_e(M_1 + \ldots + M_{r-1}, M_r) = ||p_{M_1+\ldots M_{r-1}}p_{M_r} - p_{(M_1+\ldots+M_{r-1})\cap M_r}||. \tag{4.7}$$

To this end, we have to make sure that all occurring elements belong to $\mathcal{A}$. We will use the following well-known fact.

Lemma 4.40. *Let H, K be Hilbert spaces and let $T \in \mathcal{L}(H, K)$. Then the image of T is closed if and only if 0 is not a cluster point of $\sigma(T^*T)$.*

Proof. First, we observe that the image of T is closed if and only if T is bounded below on $M = \ker(T)^\perp$, that is, if and only if there exists an $\varepsilon > 0$ such that

$$T^*T\big|_M \geq \varepsilon 1_M.$$

Now, notice that M is a reducing subspace for T^*T. Consequently, if the image of T is closed, then $\sigma(T^*T) \subset \{0\} \cup [\varepsilon, \infty)$. Conversely, if 0 is not a cluster point of $\sigma(T^*T)$, then it is not a cluster point of $\sigma(T^*T\big|_M)$ either. But $T^*T\big|_M$ is injective, so using that isolated points of the spectrum are eigenvalues, we conclude that $0 \notin \sigma(T^*T\big|_M)$. Hence $T^*T\big|_M \geq \varepsilon 1_M$ for some $\varepsilon > 0$. □

Part of the task of showing that all elements in (4.7) belong to $\mathcal{A}$ is done by the following lemma.

Lemma 4.41. *Let H be a Hilbert space and let $M, N, N_1, \ldots, N_s \subset H$ be closed subspaces.*

(a) *The algebraic sum $N_1 + \ldots + N_s$ is closed if and only if 0 is not a cluster point of the spectrum of the positive operator $P_{N_1} + \ldots + P_{N_s}$. In this case, the image of the operator $P_{N_1} + \ldots + P_{N_r}$ equals $N_1 + \ldots + N_r$.*

(b) *If $N_1 + \ldots + N_s$ is closed, then*

$$P_{N_1+\ldots+N_s} = \chi_{(0,\infty)}(P_{N_1} + \ldots + P_{N_s}),$$

where $\chi_{(0,\infty)}$ denotes the indicator function of $(0,\infty)$. In particular, the projection $P_{N_1+\ldots+N_s}$ belongs to the C^*-algebra generated by $P_{N_1}, \ldots, P_{N_s}$.

(c) *$M + N$ is closed if and only if the sequence $((P_M P_N P_M)^n)_n$ converges in norm to $P_{M\cap N}$. In particular, if $M + N$ is closed, then $P_{M\cap N}$ belongs to the C^*-algebra generated by P_M and P_N.*

Proof. (a) Consider the continuous operator

$$T : \bigoplus_{i=1}^{s} N_i \to H, \quad (x_i)_{i=1}^s \mapsto \sum_{i=1}^{s} x_i.$$

Clearly, the image of T equals $N_1 + \ldots + N_r$. Consequently, this sum is closed if and only if the image of T is closed, which, in turn, happens if and only if the image of T^* is closed. It is easy to check that T^* is given by $T^*x = (P_{N_1}x, \ldots, P_{N_s}x)$, so $TT^* = P_{N_1} + \ldots + P_{N_s}$. Hence the assertion follows from Lemma 4.40, and the additional claim is now obvious.

(b) Part (a) shows that the restriction of $\chi_{(0,\infty)}$ to $\sigma(P_{N_1} + \ldots + P_{N_s})$ is continuous, so $P = \chi_{(0,\infty)}(P_{N_1} + \ldots + P_{N_s})$ belongs to the C^*-algebra generated by $P_{N_1}, \ldots, P_{N_s}$. By standard properties of the functional calculus, P is the orthogonal projection onto the range of $P_{N_1} + \ldots + P_{N_s}$, which is $N_1 + \ldots + N_s$.

(c) For any $n \in \mathbb{N}$, we have

$$||(P_M P_N P_M)^n - P_{M\cap N}|| = ||(P_M P_N P_M - P_{M\cap N})^n|| = c(M,N)^{2n},$$

which converges to zero if and only if $c(M,N) < 1$. This, in turn, is equivalent to $M + N$ being closed by Lemma 4.23. □

Remark 4.42. Statement (c) in the preceding lemma is just part of a bigger picture: For any closed subspaces $M, N \subset H$, the sequence $((P_M P_N)^n)_n$ (and hence also $((P_M P_N P_M)^n)_n = ((P_M P_N)^n P_M)_n$) converges in the strong operator topology to $P_{M\cap N}$, and the convergence is in norm if and only if $M + N$ is closed, see for example [Deu95, Section 3].

Because of condition (a), the preceding lemma shows that $p_{M_1+\ldots+M_{r-1}} \in \mathcal{A}$. If $r \geq 3$, we define for $i = 1, \ldots, r-1$

$$S_i = M_1 + \ldots + \widehat{M_i} + \ldots + M_{r-1},$$

where $\widehat{M_i}$ stands for omission of M_i. If $r = 2$, this is understood to be the zero vector space. Note that S_i is a sum of $r-2$ subspaces for $r \geq 3$. Thus another application of Lemma 4.41 shows that p_{S_i} and $p_{S_i \cap M_r}$ belong to $\mathcal{A}$. However, care must be taken when using Proposition 4.39 to estimate $c_e(M_1 + \ldots + M_{r-1}, M_r)$ since it is not obvious a priori that $p_{(M_1+\ldots+M_{r-1})\cap M_r}$ lies in $\mathcal{A}$. Before we address this question, we record the following simple lemma for future reference.

Lemma 4.43. *Let $\pi = \bigoplus_{i=1}^r \pi_i$ be the representation from Proposition 4.39. Then for $i = 1, \ldots, r-1$,*

$$\pi_i(p_{M_1+\ldots+M_{r-1}}) = \pi_i(p_{S_i}).$$

Proof. Let $a = p_{M_1} + \ldots + p_{M_{r-1}}$ and $b = p_{M_1} + \ldots + \widehat{p_{M_i}} + \ldots + p_{M_{r-1}}$ (if $r = 2$, we set $b = 0$). By condition (a) and Lemma 4.41, the origin is neither a cluster point of $\sigma(a)$ nor one of $\sigma(b)$, and

$$\chi_{(0,\infty)}(a) = p_{M_1+\ldots+M_{r-1}} \quad \text{and} \quad \chi_{(0,\infty)}(b) = p_{S_i}.$$

The assertion therefore follows from the identity $\pi_i(a) = \pi_i(b)$ and the fact that the continuous functional calculus is compatible with $*$-homomorphisms. □

The question whether $p_{(M_1+\ldots+M_{r-1})\cap M_r}$ belongs to $\mathcal{A}$ is more difficult. We will see below that it can well happen that for subspaces M and N of a Hilbert space H, the projection $p_{M\cap N}$ does not belong to the unital C^*-algebra generated by p_M and p_N. Moreover, although there is a criterion for the closedness of $M + N$ only in terms of P_M and P_N, namely $M + N$ is closed if and only if the sequence $((P_M P_N)^n)_n$ is a Cauchy sequence in norm (see Remark 4.42), there cannot be such a criterion only in terms of p_M and p_N.

Example 4.44. A concrete example of two closed subspaces M and N of a Hilbert space H such that $M + N$ is not closed can be obtained as follows (compare the discussion preceding Problem 52 in [Hal82]): Take a continuous linear operator T on H with non-closed range, and let M be the graph of T, that is,

$$M = \{(x, Tx) : x \in H\} \subset H \oplus H.$$

Set $N = H \oplus \{0\}$. Then M and N are closed, but

$$M + N = H \oplus \operatorname{ran}(T)$$

is not closed. Suppose now that T is additionally self-adjoint and compact. It is easy to check that the projection onto M is given by

$$P_M = \begin{pmatrix} (1+T^2)^{-1} & T(1+T^2)^{-1} \\ T(1+T^2)^{-1} & T^2(1+T^2)^{-1} \end{pmatrix}.$$

Clearly,

$$P_N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

However, the equivalence classes p_M and p_N of these projections in the Calkin algebra are the same. In particular, we see that there cannot be a criterion for the closedness of $M + N$ only in terms of p_M and p_N.

Moreover, $M \cap N = \ker(T) \oplus \{0\}$, so

$$P_{M\cap N} = \begin{pmatrix} P_{\ker(T)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, if both $\ker(T)$ and $H \ominus \ker(T)$ are infinite dimensional, $p_{M\cap N}$ does not belong to the unital C^*-algebra generated by p_M and p_N. For a concrete example, set $H = \ell^2(\mathbb{N})$, choose a null sequence $(a_n)_n$ of real numbers with infinitely many zero and infinitely many non-zero terms, and let T be componentwise multiplication with $(a_n)_n$.

In the presence of conditions (a) and (b), the situation is better.

Lemma 4.45. *Under the above hypotheses, $p_{(M_1+\ldots+M_{r-1})\cap M_r} \in \mathcal{A}$. Moreover, if $\pi = \bigoplus_{i=1}^r \pi_i$ is the faithful representation from Proposition 4.39, we have*

$$\pi_i(p_{(M_1\ldots+M_{r-1})\cap M_r}) = \pi_i(p_{S_i\cap M_r}),$$

for $i = 1, \ldots, r-1$, and $\pi_r(p_{(M_1+\ldots+M_{r-1})\cap M_r}) = 0$.

Proof. If $r = 2$, condition (b) asserts that $P_{M_1}P_{M_2}$ is a compact operator. Since $P_{M_1\cap M_2} = P_{M_1}P_{M_2}P_{M_1\cap M_2}$, we conclude that $p_{M_1\cap M_2} = 0$, so the statement is trivial for $r = 2$.

Now, let us assume that $r \geq 3$ and define

$$S = M_1 + \ldots + M_{r-1}.$$

In a first step, we show that the sequence $((p_{M_r}p_S p_{M_r})^n)_n$ converges to an element $q_u \in \mathcal{A}$ with $q_u \geq p_{S\cap M_r}$. To this end, let $\pi = \bigoplus_{i=1}^r \pi_i$ be the faithful representation from Proposition 4.39. By Lemma 4.43, we have $\pi_i(p_S) = \pi_i(p_{S_i})$ for each i. Since $S_i + M_r$ is closed, Lemma 4.41 (c) shows that for $i = 1, \ldots, r-1$,

$$\pi_i\big((p_{M_r}p_S p_{M_r})^n\big) = \pi_i\big((p_{M_r}p_{S_i}p_{M_r})^n\big) \xrightarrow{n\to\infty} \pi_i(p_{S_i\cap M_r}). \tag{4.8}$$

Clearly, $\pi_r(p_{M_r}p_S p_{M_r}) = 0$. Since $\pi = \bigoplus_{i=1}^r \pi_i$ is a faithful representation, we conclude that $((p_{M_r}p_S p_{M_r})^n)_n$ is a Cauchy sequence in $\mathcal{A}$. Denoting its limit by q_u, we see from

$$(p_{M_r}p_S p_{M_r})^n - p_{S\cap M_r} = (p_{M_r}p_S p_{M_r} - p_{S\cap M_r})^n \geq 0$$

for all $n \in \mathbb{N}$ that $q_u \geq p_{S\cap M_r}$.

