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Toeplitz Extensions and Berezin Transforms

Master's Thesis

submitted by
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Contents

Introduction **v**

1 Basics **1**
1.1 Functional Hilbert spaces and the Berezin transform 1
1.2 Operators on Hilbert spaces 2

2 Compact operators in the Toeplitz algebra **7**
2.1 The generalized Bergman spaces 7
2.2 Characterization of compact Toeplitz operators 11
2.3 Compact operators in the Toeplitz algebra 13
2.4 A class of Bergman spaces on pseudoregular domains 20
2.5 Generalized Bergman spaces on convex sets 23

3 Exact Toeplitz Sequences **37**
3.1 Joint spectra 37
3.2 The Toeplitz extension 42
3.3 Essential spectrum and Shilov boundary 45

4 Appendix **51**
4.1 The $\overline{\partial}$-Neumann problem and pseudoregular sets 51

Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded open domain and $1 \leq p < \infty$. We define the Bergman space on Ω by

$$L_a^p(\Omega, \lambda) = \left\{ f \in \mathcal{O}(\Omega); \|f\|_p = \left(\int_\Omega |f|^p d\lambda \right)^{\frac{1}{p}} < \infty \right\},$$

where λ denotes the restriction of the Lebesgue measure on $\mathbb{C}^n$ to Ω. One can show that $(L_a^p(\Omega, \lambda), \|\cdot\|_p)$ is a Banach space and that for $p = 2$, the space $L_a^2(\Omega, \lambda)$ together with the $L^2(\Omega, \lambda)$-inner product is a functional Hilbert space.

In the theory of bounded operators on Bergman spaces, a topic one is particularly interested in is the study of Toeplitz operators $T_\varphi \in L(L_a^2(\Omega, \lambda))$, that is, compressions to $L_a^2(\Omega, \lambda)$ of multiplication operators M_φ on $L^2(\Omega, \lambda)$ with symbol $\phi \in L^\infty(\Omega, \lambda)$. It has long been known that on the classical Bergman space $L_a^2(\mathbb{D}, \lambda)$ on the unit disk, a Toeplitz operator T_φ with continuous symbol $\varphi \in C(\overline{\mathbb{D}})$ is compact if and only if its symbol vanishes on the unit circle. Furthermore, in every point in the unit circle, the value of the symbol coincides with the limit of the Berezin transform $\Gamma(T_\varphi)$ at this point. On a functional Hilbert space $H \subset \mathbb{C}^\Omega$ with normalized kernel function k_z, the Berezin transform of an operator $T \in L(H)$ is given by

$$\Gamma(T) : \Omega \to \mathbb{C}, \ \Gamma(T)(z) = \langle T k_z, k_z \rangle.$$

This brought up the question whether one can also use the Berezin transform to characterise compactness of operators in more general settings.
One of the most famous results in this direction was obtained by Axler and Zheng in [AZ98] for the Bergman space $L_a^2(\mathbb{D}, \lambda)$ on the unit disk. They showed that an operator which is the finite sum of finite products of Toeplitz operators with L^∞-symbols is compact if and only if the boundary limit of its Berezin transform is zero. Engliš provided the same result in [Eng99] for the weighted Bergman spaces $L_{a,\nu}^2(\Omega, \lambda)$ on an irreducible bounded symmetric domain $\Omega \subset \mathbb{C}^n$.

Although for $p \neq 2$, the Bergman space $L^p(\Omega, \lambda)$ is not a functional Hilbert space, it is also possible to define a Berezin transform for bounded operators on these spaces. In [Sua07], Suarez extended the Axler-Zheng theorem to operators in the closed subalgebra generated by the Toeplitz operators with L^∞-symbol on the Bergman spaces $L_a^p(\mathbb{B}_n, \lambda)$ $(1 < p < \infty)$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$.