The next step is to prove that 0 is not a cluster point of the spectrum of the positive element $a = p_{S_1\cap M_r} + \ldots + p_{S_{r-1}\cap M_r} \in \mathcal{A}$, and that

$$q_l = \chi_{(0,\infty)}(a) \leq p_{S\cap M_r}.$$

To this end, we fix an $i \in \{1, \ldots, r-1\}$, and for $j = 1, \ldots, r-1$ with $j \neq i$, we set

$$N_j = M_1 + \ldots + \widehat{M_i} + \ldots + \widehat{M_j} + \ldots + M_{r-1} \subset S_i,$$

which is understood as the zero vector space if $r = 3$. Clearly, N_j is closed by condition (a). Then $p_{N_j} \in \mathcal{A}$, and just as in the proof of Lemma 4.43, we see that $\pi_i(p_{S_j}) = \pi_i(p_{N_j})$. Since $N_j + M_r$ and $S_j + M_r$ are closed by condition (a), an application of Lemma 4.41 (c) yields that $p_{N_j \cap M_r}$ belongs to $\mathcal{A}$ and that $\pi_i(p_{S_j \cap M_r}) = \pi_i(p_{N_j \cap M_r})$. Therefore,

$$\pi_i(a) = \pi_i(p_{N_1 \cap M_r} + \ldots + p_{N_{i-1} \cap M_r} + p_{S_i \cap M_r} + p_{N_{i+1} \cap M_r} + \ldots + p_{N_{r-1} \cap M_r}).$$

Using the fact that the algebraic sum

$$N_1 \cap M_r + \ldots + N_{i-1} \cap M_r + S_i \cap M_r + N_{i+1} \cap M_r + \ldots + N_{r-1} \cap M_r$$

equals $S_i \cap M_r$ and is therefore evidently closed, we conclude with the help of Lemma 4.41 (a) that 0 is not a cluster point of $\sigma(\pi_i(a))$, and that

$$\chi_{(0,\infty)}(\pi_i(a)) = \pi_i(p_{S_i \cap M_r}). \tag{4.9}$$

Since $\pi_r(a) = 0$, and since $\pi = \bigoplus_{i=1}^r \pi_i$ is a faithful representation of $\mathcal{A}$, it follows that 0 is not a cluster point of $\sigma(a)$. Thus, we can define

$$q_l = \chi_{(0,\infty)}(a) \in \mathcal{A}.$$

To prove the asserted inequality, we note that $a \leq (r-1)\, p_{S \cap M_r}$, and that a and $p_{S \cap M_r}$ commute. Hence Lemma 4.32 (b) shows that

$$q_l \leq \chi_{(0,\infty)}((r-1)\, p_{S \cap M_r}) = p_{S \cap M_r}.$$

We have established the following situation so far:

$$q_l \leq p_{(M_1 + \ldots + M_{r-1}) \cap M_r} \leq q_u,$$

and q_l and q_u belong to $\mathcal{A}$. We now finish the proof of $p_{(M_1 + \ldots + M_{r-1}) \cap M_r} \in \mathcal{A}$ by showing that $q_l = q_u$. Using once again the representation from Proposition 4.39, it suffices to show that $\pi_i(q_l) = \pi_i(q_u)$ for $i = 1, \ldots, r$. This is obvious for $i = r$, because $\pi_r(q_l) = 0 = \pi_r(q_u)$. So let $i \in \{1, \ldots, r-1\}$. According to equation (4.8), we have $\pi_i(q_u) = \pi_i(p_{S_i \cap M_r})$, while equation (4.9) shows that $\pi_i(q_l) = \pi_i(p_{S_i \cap M_r})$, as desired. The additional assertion is now obvious. □

We are now in the position to prove the main theorem of this section.

Theorem 4.46. *Let H be a Hilbert space, let $r \geq 2$ and let $M_1, \ldots M_r \subset H$ be closed subspaces such that the following two conditions hold:*

(a) *Any algebraic sum of* $r-1$ *or fewer subspaces of the* M_i *is closed, that is, for any subset* $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$ *with* $k \leq r-1$*, the sum*

$$M_{i_1} + \ldots + M_{i_k}$$

is closed.

(b) *Any product of the* P_{M_i} *containing each* P_{M_i} *at least once is compact, that is, for any collection of (not necessarily distinct) indices* $i_1, \ldots, i_k$ *with* $\{i_1, \ldots, i_k\} = \{1, \ldots, r\}$*, the operator*

$$P_{M_{i_1}} P_{M_{i_2}} \ldots P_{M_{i_k}}$$

is compact.

Then the algebraic sum $M_1 + \ldots + M_r$ *is closed.*

Proof. As above, let $\mathcal{A}$ be the unital C^*-algebra generated by $p_{M_1}, \ldots, p_{M_r}$, and let $\pi = \bigoplus_{i=1}^r \pi_i$ be the faithful representation from Proposition 4.39. By the discussion preceding Lemma 4.43, the elements p_{S_i} and $p_{S_i \cap M_r}$, as well as $p_{M_1 + \ldots + M_{r-1}}$, all belong to $\mathcal{A}$ for $i = 1, \ldots, r-1$. According to Lemma 4.45, this is also true for $p_{(M_1 + \ldots + M_{r-1}) \cap M_r}$, and $\pi_i(p_{(M_1 + \ldots + M_{r-1}) \cap M_r}) = \pi_i(p_{S_i \cap M_r})$ for $i = 1, \ldots, r-1$. Moreover, for these i, we have $\pi_i(p_{M_1 + \ldots + M_{r-1}}) = \pi_i(p_{S_i})$ by Lemma 4.43. Combining these results, we obtain

$$\begin{aligned} ||\pi_i(p_{M_1 + \ldots + M_{r-1}} p_{M_r} - p_{(M_1 + \ldots + M_{r-1}) \cap M_r})|| &= ||\pi_i(p_{S_i} p_{M_r} - p_{S_i \cap M_r})|| \\ &\leq c_e(S_i, M_r). \end{aligned}$$

Since $\pi_r(p_{M_r}) = 0 = \pi_r(p_{(M_1 + \ldots + M_{r-1}) \cap M_r})$, we conclude that

$$\begin{aligned} c_e(M_1 + \ldots + M_{r-1}, M_r) &= ||p_{M_1 + \ldots + M_{r-1}} p_{M_r} - p_{(M_1 + \ldots + M_{r-1}) \cap M_r}|| \\ &\leq \max_{1 \leq i \leq r-1} c_e(S_i, M_r) < 1 \end{aligned}$$

because $S_i + M_r$ is closed for each i by condition (a). □

The desired result about sums of Fock spaces follows now by a straightforward inductive argument.

Corollary 4.47. *Let* $V_1, \ldots, V_r \subset \mathbb{C}^d$ *be subspaces. Then the algebraic sum*

$$\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r) \subset \mathcal{F}(\mathbb{C}^d)$$

is closed.

Proof. We prove the result by induction on r, noting that the case $r = 1$ is trivial. So suppose that $r \geq 2$ and that the assertion has been proved for $k \leq r - 1$. In order to show that sums of r Fock spaces $\mathcal{F}(V_1), \ldots, \mathcal{F}(V_r)$ are closed, it suffices to consider the case where $V_1 \cap \ldots \cap V_r = \{0\}$ by Lemma 4.36. Let $M_i = \mathcal{F}(V_i)$ for each i. As an application of Lemma 4.38, we see that condition (b) of the preceding theorem is satisfied, whereas condition (a) holds by the inductive hypothesis. Thus the assertion follows from the preceding theorem. □

In the terminology of the second section of this chapter, this result, combined with Lemma 4.20, shows that every radical homogeneous ideal is admissible. Hence, Proposition 4.18 and Corollary 4.19 hold without the additional hypotheses on I and J. We thus obtain the desired classification of the algebras $\mathcal{A}_I$ up to topological isomorphism. If H and K are Hilbert spaces and if $\mathcal{A} \subset \mathcal{L}(H)$ and $\mathcal{B} \subset \mathcal{L}(K)$ are norm-closed subalgebras, we say that $\mathcal{A}$ and $\mathcal{B}$ are similar if there exists an invertible continuous linear operator $A : H \to K$ such that

$$\mathcal{A} = \{A^{-1}TA : T \in \mathcal{B}\}.$$

Clearly, the operator algebras $\mathcal{A}$ and $\mathcal{B}$ are topologically isomorphic via a completely bounded isomorphism in this case.

Theorem 4.48. *Let $I, J \subset \mathbb{C}[z]$ be radical homogeneous ideals. Then the following assertions are equivalent:*

(i) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are algebraically isomorphic.*

(ii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic.*

(iii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are similar.*

(iv) *There exists an invertible linear map A on $\mathbb{C}^d$ which maps $Z(J)$ onto $Z(I)$.*

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. (i) $\Rightarrow$ (iv) is Theorem 3.28. The remaining implication (iv) $\implies$ (iii) follows from Corollary 4.19 and the fact that all radical homogeneous ideals are admissible. □

We conclude this chapter with the easiest non-trivial application of the classification results from Theorem 3.36 and Theorem 4.48.

Example 4.49. Let $d = 2$ and let both I and J be the vanishing ideal of a union of two distinct complex lines, say $V(I) = V_1 \cup V_2$ and $V(J) = W_1 \cup W_2$ for some one-dimensional subspaces $V_1, V_2, W_1, W_2 \subset \mathbb{C}^2$. Then there is always an invertible linear map on $\mathbb{C}^2$ which sends $V(J)$ onto $V(I)$ and is isometric on $V(J)$, so that $\mathcal{A}_I$ and $\mathcal{A}_J$ are always topologically isomorphic. The algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic if and only if there is a unitary on $\mathbb{C}^2$ which maps $V(J)$ onto $V(I)$. It is not hard to see that his happens if and only if the angles $c(V_1, V_2)$ and $c(W_1, W_2)$ are the same.

5. The non-radical case

5.1. Graded isomorphisms

The results established so far allow for a classification of algebras of the type $\mathcal{A}_I$ in the homogeneous radical case. One way of generalizing this setting is dropping the condition that the ideal I be homogeneous. This was done in a recent paper by Davidson, Ramsey and Shalit [DRS12], where the isomorphism problem for algebras of multipliers on Drury-Arveson space associated to not necessarily homogeneous varieties was studied.

In this chapter, we consider a different direction of generalization by dropping the condition that the ideal I be radical. Thus, we seek necessary and sufficient conditions for two algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic, where I and J are homogeneous, but not necessarily radical, ideals of polynomials. Clearly, there is no hope of obtaining a classification of the algebras $\mathcal{A}_I$ only in terms of the vanishing loci $V(I)$. What makes the non-radical setting even more difficult is that we cannot regard $\mathcal{A}_I$ as an algebra of continuous functions on $Z(I)$ as in Corollary 2.19. In fact, Corollary 3.5 shows that $\mathcal{A}_I$ will not be semi-simple unless I is a radical ideal. Nevertheless, the algebras $\mathcal{A}_I$ were classified up to isometric isomorphism in [DRS11], using the notion of a subproduct system [SS09]. The first goal is to establish this result. In the sequel, we will not use this notion, but we will employ ideas from this theory, specifically from [SS09, Section 9].

Recall from the beginning of Section 3.5 that if $I \subset \mathbb{C}[z]$ is a homogeneous ideal, then $\mathbb{C}[z]/I$ can be embedded into $\mathcal{A}_I$ via

$$\mathbb{C}[z]/I \to \mathcal{A}_I, \quad [p] \mapsto p(S^I) = P_{\mathcal{F}_I} M_p\big|_{\mathcal{F}_I}.$$

In what follows, we write $(\mathcal{A}_I)_n$ for the degree n part of $\mathbb{C}[z]/I$, viewed as a subspace of $\mathcal{A}_I$, that is,

$$(\mathcal{A}_I)_n = \{p(S^I) : p \in \mathbb{C}[z]_n\}.$$

Let $J \subset \mathbb{C}[z]$ be another homogeneous ideal. Recall that an algebra homomorphism from $\mathcal{A}_I$ to $\mathcal{A}_J$ is called graded if it restricts to a graded homomorphism from $\mathbb{C}[z]/I$ to $\mathbb{C}[z]/J$, that is, if

$$\Phi((\mathcal{A}_I)_n) \subset (\mathcal{A}_J)_n \quad \text{for all } n \in \mathbb{N}.$$

We have seen in Proposition 3.27 that if the algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic, then they are isomorphic via a vacuum-preserving isometric isomorphism (see also Definition 3.23). In the radical case, this led to a necessary criterion for $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic. An important point was that in this case, vacuum-preserving isomorphisms are always graded (see Lemma 3.31). This need no longer be true in the non-radical case.

Example 5.1. Let $d = 1$ and let $I = J = \langle z^3 \rangle \subset \mathbb{C}[z]$. Using the homogeneous expansion, it is easy to check that $\mathcal{A}_I$ and $\mathcal{A}_J$ are naturally isomorphic to $\mathbb{C}[z]/I$. Let $\Phi : \mathbb{C}[z]/I \to \mathbb{C}[z]/I$ be the unique algebra homomorphism mapping $[z]$ to $[z+z^2]$. Then Φ is a continuous algebra isomorphism whose inverse is given by the map sending $[z]$ to $[z-z^2]$. Since $Z(I) = \{0\}$, the algebra homomorphism Φ is evidently vacuum-preserving, but it does not respect the grading.

To examine the situation in the non-radical case, we need a homogeneous expansion for $\mathcal{A}_I$.

Proposition 5.2. *Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. There is a family of contractive projections $P_n^I : \mathcal{A}_I \to (\mathcal{A}_I)_n$ for $n \in \mathbb{N}$ with the following properties:*

(a) $P_n^I\big|_{(\mathcal{A}_I)_n} = \mathrm{id}_{(\mathcal{A}_I)_n}$ *and* $P_n^I\big|_{(\mathcal{A}_I)_m} = 0$ *for* $n \neq m$.

(b) *For each $T \in \mathcal{A}_I$, the series $\sum_{n=0}^\infty P_n^I(T)$ is Cesàro-convergent to T.*

Proof. In the case $I = \{0\}$, that is, $\mathcal{A}_I = \mathcal{A}_d$, Proposition 1.6 yields projections P_n as asserted. For the general case, we identify $\mathcal{A}_I$ with $\mathcal{A}_d/\widetilde{I}$ according to Theorem 2.17. Since I is homogeneous, $\widetilde{I}$ is invariant under each P_n by Lemma 1.12, so that the operators P_n induce well-defined contractive projections on $\mathcal{A}_d/\widetilde{I}$. On $\mathcal{A}_I$, they are given by

$$P_n^I : \mathcal{A}_I \to (\mathcal{A}_I)_n, \quad P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I} \mapsto P_{\mathcal{F}_I} M_{P_n(\varphi)}\big|_{\mathcal{F}_I}$$

for $\varphi \in \mathcal{A}_d$. It is clear that the family (P_n^I) satisfies the first condition, while the second one follows from the corresponding statement about the projections P_n in Proposition 1.6. □

The projections introduced in the preceding lemma allow for a simple characterization of graded homomorphisms.

Lemma 5.3. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_I$ be a continuous algebra homomorphism. Then Φ is graded if and only if*

$$P_n^J \Phi = \Phi P_n^I \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Clearly, the intertwining relations are sufficient. To prove necessity, let Φ be graded and let $T \in \mathcal{A}_I$ be arbitrary. Then $\Phi(P_n^I(T)) \in (\mathcal{A}_J)_n$ for all $n \in \mathbb{N}$. Since Φ is continuous and linear, Proposition 5.2 shows that

$$P_n^J \Phi(T) = P_n^J \Phi\Big(\sum_{k=0}^{\infty} P_k^I(T)\Big) = \Phi(P_n^I(T))$$

for all $n \in \mathbb{N}$, where the series is Cesàro-convergent. Hence the assertion follows. $\square$

An immediate consequence is the following result.

Corollary 5.4. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a continuous graded algebra isomorphism. Then also its inverse $\Phi^{-1} : \mathcal{A}_J \to A_I$ is graded.*

Proof. The assertion follows from Lemma 5.3 by multiplying the intertwining conditions by Φ^{-1} from the left and from the right. $\square$

With the help of the homogeneous expansion in $\mathcal{A}_I$, we obtain a clearer picture of the connection between vacuum-preserving and graded homomorphisms. In particular, we see that graded homomorphisms are always vacuum-preserving. To shorten notation, we define for $n \in \mathbb{N}$ closed ideals

$$(\mathcal{A}_I)_{\geq n} = \overline{\{p(S^I) : p \in \mathbb{C}[z]_{\geq n}\}}$$

in $\mathcal{A}_I$, where $\mathbb{C}[z]_{\geq n}$ denotes the homogeneous ideal $\bigoplus_{k=n}^{\infty} \mathbb{C}[z]_k$ in $\mathbb{C}[z]$.

Lemma 5.5. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a unital continuous algebra homomorphism. Then the following are equivalent:*

(i) Φ *is vacuum-preserving,*

(ii) $\Phi((\mathcal{A}_I)_1) \subset (\mathcal{A}_J)_{\geq 1}$,

(iii) $\Phi((\mathcal{A}_I)_{\geq n}) \subset (\mathcal{A}_J)_{\geq n}$ *for all* $n \in \mathbb{N}$.

Proof. (iii) trivially implies (ii), and it is easy to check that (ii) implies (iii) using the fact that Φ is a continuous algebra homomorphism.

To show that (i) and (ii) are equivalent, suppose first that (ii) holds. Then

$$\Phi^*(\delta_0)(S_i^I) = \delta_0(\Phi(S_i^I)) = 0 \quad \text{for all} \quad i = 1, \ldots, d,$$

and consequently $\Phi^*(\delta_0) = \delta_0$ by Proposition 3.4, that is, Φ is vacuum-preserving.

Conversely, assume that (i) holds and let $i \in \{1, \ldots, d\}$. An application of Proposition 5.2 shows that there are $a \in \mathbb{C}$ and $T \in (\mathcal{A}_J)_{\geq 1}$ such that

$$\Phi(S_i^I) = a1 + T.$$

By assumption, we have $a = \delta_0(\Phi(S_i^I)) = \delta_0(S_i^I) = 0$, so $\Phi(S_i^I) = T$. This observation finishes the proof. $\square$

Example 5.1 shows that vacuum-preserving isomorphisms need not be graded. However, the following lemma asserts that they are not far from being graded in a certain sense. As in the discussion preceding Lemma 3.3, let

$$R_I : \mathcal{A}_I \to \mathcal{A}_{\sqrt{I}}, \quad T \mapsto P_{\mathcal{F}_{\sqrt{I}}} T\big|_{\mathcal{F}_{\sqrt{I}}}$$

be the natural quotient map from $\mathcal{A}_I$ onto $\mathcal{A}_{\sqrt{I}}$.

Lemma 5.6. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ be a vacuum-preserving algebra isomorphism. Then*

$$R^J(P_n^J \Phi - \Phi P_n^I) = 0.$$

Proof. Consider

$$\Psi = R^J \circ \Phi : \mathcal{A}_I \to \mathcal{A}_{\sqrt{J}}.$$

Then Ψ is a vacuum-preserving algebra homomorphism, in fact, with the identifications explained in Proposition 3.4, we have $\Phi^* = \Psi^*$. Thus, Lemma 3.18 and Corollary 3.20 imply the existence of a linear map A on $\mathbb{C}^d$ such that $\Psi^* = A\big|_{Z(J)}$. Regarding $\mathcal{A}_{\sqrt{J}}$ as an algebra of functions on $Z(J)$, we obtain for $T \in \mathcal{A}_I$ and $\lambda \in Z(J)$ the identity

$$\Psi(T)(\lambda) = (\delta_\lambda \circ \Psi)(T) = \delta_{A\lambda}(T) = (R_I(T))(A\lambda),$$

where we have also viewed $\mathcal{A}_{\sqrt{I}}$ as an algebra of functions on $Z(I)$ in the last step. We conclude that $\Psi(T) = R_I(T) \circ A$, from which it easily follows that Ψ is graded. Since $\mathcal{A}_{\sqrt{J}}$ is semi-simple by Corollary 3.5, the algebra homomorphism Ψ is continuous, so that $P_n^{\sqrt{J}} \circ \Psi = \Psi \circ P_n^I$ holds for all $n \in \mathbb{N}$ by Lemma 5.3. The assertion now follows by another application of Lemma 5.3 to the continuous graded homomorphism R^J. □

5.2. Isometric isomorphisms in the non-radical case

In this section, we classify the algebras $\mathcal{A}_I$ for not necessarily radical homogeneous ideals I, up to isometric isomorphism. A sufficient condition for two algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic is given by Lemma 2.22: it suffices that there exists a unitary map U on $\mathbb{C}^d$ such that $J = \{p \circ U : p \in I\}$. We will see that this condition is also necessary.

If $I \subsetneq \mathbb{C}[z]$ is a homogeneous ideal, then $1 \in \mathcal{F}_I$. The following property of operators in $\mathcal{A}_I$ is reminiscent of the behaviour of multiplication operators, even though the elements of $\mathcal{A}_I$ are not multiplication operators in a natural way.

Lemma 5.7. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal and let $T \in \mathcal{A}_I$. Let $\varphi \in \mathcal{A}_d$ such that $T = P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$ (see Theorem 2.17). Then the following are equivalent:*

(i) $T = 0$,

(ii) $T1 = 0$,

(iii) $\varphi \in \widetilde{I}$.

Proof. (i) $\Rightarrow$ (ii) is trivial. If (ii) holds, then $\varphi \in \overline{I}$. A straightforward application of the homogeneous decomposition (Proposition 1.6) and Lemma 1.12 shows that $\varphi \in \widetilde{I}$, that is, (iii) holds. The implication (iii) $\Rightarrow$ (i) follows at once from Lemma 2.16 (b). □

The following lemma shows that isometric isomorphisms do not show the bahaviour from Example 5.1 (compare [SS09, Lemma 9.6]).

Lemma 5.8. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals. A unital isometric algebra isomorphism $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is vacuum-preserving if and only if it is graded.*

Proof. Clearly, graded continuous algebra homomorphisms are vacuum-preserving by Lemma 5.5. Conversely, let Φ be an isometric vacuum-preserving isomorphism. To prove that Φ is graded, it suffices to show that $\Phi(S_i^I) \in (A_I)_1$ for $i = 1, \ldots, d$. So let $i \in \{1, \ldots, d\}$. According to Lemma 5.5 and Proposition 5.2, there are $S \in (\mathcal{A}_I)_1$ and $T \in (\mathcal{A}_I)_{\geq 2}$ such that

$$\Phi(S_i^I) = S + T.$$

We have to show that $T = 0$. Note that $S1$ is orthogonal to $T1$ in $\mathcal{F}_J$, so using Lemma 3.32 and the fact that Φ is isometric, we obtain the estimate

$$||S||^2 = ||S1||^2 \leq ||S1||^2 + ||T1||^2 = ||\Phi(S_i^I)1||^2 \leq ||\Phi(S_i^I)||^2 = ||S_i^I||^2.$$

Since Φ^{-1} is vacuum-preserving as well, Lemma 5.5 shows that $\Phi^{-1}(T) \in (\mathcal{A}_I)_{\geq 2}$, hence $S_i^I 1$ and $\Phi^{-1}(T)1$ are orthogonal in $\mathcal{F}_I$. Using Lemma 3.32 once again, we infer that

$$\begin{aligned} ||S_i^I||^2 = ||S_i^I 1||^2 \leq ||S_i^I 1||^2 + ||\Phi^{-1}(T)1||^2 &= ||(S_i^I - \Phi^{-1}(T))1||^2 \\ &\leq ||(S_i^I - \Phi^{-1}(T))||^2 = ||S||^2, \end{aligned}$$

where the last equality holds because Φ is isometric. Combining both estimates, we see that equality holds everywhere. In particular, $T1 = 0$ and hence $T = 0$ by Lemma 5.7. □

Just as in the radical case (see Theorem 3.36), we can now show that the sufficient condition for $\mathcal{A}_I$ and $\mathcal{A}_J$ being isomorphic from Lemma 2.22 is also necessary.

Theorem 5.9. *Let $I, J \subset \mathbb{C}^d$ be (not necessarily radical) homogeneous ideals. Then the following assertions are equivalent:*

(i) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic.*

(ii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are unitarily equivalent.*

(iii) *There is a unitary U on $\mathbb{C}^d$ such that $I = \{p \circ U : p \in J\}$.*

Proof. The trivial case where $I = \mathbb{C}[z]$ or $J = \mathbb{C}[z]$ is readily disposed of by Remark 3.1. Hence, we may assume that I and J are proper ideals. Taking Lemma 2.22 into account, we only have to show that (i) implies (iii). If (i) holds, we find a vacuum-preserving isometric isomorphism $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ according to Proposition 3.27. Lemma 5.8 shows that Φ is graded, so that the existence of the unitary U follows from Lemma 3.33. □

5.3. Topological isomorphisms in the non-radical case

Classifying the algebras $\mathcal{A}_I$ up to topological isomorphism is more difficult than classifying them up to isometric isomorphism. The main result in the radical case was Theorem 4.48. In this section, we study the same problem for not necessarily radical ideals.

As in the radical case, maps on spaces of polynomials given by $p \mapsto p \circ A^*$, where A is a linear map on $\mathbb{C}^d$, play an important role. In the non-radical case, we do not have a generating set for $\mathcal{D}_I = \mathcal{F}_I \cap \mathbb{C}[z]$ as in Lemma 4.13. Nevertheless, the existence of densely defined maps given by composition with a linear map easily follows from the following lemma.

Lemma 5.10. *Let n be a natural number. For a linear map A on $\mathbb{C}^d$, we define*

$$C_A : \mathbb{C}[z]_n \to \mathbb{C}[z]_n, \quad p \mapsto p \circ A.$$

Then the adjoint of C_A is given by composition with A^*, that is, $(C_A)^* = C_{A^*}$.

Proof. For $\lambda, \mu \in \mathbb{B}_d$, we have

$$\Big\langle \langle \cdot, \lambda \rangle^n, \langle \cdot, \mu \rangle^n \circ A^* \Big\rangle_{H^2_d} = \Big\langle \langle \cdot, \lambda \rangle^n, \langle \cdot, A\mu \rangle^n \Big\rangle_{H^2_d} = \overline{\Big\langle \langle \cdot, A\mu \rangle^n, K(\cdot, \lambda) \Big\rangle}_{H^2_d} = \langle A\mu, \lambda \rangle^n.$$

Similarly,

$$\Big\langle \langle \cdot, \lambda \rangle^n \circ A, \langle \cdot, \mu \rangle^n \Big\rangle_{H^2_d} = \langle \mu, A^* \lambda \rangle^n = \langle A\mu, \lambda \rangle^n,$$

so that the assertion follows from the fact that $\mathbb{C}[z]_n$ is the linear span of the polynomials of the form $\langle \cdot, \lambda \rangle^n$ for $\lambda \in \mathbb{C}^d$. □

Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals. If A is an invertible linear map on $\mathbb{C}^d$ such that $J = \{p \circ A : p \in I\}$, then for each natural number n, composition with A maps $I \cap \mathbb{C}[z]_n$ into $J \cap \mathbb{C}[z]_n$. The preceding lemma thus implies that composition with A^* maps $\mathcal{F}_J \cap \mathbb{C}[z]_n$ into $\mathcal{F}_I \cap \mathbb{C}[z]_n$, so that we obtain a densely defined map

$$\mathcal{F}_J \supset \mathcal{D}_J \to \mathcal{F}_I, \quad p \mapsto p \circ A^*,$$

where $\mathcal{D}_J = \mathcal{F}_J \cap \mathbb{C}[z]$. Just as in the radical case, boundedness of this map is essential.

Lemma 5.11. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals and let A be an invertible linear map on $\mathbb{C}^d$ satisfying $J = \{p \circ A : p \in I\}$. Regarding $\mathbb{C}[z]/I$ and $\mathbb{C}[z]/J$ as subspaces of $\mathcal{A}_I$ and $\mathcal{A}_J$, respectively, the following assertions are equivalent:*

(i) *The algebra isomorphism*

$$\mathbb{C}[z]/I \to \mathbb{C}[z]/J, \quad [p] \mapsto [p \circ A]$$

extends to a topological isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$.

(ii) *The linear maps*

$$\mathcal{F}_J \supset \mathcal{D}_J \to \mathcal{F}_I, \quad p \mapsto p \circ A^*$$

and

$$\mathcal{F}_I \supset \mathcal{D}_I \to \mathcal{F}_J, \quad p \mapsto p \circ (A^{-1})^*$$

are bounded.

(iii) *There exist constants $M_1, M_2 > 0$ such that for all $n \in \mathbb{N}$ and all homogeneous polynomials p of degree n, we have*

$$\frac{1}{M_1} ||[p]||_{(\mathcal{A}_I)_n} \leq ||[p \circ A]||_{(\mathcal{A}_J)_n} \leq M_2 ||[p]||_{(\mathcal{A}_I)_n}.$$

In this case, $\mathcal{A}_I$ and $\mathcal{A}_J$ are similar.

Proof. (i) $\Rightarrow$ (iii). Set $M_2 = ||\Phi||$ and $M_1 = ||\Phi^{-1}||$.

(iii) $\Rightarrow$ (ii). With the help of Lemma 3.32, we see that the inequality on the right-hand side implies that the norms of the maps

$$C_n^* : \mathcal{F}_I \cap \mathbb{C}[z]_n \to \mathcal{F}_J \cap \mathbb{C}[z]_n, \quad p \mapsto P_{\mathcal{F}_J}(p \circ A)$$

are bounded by M_2. Lemma 5.10 shows that the adjoints are given by

$$C_n : \mathcal{F}_J \cap \mathbb{C}[z]_n \to \mathcal{F}_I \cap \mathbb{C}[z]_n, \quad p \mapsto p \circ A^*.$$

Using that homogeneous polynomials of different degree are orthogonal, we conclude that the first map is bounded. Boundedness of the second map follows by a similar argument from the inequality on the left-hand side.

(ii) $\Rightarrow$ (i). Let C denote the continuous extension of the first map in (ii) to $\mathcal{F}_J$. Clearly, the continuous extension of the second map in (ii) is the inverse of C. We claim that

$$\Phi : \mathcal{A}_I \to \mathcal{A}_J, \quad T \mapsto C^* T (C^{-1})^*$$

is a topological isomorphism which extends the map $[p] \mapsto [p \circ A^*]$. To this end, let p be a homogeneous polynomial, and let $f \in \mathcal{F}_J \cap \mathbb{C}[z]_n$ for some natural number n. An application of Lemma 5.10 yields the identity

$$(C^{-1})^* f = P_{\mathcal{F}_I}(f \circ A^{-1}). \tag{5.1}$$

Using the co-invariance of $\mathcal{F}_I$ under multiplication by p, we deduce that

$$C^*(P_{\mathcal{F}_I} M_p\big|_{\mathcal{F}_I})(C^{-1})^* f = C^* P_{\mathcal{F}_I} M_p (f \circ A^{-1}).$$

By an obvious analogue of (5.1), and since composition with A maps I into J, this is equal to

$$P_{\mathcal{F}_J}((P_{\mathcal{F}_I} M_p (f \circ A^{-1})) \circ A) = P_{\mathcal{F}_J}((p \cdot (f \circ A^{-1})) \circ A) = P_{\mathcal{F}_J} M_{p \circ A} f.$$

Hence

$$C^*(P_{\mathcal{F}_I} M_p\big|_{\mathcal{F}_I})(C^{-1})^* = P_{F_J} M_{p \circ A}\big|_{\mathcal{F}_J}.$$

This observation shows that Φ indeed maps $\mathcal{A}_I$ into $\mathcal{A}_J$, and that it extends the map $[p] \mapsto [p \circ A]$. Arguing in the other direction, we find that Φ is a topological isomorphism, as asserted. In fact, Φ is a similarity, which establishes the additional assertion. $\square$

Remark 5.12. If $A = U$ is a unitary map on $\mathbb{C}^d$ in the preceding lemma, then composition with U induces a unitary C_U on H^2_d (see Lemma 2.21). Note that C_U maps $\overline{I}$ onto $\overline{J}$, so it maps $\mathcal{F}_I$ onto $\mathcal{F}_J$. Thus, the analogue of (5.1) for the operator C shows that in this case, the operator C^* from the last part of the above proof is the restriction of C_U to $\mathcal{F}_I$. In particular, the algebra isomorphism constructed above is just the algebra isomorphism from Lemma 2.22 if $A = U$ is a unitary.

We can now deduce a necessary criterion for $\mathcal{A}_I$ and $\mathcal{A}_J$ being topologically isomorphic. Moreover, we show that if $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic, then they are isomorphic via a graded isomorphism.

Theorem 5.13. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals such that $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic. Then there exists an invertible linear map A on $\mathbb{C}^d$ with the following properties:*

(a) $J = \{p \circ A : p \in I\}$.

(b) *A maps* $Z(J)$ *bijectively onto* $Z(I)$.

In addition, it can be achieved that $\mathbb{C}[z]/I \to \mathbb{C}[z]/J, [p] \mapsto [p \circ A]$, *extends to a topological isomorphism between* $\mathcal{A}_I$ *and* $\mathcal{A}_J$.

Proof. By Proposition 3.27, there exists a vacuum-preserving topological isomorphism $\Phi : \mathcal{A}_I \to \mathcal{A}_J$. The isomorphism Φ need not be graded, but an application of Lemma 5.5 and Proposition 5.2 shows that there are $T_i \in (\mathcal{A}_J)_1$ and $R_i \in (\mathcal{A}_J)_{\geq 2}$ for $i = 1, \ldots, d$ such that

$$\Phi(S_i^I) = T_i + R_i \quad (i = 1, \ldots, d).$$

Set $T = (T_1, \ldots, T_d)$. If p is a homogeneous polynomial of degree n, then

$$\Phi(p(S^I)) = p(\Phi(S_1^I), \ldots, \Phi(S_d^I)) = p(T_1, \ldots, T_d) + R, \tag{5.2}$$

for some $R \in (\mathcal{A}_J)_{\geq n+1}$, where we have used that $(\mathcal{A}_J)_{\geq m} \cdot (\mathcal{A}_J)_{\geq k} \subset (\mathcal{A}_J)_{\geq (m+k)}$ for natural numbers m and k. In particular, if $p \in I$, then the left-hand side of the last identity is zero, so by applying the projection P_n^J, we conclude that $p(T) = 0$. Consequently, if we regard $\mathbb{C}[z]/J$ as a subspace of $\mathcal{A}_J$, we obtain a well-defined graded algebra homomorphism

$$\Psi : \mathbb{C}[z]/I \to \mathbb{C}[z]/J, \quad [p] \mapsto p(T).$$

Equation (5.2) shows that

$$\Psi([p]) = (P_n^J \circ \Phi)(p(S^I)) \tag{5.3}$$

for all $p \in \mathbb{C}[z]_n$ and all $n \in \mathbb{N}$. We claim that Ψ is given by composition with a linear map, and that it extends to a topological isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$.

First, notice that Φ^{-1} is vacuum-preserving as well, so that by the same argument, applied to Φ^{-1}, we obtain a graded algebra homomorphism $\widetilde{\Psi} : \mathbb{C}[z]/J \to \mathbb{C}[z]/I$ satisfying

$$\widetilde{\Psi}([p]) = (P_n^I \circ \Phi^{-1})(p(S^J)) \tag{5.4}$$

for all $p \in \mathbb{C}[z]_n$ and all $n \in \mathbb{N}$. Using Lemma 5.5, it is easy to check that for $n \in \mathbb{N}$, the maps

$$P_n^I \circ \Phi^{-1} \circ P_n^J \circ \Phi \quad \text{and} \quad P_n^J \circ \Phi \circ P_n^I \circ \Phi^{-1}$$

are the identity maps on $(\mathcal{A}_I)_n$ and $(\mathcal{A}_J)_n$, respectively, from which we infer that Ψ and $\widetilde{\Psi}$ are inverse to each other. In this situation, Lemma 3.33 yields an invertible linear map A on $\mathbb{C}^d$ satisfying statement (a) of the theorem such that Ψ is given by composition with A. Since the projections P_n^I and P_n^J are contractive, and since Φ

is a topological isomorphism, equations (5.3) and (5.4) show that condition (iii) of Lemma 5.11 is fulfilled. Thus, Ψ extends to a topological isomorphism as asserted.

It remains to prove statement (b) of the theorem. To this end, let $\lambda \in Z(J)$. Then

$$\Psi^*(\delta_\lambda)(S_i^I) = \delta_\lambda(P_{\mathcal{F}_J} M_{z_i \circ A}\big|_{\mathcal{F}_J}) = z_i(A\lambda),$$

where we have used Lemma 3.3 (a). By Proposition 3.4, this means that $A\lambda \in Z(I)$ and $\Psi^*(\delta_\lambda) = \delta_{A\lambda}$, so that Ψ^* coincides with A on $Z(J)$. This observation finishes the proof. $\square$

Remark 5.14. Let Φ and Ψ be as in the preceding proof. Equation (5.3), combined with Lemma 5.6, shows that

$$R^J \circ \Phi = R^J \circ \Psi,$$

that is, the maps Φ and Ψ coincide modulo the kernel of the Gelfand transform on $\mathcal{A}_J$. In particular, we have $\Phi^* = \Psi^*$, so that also Φ^* is the restriction of A to $Z(J)$.

We record an immediate consequence of the preceding theorem and the additional assertion in Lemma 5.11.

Corollary 5.15. *Let $I, J \subset \mathbb{C}[z]$ be homogeneous ideals. Then $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic if and only if they are similar.* $\square$

In general, conditions (a) and (b) of Theorem 5.13 are not sufficient for $\mathcal{A}_I$ and $\mathcal{A}_J$ being topologically isomorphic.

Example 5.16. Let $I = \langle x^4, x^2(x+y)\rangle$ and $J = \langle x^4, x^2 y\rangle$. Clearly, $V(I) = V(J)$ is the y-axis. We set

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then A is invertible, and we have

(a) $J = \{p \circ A : p \in I\}$.

(b) A maps $Z(J)$ bijectively onto $Z(I)$.

We claim that $\mathcal{A}_I$ and $\mathcal{A}_J$ are not topologically isomorphic. Suppose, for a contradiction, that they are. Then, by Theorem 5.13, there is an invertible linear map B on $\mathbb{C}^d$, satisfying conditions (a) and (b) above with B in place of A, such that composition with B induces a topological isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$. In particular, the map

$$\mathcal{F}_J \supset \mathcal{D}_J \to \mathcal{F}_I, \quad p \mapsto p \circ B^* \tag{5.5}$$

is bounded in this case by Lemma 5.11.

First, we show that B must essentially look like A. From condition (b) and Lemma 4.2, we infer that B maps the y-axis isometrically onto itself, so that there are complex numbers a, b and λ, the latter of modulus 1, such that

$$B = \begin{pmatrix} a & 0 \\ b & \lambda \end{pmatrix}.$$

If we replace B with $\overline{\lambda} B$, then assumptions (a) and (b) are still valid, and since multiplication by λ is a unitary on $\mathbb{C}^2$, the map in (5.5) is still bounded by Lemma 2.22. Hence, we may assume without loss of generality that $\lambda = 1$. Condition (a) implies in particular that the polynomial

$$(x^2(x+y)) \circ B = (ax)^2(ax + bx + y) = a^2(a+b)x^3 + ax^2y$$

is contained in J, that is, $a^2(a+b)x^3 \in J$. Since the monomial x^3 does not belong to J, this is only possible if $a^2(a+b) = 0$. From invertibility of B, we deduce that $a \neq 0$, hence $b = -a$. Consequently, we have

$$B = \begin{pmatrix} a & 0 \\ -a & 1 \end{pmatrix}.$$

To obtain the desired contradiction, we show that the map in (5.5) is not bounded. Note that for all natural numbers n, the monomial xy^n belongs to $\mathcal{D}_J$, and that

$$||xy^n||^2 = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

On the other hand,

$$\begin{aligned} ||(xy^n) \circ B^*||^2 &= ||(\overline{a}x - \overline{a}y)y^n||^2 = |a|^2(||xy^n||^2 + ||y^{n+1}||^2) \\ &= |a|^2\Big(\frac{1}{n+1} + 1\Big) = |a|^2(n+2)||xy^n||^2, \end{aligned}$$

which finishes the proof since $a \neq 0$.

In a very particular situation, conditions (a) and (b) of Theorem 5.13 are also sufficient. To explain this situation, let I be a homogeneous ideal, and suppose that $V(I) = V_1 \cup \ldots \cup V_r$ is the irreducible decomposition of $V(I)$. Let $\widehat{I}$ be the vanishing ideal of $\operatorname{span}(V_1) \cup \ldots \cup \operatorname{span}(V_r)$. We say that I is *good* if $\widehat{I} \subset I$. Roughly speaking, this means that the non-reduced structure is entirely contained in the union of the linear spans of the irreducible components. Note that all radical homogeneous ideals are good. Another example are homogeneous ideals I such that $V(I)$ is a non-linear hypersurface. As for a non-example, let I be the vanishing ideal of a proper subspace of $\mathbb{C}^d$. Then I^k is not good for $k \geq 2$.

Theorem 5.17. *Let $I, J \subset \mathbb{C}[z]$ be good homogeneous ideals. Then $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic if and only if there exists an invertible linear map A on $\mathbb{C}^d$ with the following properties:*

(a) $J = \{p \circ A : p \in I\}$.

(b) *A maps $Z(J)$ onto $Z(I)$.*

Proof. Necessity follows from Theorem 5.13. To establish sufficiency, we apply Proposition 4.18 to $\sqrt{I}$ and $\sqrt{J}$. Recall from Lemma 4.20 and Theorem 4.46 that all radical homogeneous ideals are admissible, so that $p \mapsto p \circ A^*$ is bounded on $\mathcal{D}_{\sqrt{J}}$. In fact, the proof of Proposition 4.18 shows that $p \mapsto p \circ A^*$ is bounded on $\mathcal{D}_{\widehat{J}}$ because $\widehat{\sqrt{J}} = \widehat{J}$. Since J is good, we have $\mathcal{D}_J \subset \mathcal{D}_{\widehat{J}}$, so that $p \mapsto p \circ A^*$ is bounded on $\mathcal{D}_J$. Similarly, we see that $p \mapsto p \circ (A^{-1})^*$ is bounded on $\mathcal{D}_I$. The assertion thus follows from Lemma 5.11. $\square$

6. The WOT-closures of the algebras $\mathcal{A}_I$

6.1. The algebras $\mathcal{L}_I$

Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal, and let $\mathcal{L}_I$ denote the WOT-closure of the algebra $\mathcal{A}_I$. In this chapter, we study the isomorphism problem for algebras of the type $\mathcal{L}_I$, where I is radical. We follow again [DRS11]. The main result is that two such algebras $\mathcal{L}_I$ and $\mathcal{L}_J$ are (isometrically) isomorphic if and only if $\mathcal{A}_I$ and $\mathcal{A}_J$ are (isometrically) isomorphic, so that the results deduced in the preceding chapters also lead to a classification of the algebras $\mathcal{L}_I$. In fact, it is not hard to see from the classification of the algebras $\mathcal{A}_I$ that two isomorphic algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ also have isomorphic WOT-closures $\mathcal{L}_I$ and $\mathcal{L}_J$. Thus, the goal of this chapter is to establish the converse.

As it was pointed out at the beginning of Section 11 in [DRS11], it can well happen that two operator algebras are isomorphic, but that their WOT-closures are not, or conversely, that non-isomorphic operator algebras have isomorphic WOT-closures. An elementary example of the latter phenomenon are the C^*-algebras $\mathbb{C} \oplus \mathcal{K}(H)$ and $\mathcal{L}(H)$, where H is an infinite dimensional Hilbert space and $\mathcal{K}(H)$ denotes the ideal of compact operators. It is essentially due to the homogeneity of the ideals that this problem does not occur in our case.

Recall that $\mathcal{A}_I$ can be identified with $\mathcal{A}_d/\widetilde{I}$ by Theorem 2.17. If I is radical, $\mathcal{A}_I$ can be regarded as an algebra of functions on $Z(I)$ by Corollary 2.19. We begin by establishing analogous results for the algebras $\mathcal{L}_I$.

Theorem 6.1. *The map*

$$\operatorname{Mult}(H^2_d)/\overline{I}^{WOT} \to \mathcal{L}_I, \quad [M_\varphi] \mapsto P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$$

is an isometric algebra isomorphism.

Proof. Since the map $\mathcal{L}(H^2_d) \to \mathcal{L}(\mathcal{F}_I), T \mapsto P_{\mathcal{F}_I} T\big|_{\mathcal{F}_I}$, is WOT-continuous, and since $\operatorname{Mult}(H^2_d)$ is WOT-closed by Lemma A.10, we have a map

$$\Phi : \operatorname{Mult}(H^2_d) \to \mathcal{L}_I, \quad M_\varphi \mapsto P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}.$$

This mapping is an algebra homomorphism because $\mathcal{F}_I$ is co-invariant for all multipliers by Lemma 2.16 (a), and Lemma 2.16 (b) shows that its kernel is equal to

$\overline{I}^{WOT} = \overline{I}^{SOT}$, so that the map in the statement of the theorem is a well-defined, injective and contractive algebra homomorphism.

To establish surjectivity of Φ, note that for all polynomials $p \in \mathbb{C}[z]$, the operator $p(S^I)$ commutes with all compressed multipliers since $\mathcal{F}_I$ is co-invariant for all multiplication operators, that is,

$$p(S^I)(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) = (P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) p(S^I) \quad \text{for all } \varphi \in \mathrm{Mult}(H^2_d).$$

It follows that

$$T(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) = (P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) T \quad \text{for all } \varphi \in \mathrm{Mult}(H^2_d)$$

and all $T \in \mathcal{L}_I$. Hence an application of Theorem 2.5 shows that given $T \in \mathcal{L}_I$ with $||T|| \leq 1$, we find a contractive multiplier $\varphi \in \mathrm{Mult}(H^2_d)$ such that $\Phi(M_\varphi) = T$. This shows that Φ is surjective, and that the induced map in the statement of the theorem is an isometric algebra isomorphism. $\square$

If $I \subsetneq \mathbb{C}[z]$ is a radical homogeneous ideal, then according to Lemma 2.18, $\mathcal{F}_I$ can be identified with $H^2_d\big|_{Z^0(I)}$. Under this identification, $\mathcal{L}_I$ corresponds to $\mathrm{Mult}(H^2_d\big|_{Z^0(I)})$. In particular, the algebra $\mathcal{L}_I$ is semi-simple if I is radical.

Corollary 6.2. *Let U be the unitary operator*

$$\mathcal{F}_I \to H^2_d\big|_{Z^0(I)}, \quad f \mapsto f\big|_{Z^0(I)}$$

from Lemma 2.18. Then

$$\mathcal{L}_I \to \mathrm{Mult}(H^2_d\big|_{Z^0(I)}), \quad T \mapsto UTU^*$$

is an algebra isomorphism sending $P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$ to $\varphi\big|_{Z^0(I)}$ for $\varphi \in \mathrm{Mult}(H^2_d)$.

Proof. Corollary 2.19 shows that $US_i^I U^* = z_i\big|_{Z^0(I)}$ for $i = 1, \ldots, d$, from which we infer that $T \mapsto UTU^*$ defines an isometric algebra homomorphism from $\mathcal{L}_I$ into the multiplier algebra of $H^2_d\big|_{Z^0(I)}$ (recall that the multiplier algebra is WOT-closed by Lemma A.10). Thus, it remains to show that given $\varphi \in \mathrm{Mult}(H^2_d\big|_{Z^0(I)})$, the operator $U^* M_\varphi U$ is contained in $\mathcal{L}_I$. To this end, note that M_φ commutes with $U(P_{\mathcal{F}_I} M_\psi\big|_{\mathcal{F}_I})U^*$ for each $\psi \in \mathrm{Mult}(H^2_d)$, thus an application of Theorem 2.5 shows that there is a multiplier ψ on H^2_d such that $U^* M_\varphi U = P_{\mathcal{F}_I} M_\psi\big|_{\mathcal{F}_I}$. In particular, this operator is contained in $\mathcal{L}_I$, which finishes the proof. $\square$

6.2. The maximal ideal space of $\mathcal{L}_I$

Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal. In contrast to the norm-closed case, we cannot hope to write down explicitly all multiplicative linear functionals on $\mathcal{L}_I$. For example, if $d = 1$ and $I = \{0\}$, we have $\mathcal{L}_I = H^\infty(\mathbb{D})$, an algebra whose maximal ideal space is rather complicated (see [Hof62, Chapter 10]). Nevertheless, just as with $H^\infty(\mathbb{D})$, some multiplicative linear functionals on $\mathcal{L}_I$ are easy to come by: In the discussion preceding Lemma 3.3, we have seen that every $\lambda \in Z^0(I)$ gives rise to a multiplicative linear functional δ_λ^I on $\mathcal{A}_I$, and according to the lemma itself, it is given by

$$\delta_\lambda^I(T) = \langle T1, K(\cdot, \lambda)\rangle \quad \text{for all } T \in \mathcal{A}_I,$$

where K is the reproducing kernel of H^2_d. In particular, δ_λ^I is WOT-continuous, and hence uniquely extends to a WOT-continuous linear functional $\widehat{\delta}_\lambda^I$ on $\mathcal{L}_I$. As in the norm-closed case, we will drop the superscript I when the ideal is understood. It is easy to see that the extension is again multiplicative, for example using separate continuity of multiplication in the weak operator topology. For future reference, we record some properties of the functionals $\widehat{\delta}_\lambda$.

Lemma 6.3. *Let $I \subsetneq \mathbb{C}[z]$ be a homogeneous ideal, and let $\lambda \in Z^0(I)$.*

(a) $\widehat{\delta}_\lambda(T) = \langle T1, K(\cdot, \lambda)\rangle$ *for all $T \in \mathcal{L}_I$.*

(b) *For $\varphi \in \operatorname{Mult}(H^2_d)$, we have $\widehat{\delta}_\lambda(P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}) = \varphi(\lambda)$.*

(c) *If $J \subset I$ is another homogeneous ideal, then*

$$\widehat{\delta}_\lambda^I(P_{\mathcal{F}_I} T\big|_{\mathcal{F}_I}) = \widehat{\delta}_\lambda^J(T)$$

holds for all $T \in \mathcal{L}_J$.

(d) *Suppose that I is a radical ideal. Then modulo the identification of $\mathcal{L}_I$ with $\operatorname{Mult}(H^2_d\big|_{Z^0(I)})$ explained in Corollary 6.2, the functional $\widehat{\delta}_\lambda$ equals point evaluation at λ. In particular, if $T \in \mathcal{L}_I$ with $\widehat{\delta}_\lambda(T) = 0$ for all $\lambda \in Z^0(I)$, then $T = 0$.*

Proof. Part (a) is immediate from the definition, while (b) and (c) easily follow from (a), since $K(\cdot, \lambda) \in \mathcal{F}_I$ for $\lambda \in Z^0(I)$. Part (d) finally follows from (b). $\square$

Part (d) of the preceding lemma could also serve as the definition of $\widehat{\delta}_\lambda$ in the radical case, while the statement of part (c) could then be used to extend the definition to arbitrary ideals, taking $I = \sqrt{J}$.

It turns out that all WOT-continuous linear functionals on $\mathcal{L}_I$ arise in the way described above. We write $\Delta_0(\mathcal{L}_I)$ for the set of all WOT-continuous linear functionals, endowed with the weak-* topology.

Proposition 6.4. *The map*

$$Z^0(I) \to \Delta_0(\mathcal{L}_I) \quad \lambda \mapsto \widehat{\delta}_\lambda$$

is a homeomorphism, whose inverse is given by

$$\Delta_0(\mathcal{L}_I) \to Z^0(I), \quad \rho \mapsto \rho(S_1^I, \dots, S_d^I).$$

Proof. By the above discussion, each $\lambda \in Z^0(I)$ gives rise to a multiplicative linear functional $\widehat{\delta}_\lambda \in \Delta_0(\mathcal{L}_I)$, and it is clear that $\widehat{\delta}_\lambda(S_1^I, \dots, S_d^I) = \lambda$. Conversely, given $\rho \in \Delta_0(\mathcal{L}_I)$, Proposition 3.4 shows that $\rho\big|_{\mathcal{A}_I} = \delta_\lambda$ where $\lambda = \rho(S_1^I, \dots, S_d^I)$. We have to show that $\lambda \in Z^0(I)$. Assume, to the contrary, that $\lambda \in \partial\mathbb{B}_d$, and let U be a unitary on $\mathbb{C}^d$ which maps λ to the point $(1, 0, \dots, 0)$. Since

$$\Phi : \mathcal{A}_d \to \mathcal{A}_I, \quad M_\varphi \mapsto P_{\mathcal{F}_I} M_\varphi\big|_{\mathcal{F}_I}$$

and

$$\Phi_U : \mathcal{A}_d \to \mathcal{A}_d, \quad M_\varphi \mapsto M_{\varphi \circ U}$$

are WOT-continuous algebra homomorphisms, the latter by Lemma 2.21, the multiplicative linear functional $\widetilde{\rho} = \rho \circ \Phi \circ \Phi_U$ is WOT-continuous and satisfies

$$\widetilde{\rho}(M_{z_1}) = \rho(P_{\mathcal{F}_I} M_{z_1 \circ U}\big|_{\mathcal{F}_I}) = (z_1 \circ U)(\lambda) = 1$$

by Lemma 6.3 (b). But the sequence $\left(M_{z_1}^n\right)_n$ converges to zero in the weak operator topology, so

$$(\widetilde{\rho}(M_{z_1}))^n = \widetilde{\rho}(M_{z_1}^n) \xrightarrow{n \to \infty} 0,$$

which is the desired contradiction. We conclude that $\rho = \widehat{\delta}_\lambda$.

To finish the proof, note that continuity of the second map is clear, while continuity of the first map follows from the identity

$$\delta_\lambda(T) = (T1)(\lambda)$$

for $T \in \mathcal{L}_I$ and the fact that the functions in $\mathcal{F}_I \subset H_d^2$ are continuous on $\mathbb{B}_d$. $\square$

If ρ is a multiplicative linear functional on $\mathcal{L}_I$, then Proposition 3.4, applied to the restriction of ρ to $\mathcal{A}_I$, shows that $\lambda = \rho(S_1^I, \dots, S_d^I) \in Z(I)$. The preceding proposition implies that $\lambda \in Z^0(I)$ if ρ is WOT-continuous. It turns out that this actually characterises WOT-continuity among the characters. In other words, if ρ is any multiplicative linear functional on $\mathcal{L}_I$ which coincides with δ_λ on $\mathcal{A}_I$ for some $\lambda \in Z^0(I)$, then ρ is WOT-continuous and equals $\widehat{\delta}_\lambda$. This assertion will follow as an application of the following result, which is known as the solution to Gleason's problem for $\mathrm{Mult}(H_d^2)$.

Theorem 6.5. *Every $\varphi \in \mathrm{Mult}(H^2_d)$ can be written as*

$$\varphi = \varphi(0) + \sum_{i=1}^{d} z_i \varphi_i$$

for some $\varphi_i \in \mathrm{Mult}(H^2_d)$.

Proof. See [GRS05, Corollary 4.2] for a proof using an interpolation result ([EP02, Theorem 1.6]) for reproducing kernel Hilbert spaces. The theorem is also a consequence of a corresponding result for free semigroup algebras (see [DP98a, Lemma 2.5] and the discussion below) and the fact that $\mathrm{Mult}(H^2_d)$ is isomorphic to a quotient of a free semigroup algebra [DP98b, Theorem 2.3]. $\square$

Corollary 6.6. *Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal, and let ρ be a multiplicative linear functional on $\mathcal{L}_I$. If $\lambda = \rho(S^I_1, \dots, S^I_d) \in Z^0(I)$, then ρ equals $\widehat{\delta}_\lambda$, and in particular, ρ is WOT-continuous.*

Proof. Theorem 6.5 immediately implies the assertion for $I = \{0\}$ and $\lambda = 0$. More generally, if $I = \{0\}$ and $\lambda \in \mathbb{B}_d$ is arbitrary, we can find an automorphism F of $\mathbb{B}_d$ which maps λ to 0. By Lemma 3.15,

$$\Phi : \mathrm{Mult}(H^2_d) \to \mathrm{Mult}(H^2_d), \quad M_\varphi \mapsto M_{\varphi \circ F}$$

is an algebra isomorphism, so the linear functional $\widetilde{\rho} = \rho \circ \Phi$ is multiplicative, and it satisfies

$$\widetilde{\rho}(M_{z_i}) = \rho(M_{z_i \circ F}) = (z_i \circ F)(\lambda) = 0$$

for $i = 1, \dots, d$. From the first part, we infer that $\widetilde{\rho} = \widehat{\delta}_0$, and hence

$$\rho(M_\varphi) = (\delta_0 \circ \Phi^{-1})(M_\varphi) = \delta_0(M_{\varphi \circ F^{-1}}) = \varphi(\lambda)$$

for all $\varphi \in \mathrm{Mult}(H^2_d)$, that is, $\rho = \widehat{\delta}_\lambda$.

To deduce the assertion for an arbitrary homogeneous ideal I, we consider the natural quotient homomorphism

$$\Psi : \mathrm{Mult}(H^2_d) \to \mathcal{L}_I, \quad M_\varphi \mapsto P_{\mathcal{F}_I} M_\varphi \big|_{\mathcal{F}_I}.$$

Given a multiplicative linear functional ρ on $\mathcal{L}_I$ with $\rho(S^I_1, \dots, S^I_d) = \lambda \in Z^0(I)$, we obtain a multiplicative linear functional $\rho \circ \Psi$ on $\mathrm{Mult}(H^2_d)$. This character satisfies $(\rho \circ \Psi)(M_{z_i}) = \lambda_i$ for all $i = 1, \dots, d$. From what we have shown so far, it follows that

$$\rho(\Psi(M_\varphi)) = \varphi(\lambda) = \widehat{\delta}_\lambda(\Psi(M_\varphi)),$$

for all $\varphi \in \mathrm{Mult}(H^2_d)$, where the last equality follows from Lemma 6.3 (b). We conclude that ρ and δ_λ coincide on the range of Ψ. Hence the observation that Ψ is surjective (see Theorem 6.1) finishes the proof. $\square$

The preceding corollary and Proposition 6.4 allow for a disjoint decomposition of the maximal ideal space $\Delta(\mathcal{L}_I)$ of $\mathcal{L}_I$ of the form

$$\Delta(\mathcal{L}_I) = Z^0(I) \cup \Delta_r(\mathcal{L}_I),$$

where we have identified $Z^0(I)$ with $\Delta_0(\mathcal{L}_I)$. The characters in $Z^0(I)$ are precisely the WOT-continuous ones, and a character ρ belongs to $\Delta_r(\mathcal{L}_I)$ if and only if the point $(\rho(S_1^I), \dots \rho(S_d^I))$ lies in $Z(I) \setminus Z^0(I)$. Note, however, that although we understand the part $Z^0(I)$ of $\Delta(\mathcal{L}_I)$ quite well, the example of $H^\infty(\mathbb{D})$ shows that the "residual part" $\Delta_r(\mathcal{L}_I)$ can be rather complicated. In particular, the map from $\Delta_r(\mathcal{L}_I)$ into $Z(I) \setminus Z^0(I)$, given by evaluating at S^I, need not be injective.

Similarly to the norm-closed case, we will use the description of $\Delta(\mathcal{L}_I)$ to deduce necessary conditions for the existence of isomorphisms between algebras of the type $\mathcal{L}_I$. It is the presence of the part $\Delta_r(\mathcal{L}_I)$ of the maximal ideal space which makes things more complicated for the WOT-closed algebras. Our scheme is to restrict a given algebra homomorphism from $\mathcal{L}_I$ into $\mathcal{L}_J$ to the subalgebra $\mathcal{A}_I$, so that we get a map from $Z^0(J) \subset \Delta(\mathcal{L}_J)$ into $Z(I) = \Delta(\mathcal{A}_I)$. We begin with an analogue of Proposition 3.7.

Proposition 6.7. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals, and let $\Phi : \mathcal{L}_I \to \mathcal{L}_J$ be a unital algebra homomorphism. Let $\Phi_0 : \mathcal{A}_I \to \mathcal{L}_J$ denote the restriction of Φ to the subalgebra $\mathcal{A}_I$, and write $\Phi_0^* : \Delta(\mathcal{L}_J) \to \Delta(\mathcal{A}_I)$ for the induced map between the maximal ideal spaces. Then there is a tuple $F \in \mathrm{Mult}(H^2_d)^d$ such that, with the identifications of Proposition 3.4 and Proposition 6.4, we have*

$$\Phi_0^*\big|_{Z^0(J)} = F\big|_{Z^0(J)}.$$

Proof. Theorem 6.1 allows us to choose $\varphi_i \in \mathrm{Mult}(H^2_d)$ such that

$$P_{\mathcal{F}_J} M_{\varphi_i}\big|_{\mathcal{F}_J} = \Phi(S_i^I)$$

for $i = 1, \dots, d$. Put $F = (\varphi_1, \dots, \varphi_d)$. Then for all $\lambda \in Z^0(J)$ and $i = 1, \dots, d$, we have

$$\Phi_0^*(\widehat{\delta}_\lambda)(S_i^I) = \widehat{\delta}_\lambda(\Phi(S_i^I)) = \varphi_i(\lambda),$$

so $\Phi_0^*(\widehat{\delta}_\lambda) = \widehat{\delta}_{F(\lambda)}$ by Proposition 3.4. □

We will show that if the algebra homomorphism Φ in the preceding proposition is an isomorphism, then Φ_0^* maps $Z^0(J)$ into $Z^0(I)$. The following analogue of Lemma 3.6 allows us to include the non-radical case as well.

Lemma 6.8. *Let $I \subset \mathbb{C}[z]$ be a homogeneous ideal. Then there is a natural number N such that*

$$\bigcap_{\lambda \in Z^0(I)} \ker(\widehat{\delta}_\lambda) = \{T \in \mathcal{L}_I : T^N = 0\}.$$

Proof. It is trivial that nilpotent elements belong to the kernel of every multiplicative linear functional. To show the converse, note that the Noetherian property of $\mathbb{C}[z]$ allows us to choose a natural number N with $J^N \subset I$, where $J = \sqrt{I}$. Suppose now that $T \in \mathcal{L}_I$ lies in the kernel of each $\widehat{\delta}_\lambda$. By Theorem 6.1, there is a multiplier $\varphi \in \operatorname{Mult}(H^2_d)$ with $T = P_{\mathcal{F}_I} M_\varphi \big|_{\mathcal{F}_I}$. Using Lemma 6.3 (c), we see that

$$\widehat{\delta}_\lambda(P_{F_J} M_\varphi \big|_{\mathcal{F}_J}) = \widehat{\delta}_\lambda(T) = 0$$

for each $\lambda \in Z^0(I)$, so that $P_{\mathcal{F}_J} M_\varphi \big|_{\mathcal{F}_J} = 0$ by Lemma 6.3 (d). According to Lemma 2.16 (b), this is only possible if φ is contained in the SOT-closure of J. Since multiplication is separately continuous in the strong operator topology, and since $J^N \subset I$, an obvious inductive application of Lemma 1.17 shows that $\varphi^N \in \overline{I}^{SOT}$. Invoking Lemma 2.16 (b) once again, we infer that $T^N = P_{\mathcal{F}_I} M_{\varphi^N} \big|_{\mathcal{F}_I} = 0$, as asserted. $\square$

We can now show that an isomorphism between $\mathcal{L}_I$ and $\mathcal{L}_J$ induces a biholomorphic map between $Z^0(J)$ and $Z^0(I)$.

Lemma 6.9. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals and let $\Phi : \mathcal{L}_I \to \mathcal{L}_J$ be a unital algebra isomorphism. Then Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$.*

Proof. Let Φ_0 be the restriction of Φ to $\mathcal{A}_I$. Then Φ_0^* is a holomorphic map from $Z^0(J)$ into $Z(I)$ by Proposition 6.7. By Corollary 6.6, it suffices to show that Φ_0^* maps $Z^0(J)$ into $Z^0(I)$. So assume to the contrary that there is a point $q_0 \in Z^0(J)$ such that $\Phi_0^*(q_0) = p \in Z(I) \setminus Z^0(I)$. Then Φ_0^* is constant on $Z^0(J)$ by Lemma 3.17, that is, $\Phi_0^*(\widehat{\delta}_q)\big|_{\mathcal{A}_I} = \delta_p$ for all $q \in Z^0(J)$. In particular, we find that

$$\delta_q(\Phi(S_i^I - p_i \cdot 1)) = \Phi_0^*(\delta_q)(S_i^I - p_i \cdot 1) = 0$$

for $i = 1, \dots, d$ and all $q \in Z^0(J)$. Consequently, Lemma 6.8 yields a natural number N such that

$$(\Phi(S_i^I - p_i \cdot 1))^N = 0 \quad \text{for } i = 1, \dots, d.$$

Since Φ is injective, this means that $(S_i^I - p_1 \cdot 1)^N = 0$ for all i, which, in turn, implies that

$$(z_i - p_i)^N \in I \quad \text{for } i = 1, \dots, d$$

by Lemma 2.16 (c). However, because $p \in \partial\mathbb{B}_d$, this means that that $Z^0(I) = \emptyset$, which is not possible. $\square$

Remark 6.10. The preceding lemma is essentially Lemma 11.5 in [DRS11]. The proof given here differs from the one in [DRS11] in that we did not use the notion of Gleason parts. The crucial point in the approach presented here is holomorphicity of the map Φ_0^*, and the maximum modulus principle for homogeneous varieties.

Recall from Corollary 6.2 that in the radical case, the algebra $\mathcal{L}_I$ can be identified with the multiplier algebra of $H^2_d\big|_{Z^0(I)}$. Just as in the norm-closed case (see Lemma 3.9), isomorphisms act as composition operators under this identification.

Lemma 6.11. *Let $I, J \subsetneq \mathbb{C}[z]$ be radical homogeneous ideals and let $\Phi : \mathcal{L}_I \to \mathcal{L}_J$ be a unital algebra isomorphism. Regarding $\mathcal{L}_I$ and $\mathcal{L}_J$ as algebras of functions on $Z^0(I)$ and $Z^0(J)$, respectively, Φ is given by composition with Φ^*, that is,*

$$\Phi(\varphi) = \varphi \circ (\Phi^*\big|_{Z^0(J)})$$

for $\varphi \in \mathcal{L}_I$.

Proof. Write F for Φ^*, regarded as a map from $Z^0(J)$ to $Z^0(I)$ (see Lemma 6.9), and let $\varphi \in \mathcal{L}_I$, viewed as an algebra of functions on $Z^0(I)$. Then for all $\lambda \in Z^0(J)$, we have

$$\Phi(\varphi)(\lambda) = \widehat{\delta}_\lambda(\Phi(\varphi)) = \Phi^*(\widehat{\delta}_\lambda)(\varphi) = \widehat{\delta}_{F(\lambda)}(\varphi) = (\varphi \circ F)(\lambda). \qquad \square$$

6.3. Isomorphisms between the algebras $\mathcal{L}_I$

As in the norm-closed case, an algebra homomorphism $\Phi : \mathcal{A}_I \to \mathcal{A}_J$ is said to be *vacuum-preserving* if $\Phi^*(\delta_0) = \delta_0$. The following result is an analogue of Proposition 3.27.

Proposition 6.12. *Let $I, J \subsetneq \mathbb{C}[z]$ be homogeneous ideals, and suppose that $\mathcal{L}_I$ and $\mathcal{L}_J$ are topologically isomorphic. Then there exists a vacuum-preserving topological isomorphism $\Phi : \mathcal{L}_I \to \mathcal{L}_J$. If $\mathcal{L}_I$ and $\mathcal{L}_J$ are isometrically isomorphic, then Φ can be chosen to be isometric.*

Proof. The construction is the same as in the proof of Proposition 3.27, we only need to check that everything remains valid when passing to the WOT-closure. Here are the details:

For $t \in \mathbb{R}$, the isometric automorphism Φ^I_t of $\mathcal{A}_I$ from Remark 3.26, induced by the unitary map $z \mapsto e^{it}z$ on $\mathbb{C}^d$, is given by conjugation with a unitary on $\mathcal{F}_I$. Thus, it extends to an isometric automorphism of $\mathcal{L}_I$. It is easy to check that the induced map on $Z^0(I) \subset \Delta(\mathcal{L}_I)$ is given by multiplication with e^{it} (compare Example 3.8). Moreover, Lemma 6.9 shows that Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$. Thus the assertion follows, just as in Proposition 3.27, as an application of Lemma 3.22. $\square$

We can now prove the theorem which was alluded to at the beginning of this chapter.

Theorem 6.13. *Let $I, J \subset \mathbb{C}[z]$ be radical homogeneous ideals. Then $\mathcal{L}_I$ is isometrically (respectively topologically) isomorphic to $\mathcal{L}_J$ if and only if $\mathcal{A}_I$ is isometrically (respectively topologically) isomorphic to $\mathcal{A}_J$.*

Proof. If $\mathcal{A}_I$ is isometrically isomorphic to $\mathcal{A}_J$, then Theorem 3.36 shows that there is an isometric isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$ which is given by conjugation with a unitary map from $\mathcal{F}_J$ onto $\mathcal{F}_I$, and hence extends to an isometric isomorphism between the WOT-closures. Similarly, assuming that $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic, we can find a similarity between $\mathcal{A}_I$ and $\mathcal{A}_J$ by Theorem 4.48, which again extends to a topological isomorphism between $\mathcal{L}_I$ and $\mathcal{L}_J$. This establishes one direction.

To prove the other one, we will show that if $\mathcal{L}_I$ and $\mathcal{L}_J$ are topologically (respectively isometrically) isomorphic, then there is a topological (respectively isometric) isomorphism between $\mathcal{L}_I$ and $\mathcal{L}_J$ which restricts to an isomorphism between $\mathcal{A}_I$ and $\mathcal{A}_J$. So assume that $\mathcal{L}_I$ and $\mathcal{L}_J$ are topologically isomorphic. Proposition 6.12 yields a topological isomorphism

$$\Phi : \mathcal{L}_I \to \mathcal{L}_J \quad \text{with} \quad \Phi^*(0) = 0.$$

As an application of Lemma 6.9, we see that Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$, and since it fixes the origin, we infer from Corollary 3.20 that there exists an invertible linear map A on $\mathbb{C}^d$ which coincides with Φ^* on $Z^0(J)$. Moreover, Lemma 6.11 asserts that if we regard $\mathcal{L}_I$ and $\mathcal{L}_J$ as algebras of functions on $Z^0(I)$ and $Z^0(J)$, respectively, then Φ is given by

$$\Phi(\varphi) = \varphi \circ A \qquad \text{for all } \varphi \in \mathcal{L}_I.$$

Since $\mathcal{A}_I$ and $\mathcal{A}_J$ correspond to the norm-closures of the polynomials in these algebras (see Corollary 2.19), it is clear that the topological isomorphism Φ maps $\mathcal{A}_I$ onto $\mathcal{A}_J$. This proves the assertion concerning topological isomorphisms. The isometric case follows from the fact that the vacuum-preserving isomorphism Φ from above can be chosen to be isometric if $\mathcal{L}_I$ and $\mathcal{L}_J$ are isometrically isomorphic (see Proposition 6.12). $\square$

A. Reproducing kernel Hilbert spaces

A.1. Hilbert function spaces and kernels

The purpose of this section is to recall some results from the theory of reproducing kernel Hilbert spaces. General references for this topic include the classical paper by Aronszajn [Aro50], the books [AM02] and [BM84] and the thesis [Bar07].

Definition A.1. Let X be a set. A Hilbert space H of complex-valued functions on X is called a *reproducing kernel Hilbert space* or *Hilbert function space* if for each $\lambda \in X$, the point evaluation functional

$$\delta_\lambda : H \to \mathbb{C}, \quad f \mapsto f(\lambda)$$

is continuous.

Suppose that H is a reproducing kernel Hilbert space on X. By the Riesz representation theorem, there is for any $\lambda \in X$ a function $k_\lambda \in H$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \text{for all } f \in H.$$

The function

$$K : X \times X \to \mathbb{C}, \quad (\mu, \lambda) \mapsto k_\lambda(\mu) = \langle k_\lambda, k_\mu \rangle$$

is called the reproducing kernel of H. It is easy to check that K indeed is a kernel in the following sense.

Definition A.2. Let X be a set. A function $K : X \times X \to \mathbb{C}$ is called *positive definite* or a *kernel* if for any finite sequence $(\lambda_i)_{i=1}^n$ of points in X, the matrix

$$\Big(K(\lambda_i, \lambda_j) \Big)_{i,j=1}^n$$

is positive semidefinite.

A theorem of E.H. Moore asserts that any kernel on a set is the reproducing kernel of a Hilbert function space.

Theorem A.3. *Let X be a set, and let $K : X \times X \to \mathbb{C}$ be positive definite. Then there exists a unique Hilbert function space H on X whose reproducing kernel is K.*

Proof. See, for example, [AM02, Theorem 2.23]. $\square$

We record a few elementary properties of the kernel of a Hilbert function space.

Lemma A.4. *Let H be a reproducing kernel Hilbert space on a set X with kernel K. Then the following assertions are true:*

(a) $K(\mu, \lambda) = \overline{K(\lambda, \mu)}$ *for all* $\lambda, \mu \in X$.

(b) $||K(\cdot, \lambda)||^2 = K(\lambda, \lambda)$ *for all* $\lambda \in X$.

(c) *The linear span of the kernel functions* $K(\cdot, \lambda)$ *for* $\lambda \in X$ *is dense in* H.

(d) *Let* $\lambda \in X$. *Then* $f(\lambda) = 0$ *for all* $f \in H$ *if and only if* $K(\lambda, \lambda) = 0$.

Proof. Part (a) follows from the fact that K is positive definite, and (b) is immediate. To prove (c), note that if f is orthogonal to all kernel functions $K(\cdot, \lambda)$, then

$$f(\lambda) = \langle f, K(\cdot, \lambda) \rangle = 0 \quad \text{for all } \lambda \in X.$$

The non-trivial implication of (d) finally follows from (b) and the defining property of $K(\cdot, \lambda)$. $\square$

Theorem A.3 shows that the structure of a Hilbert function space is completely determined by its reproducing kernel. We will need the following result along these lines.

Lemma A.5. *Let X and Y be sets and let H_X and H_Y be reproducing kernel Hilbert spaces on X and Y with kernels K_X and K_Y, respectively. For an invertible map $\varphi : Y \to X$ and a function $\psi : Y \to \mathbb{C}$ with $0 \notin \operatorname{ran} \psi$, the following are equivalent:*

(i) *The map* $V : H_X \to H_Y, f \mapsto \psi \cdot (f \circ \varphi)$ *is a well-defined unitary.*

(ii) $K_Y(\lambda, \mu) = K_X(\varphi(\lambda), \varphi(\mu))\, \psi(\lambda)\overline{\psi(\mu)}$ *for all* $\lambda, \mu \in Y$.

In particular, if H is a reproducing kernel Hilbert space on X with kernel K, and if $\varphi : X \to X$ is an invertible map, then $f \mapsto f \circ \varphi$ defines a unitary on H if and only if K is invariant under composition with φ.

Proof. The proof of (i) implies (ii) is a simple calculation. Indeed, suppose that $\lambda, \mu \in Y$. The assumptions imply that the adjoint of V is given by

$$V^*(f) = \left(\frac{f}{\psi}\right) \circ \varphi^{-1} \quad \text{for all } f \in H_Y.$$

Consequently,

$$\begin{aligned}
K_X(\varphi(\lambda), \varphi(\mu))\, \psi(\lambda)\overline{\psi(\mu)} &= \big\langle V K_X(\cdot, \varphi(\mu)), K_Y(\cdot, \lambda)\big\rangle\, \overline{\psi(\mu)} \\
&= \overline{\big\langle V^* K_Y(\cdot, \lambda), K_X(\cdot, \varphi(\mu))\big\rangle\, \psi(\mu)} \\
&= \overline{K_Y(\mu, \lambda)} = K_Y(\lambda, \mu).
\end{aligned}$$

To establish the converse, we first examine the action of $f \mapsto \psi \cdot (f \circ \varphi)$ on kernel functions $K(\cdot, \xi)$ for $\xi \in X$. By assumption, we have for $\lambda \in Y$ the identity

$$K_Y(\lambda, \varphi^{-1}(\xi)) = K_X(\varphi(\lambda), \xi)\, \psi(\lambda)\overline{\psi(\varphi^{-1}(\xi))},$$

hence

$$\psi \cdot (K_X(\cdot, \xi) \circ \varphi) = \frac{1}{\overline{\psi(\varphi^{-1}(\xi))}} K_Y(\cdot, \varphi^{-1}(\xi)) \in H_Y.$$

Let V_0 denote the map from the linear span of the kernel functions $K_X(\cdot, \xi)$ for $\xi \in X$ into H_Y, given by $f \mapsto \psi \cdot (f \circ \varphi)$. From

$$\begin{aligned}
\langle V_0 K_X(\cdot, \xi), V_0 K_X(\cdot, \nu)\rangle &= \frac{1}{\psi(\varphi^{-1}(\nu))\overline{\psi(\varphi^{-1}(\xi))}} \langle K_Y(\cdot, \varphi^{-1}(\xi)), K_Y(\cdot, \varphi^{-1}(\nu))\rangle \\
&= \frac{1}{\psi(\varphi^{-1}(\nu))\overline{\psi(\varphi^{-1}(\xi))}} K_Y(\varphi^{-1}(\nu), \varphi^{-1}(\xi)) \\
&= K_X(\nu, \xi) = \langle K_X(\cdot, \xi), K_X(\cdot, \nu)\rangle
\end{aligned}$$

for $\xi, \nu \in X$, we deduce that V_0 is an isometry. Since all kernel functions $K_Y(\cdot, \lambda)$ for $\lambda \in Y$ are contained in the range of V_0, and since the kernel functions form a total set in every reproducing kernel Hilbert space by Lemma A.4 (c), V_0 extends to a unitary $V : H_X \to H_Y$. It is easy to check that V is again given by $f \mapsto \psi \cdot (f \circ \varphi)$, which finishes the proof of the first assertion. The second one is now obvious. □

Remark A.6. Using more sophisticated tools from the theory of reproducing kernel Hilbert spaces, a less technical proof for the implication from (ii) to (i) in the preceding lemma can be given. We only sketch the argument.

Define

$$K_\varphi : Y \times Y \to \mathbb{C}, \quad (\lambda, \mu) \mapsto K_X(\varphi(\lambda), \varphi(\mu)).$$

Then K_φ is positive-definite, so there is a unique Hilbert function space H_φ on Y with kernel K_φ by Theorem A.3. It is known that $H_\varphi = \{f \circ \varphi : f \in H_X\}$ and that the map

$$H_X \to H_\varphi, \quad f \mapsto f \circ \varphi,$$

is a unitary operator (here, we need that φ is invertible). The assumption (ii) in the preceding lemma now becomes

$$K_Y(\lambda, \mu) - \psi(\lambda)\overline{\psi(\mu)} K_\varphi(\lambda, \mu) = 0.$$

This observation, together with the fact that $0 \notin \operatorname{ran} \psi$, is known to imply that

$$H_\varphi \to H_Y, \quad f \mapsto \psi \cdot f$$

is a unitary operator as well, which finishes the outline of the proof.

If H is a reproducing kernel Hilbert space on a set X with kernel K, then for any subset $Y \subset X$, the restriction of K to Y is again positive definite. The corresponding reproducing kernel Hilbert space on Y has a very natural description:

Lemma A.7. *Let H be a reproducing kernel Hilbert space on a set X with kernel K, let $Y \subset X$ be a subset, and let K_Y denote the restriction of K to $Y \times Y$. If H_Y is the reproducing kernel Hilbert space on Y with kernel K_Y, then*

$$H_Y = \{f\big|_Y : f \in H\},$$

and for $g \in H_Y$, we have

$$||g||_{H_Y} = \inf\{||f||_H : f \in H \text{ with } f\big|_Y = g\}.$$

Proof. See [Aro50, Part I, Section 5]. □

A.2. Multipliers

Definition A.8. Let H be a Hilbert function space on a set X. A function $\varphi : X \to \mathbb{C}$ is called a multiplier if $\varphi \cdot H \subset H$. The algebra of all multipliers is called the multiplier algebra and denoted by $\mathrm{Mult}(H)$.

It is elementary to see that the multiplier algebra is indeed an algebra. Moreover, a standard application of the closed graph theorem shows that every $\varphi \in \mathrm{Mult}(H)$ induces a bounded linear operator

$$M_\varphi : H \to H, \quad f \mapsto \varphi \cdot f.$$

The multiplier norm of a multiplier φ is defined as $||\varphi||_M = ||M_\varphi||$.

We will only consider the case where H has no common zeros. By Lemma A.4 (d), this is equivalent to demanding that the kernel of K be non-zero on the diagonal. In this case, the assignment $\varphi \mapsto M_\varphi$ is one-to-one, so that we can regard $\mathrm{Mult}(H)$ as a subalgebra of $\mathcal{L}(H)$.

Lemma A.9. *Let H be a reproducing kernel Hilbert space on a set X with kernel K without common zeros, and let φ be a multiplier on H.*

(a) $M_\varphi^* K(\cdot, \lambda) = \overline{\varphi(\lambda)} K(\cdot, \lambda)$ *for $\lambda \in X$.*

(b) $||\varphi||_M \geq ||\varphi||_\infty$.

(c) *The linear functional* $\text{Mult}(H) \to \mathbb{C}, \varphi \mapsto \varphi(\lambda)$*, is WOT-continuous.*

Proof. (a) For all $f \in H$, we have

$$\langle f, M_\varphi^* K(\cdot,\lambda)\rangle = \langle \varphi \cdot f, K(\cdot,\lambda)\rangle = \varphi(\lambda) f(\lambda) = \langle f, \overline{\varphi(\lambda)} K(\cdot,\lambda)\rangle.$$

(b) If H has no common zeros, then $K(\cdot,\lambda) \neq 0$ for all $\lambda \in X$, so part (a) shows that

$$||\varphi||_M = ||M_\varphi^*|| \geq |\varphi(\lambda)|$$

for all $\lambda \in X$.

(c) This follows from the identity

$$\varphi(\lambda) = \frac{\langle M_\varphi K(\cdot,\lambda), K(\cdot,\lambda)\rangle}{K(\lambda,\lambda)}$$

for $\varphi \in \text{Mult}(H)$. □

We also require the following standard fact about the multiplier algebra.

Lemma A.10. *If H is a reproducing kernel Hilbert space on X without common zeros, then* $\text{Mult}(H)$ *is a WOT-closed subalgebra of* $\mathcal{L}(H)$.

Proof. It is clear that $\text{Mult}(H)$ is a subalgebra of $\mathcal{L}(H)$. To show that it is WOT-closed, suppose that $(M_{\varphi_\alpha})_\alpha$ is a net of multiplication operators on H which converges to T in the weak operator topology. An application of Lemma A.9 (c) shows that the net $(\varphi_\alpha)_\alpha$ converges pointwise on X to a function φ. Consequently, we have for all $f \in H$ and $\lambda \in X$ the identity

$$(Tf)(\lambda) = \lim_\alpha((M_{\varphi_\alpha} f)(\lambda)) = \varphi(\lambda) f(\lambda),$$

so that $\varphi \in \text{Mult}(H)$ and $T = M_\varphi$. □

A.3. Vector-valued Hilbert function spaces

It is possible to generalize the theory of Hilbert function spaces of scalar-valued functions to a vector-valued setting. The reader is referred to the Sections 2.5 and 2.8 in [AM02] for an introduction, and to the first chapter of the thesis [Bar07] for a comprehensive treatment of this subject. We will only need the most basic case, which is the tensor product of a scalar-valued Hilbert function space with another Hilbert space.

Suppose that H is a (scalar-valued) Hilbert function space on a set X, and let $\mathcal{E}$ be Hilbert space of dimension n, where n is finite. Identifying an elementary tensor $f \otimes v$ with the function

$$X \to \mathcal{E}, \quad \lambda \mapsto f(\lambda)v,$$

the space $H \otimes \mathcal{E}$ can be regarded as a space of $\mathcal{E}$-valued functions on X. Alternatively, $H \otimes \mathcal{E}$ can be viewed as a direct sum of n copies of H. As in the scalar-valued case, there is a notion of multipliers.

Definition A.11. Let H be a Hilbert function space on a set X and let $\mathcal{E}$ be a finite-dimensional Hilbert space. A multiplier on $H \otimes \mathcal{E}$ is a map $\Phi : X \to \mathcal{L}(\mathcal{E})$ such that for each $F \in H$, the map

$$X \to \mathcal{E}, \quad \lambda \mapsto \Phi(\lambda)F(\lambda)$$

is contained in H.

If $H \otimes \mathcal{E}$ is identified with H^n, operators on $H \otimes \mathcal{E}$ can be identified with $n \times n$-matrices of operators on H. Thus, multipliers on $H \otimes \mathcal{E}$ can be thought of as $n \times n$-matrices of functions on H. Just as in the scalar-valued case, every multiplier Φ induces a bounded multiplication operator M_Φ on $H \otimes \mathcal{E}$, and the multiplier norm of Φ is defined by $||\Phi||_M = ||M_\Phi||$.

We require a result that characterises multipliers in terms of the reproducing kernel of the Hilbert function spaces. A map $K : X \times X \to \mathcal{L}(E)$ is called *positive definite* if for any finite sequence $(\lambda_i)_{i=1}^n$ of points in X, the operator matrix

$$\Big(K(\lambda_i, \lambda_j)\Big)_{i,j=1}^n$$

is positive.

Lemma A.12. *Let H be a reproducing kernel Hilbert space on X with reproducing kernel K, and let $\mathcal{E}$ be a finite-dimensional Hilbert space. For a map $\Phi : X \to \mathcal{L}(\mathcal{E})$, the following assertions are equivalent:*

(i) $\Phi \in \mathrm{Mult}(H \otimes \mathcal{E})$ *with* $||\Phi||_M \leq 1$.

(ii) *The map*

$$L : X \times X \to \mathcal{L}(\mathcal{E}), \quad (\lambda, \mu) \mapsto \big(\mathrm{id}_\mathcal{E} - \Phi(\lambda)\Phi(\mu)^*\big)K(\lambda, \mu)$$

is positive definite.

Proof. The scalar-valued case of this lemma is, for example, [AM02, Corollary 2.37]. The vector-valued case can be found, for example, in [Esc11, Satz 2.2]. Because this reference is not widely available, we briefly sketch the argument.

Let Φ be a multiplier on $H \otimes \mathcal{E}$. The vector-valued analogue of Lemma A.9 (a) asserts that $M_\Phi^*(K(\cdot, \lambda)x) = K(\cdot, \lambda)\Phi(\lambda)^*x$ for all $\lambda \in X$ and $x \in \mathcal{E}$. Moreover, the defining property of K is easily seen to imply that

$$\langle f, K(\cdot, \lambda)x\rangle_{H \otimes \mathcal{E}} = \langle f(\lambda), x\rangle_\mathcal{E}$$

for $f \in H \otimes \mathcal{E}$, $\lambda \in X$ and $x \in \mathcal{E}$. Using these facts, it is straightforward to deduce for $\lambda_1, \ldots, \lambda_n \in X$ and $x_1, \ldots, x_n \in \mathcal{E}$ the identity

$$\begin{aligned}\sum_{i,j=1}^{n} \langle L(\lambda_i, \lambda_j) x_j, x_i \rangle &= || \sum_{j=1}^{n} K(\cdot, \lambda_j) x_j ||^2 - || \sum_{j=1}^{n} K(\cdot, \lambda_j) \Phi(\lambda_j)^* x_j ||^2 \\ &= || \sum_{j=1}^{n} K(\cdot, \lambda_j) x_j ||^2 - || M_\Phi^* \sum_{j=1}^{n} K(\cdot, \lambda_j) x_j ||^2.\end{aligned}$$

The implication (i) $\Rightarrow$ (ii) readily follows from this observation. Conversely, Lemma A.4 (c) implies that the set of all elements $K(\cdot, \lambda)x$, where $\lambda \in X$ and $x \in \mathcal{E}$, is total in $H \otimes \mathcal{E}$. Thus, if L is positive definite, then the last identity shows the existence of a contraction T on $H \otimes \mathcal{E}$ with $T(K(\cdot, \lambda)x) = K(\cdot, \lambda)\Phi(\lambda)^* x$ for $\lambda \in X$ and $x \in \mathcal{E}$. It is easy to check that

$$\langle (T^* f)(\lambda), x \rangle = \langle \Phi(\lambda) f(\lambda), x \rangle$$

holds for all $f \in H \otimes \mathcal{E}$, $\lambda \in X$ and $x \in \mathcal{E}$. Hence Φ is a contractive multiplier. □

The classical Schur product theorem [Sch11] asserts that the entrywise product of positive matrices is again positive. We need the following generalization.

Theorem A.13. *Let $K_1 : X \times X \to \mathbb{C}$ and $K_2 : X \times X \to \mathcal{L}(\mathcal{E})$ be positive definite maps on a set X. Then the pointwise product*

$$K_1 K_2 : X \times X \to \mathbb{C}, \quad (\lambda, \mu) \mapsto K_1(\lambda, \mu) K_2(\lambda, \mu)$$

is again positive definite.

Proof. See, for example, [Bar07, Propostion 1.1.9], or [Ful11, Theorem 2.3] for an even more general version. □

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