Together with Mitkovski and Wick, in [MSW13], he also proved a version for the weighted Bergman spaces $L^p_{a,\nu}(\mathbb{B}_n, \lambda)$ on the unit ball, where $1 < p < \infty$. Moreover, Mitkovski and Wick showed in [MW14] that the same theorem also holds on the unweighted Bergman spaces $L^p_a(\mathbb{D}^n, \lambda)$ for $1 < p < \infty$ on the unit polydisk.

In case that one considers only operators in the Toeplitz algebra, that is, the closed subalgebra of $L(L^p_a(\Omega, \lambda))$ generated by the Toeplitz operators T_φ with continuous symbol $\varphi \in C(\overline{\Omega})$, even more can be said.
Čučković and Şahutoğlu proved in [ČŞ13] that for operators in the Toeplitz algebra, on the unweighted Bergman space $L^2_a(\Omega, \lambda)$ on a smooth bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ on which the $\overline{\partial}$-Neumann operator is compact, the statement of the Axler-Zheng theorem still holds.
Another interesting result on the compactness of Toeplitz operators with continuous symbols was given by Trieu Le in [Le09]. One can prove that the set of holomorphic polynomials on $\mathbb{B}_n$ is a dense subset of the unweighted Bergman space on the unit ball. Le then defined Toeplitz operators on the $L^2(\Omega, \nu)$-closure of the holomorphic polynomials with respect to a wider class of measures ν and showed that, also in this case, a Toeplitz operator is compact if its symbol vanishes on the unit sphere.

We will define Toeplitz operators on an even larger class of functional Hilbert spaces $\mathcal{H}^2_A(\mu)$ that can be seen as a generalization of the Bergman spaces $L^2_a(\mathbb{B}_n, \lambda)$ with respect to both the set Ω and the underlying measure. In the first part of this thesis, we give a brief overview on the theory of functional Hilbert spaces and the Berezin transform. We also prove some results for operators on Hilbert spaces that will be useful later on. At the beginning of Chapter 2, we show that a version of Le's theorem also holds in our framework. We then use this result to prove our main theorem which states that if the multiplication tuple T_z on $\mathcal{H}^2_A(\mu)$ is essentially normal, then under certain technical conditions, an operator T in the Toeplitz algebra is compact if and only if

$$\lim_{z \to \partial\Omega} \Gamma(T)(z) = 0.$$

In addition, we obtain corollaries for the case when Ω is convex and for pseudoregular open domains $\Omega \subset \mathbb{C}^n$, that is, bounded pseudoconvex domains on which the $\overline{\partial}$-Neumann operator is compact. In particular, we show that our main theorem includes both Theorem 1.1 in [Le09] and Theorem 1 in [ČŞ13] as special cases. For an overview on the $\overline{\partial}$-Neumann problem, we refer the reader to the appendix, where we provide a survey on the most important results in this topic.
In the last chapter, we use Gelfand theory to define the Toeplitz extension of the ideal of compact operators on $\mathcal{H}^2_A(\mu)$. We use the essential spectrum of the multiplication tuple T_z to replace some of the prerequisites in the main theorem by weaker conditions.

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1 Basics

1.1 Functional Hilbert spaces and the Berezin transform

In this thesis, we will characterize the compact operators in the Toeplitz algebra with help of their Berezin transform. The setting we will work in is a certain class of functional Hilbert spaces that can be considered as generalized Bergman spaces. Before we start introducing these spaces, we would like to remind the reader of some basic definitions in the theory of functional Hilbert spaces.

Definition 1.1. *Let Ω be an arbitrary set and let $\mathbb{C}^\Omega$ be the set of all maps from Ω to $\mathbb{C}$. We call a Hilbert space $H \subset \mathbb{C}^\Omega$ a functional Hilbert space if, for every $z \in \Omega$, the point evaluation*

$$\delta_z : H \longrightarrow \mathbb{C}, \ f \mapsto f(z)$$

is continuous.

With each functional Hilbert space on a set Ω, we associate its reproducing kernel defined as follows.

Definition 1.2. *Let Ω be an arbitrary set and let $H \subset \mathbb{C}^\Omega$ be a functional Hilbert space. We call the map*

$$K : \Omega \times \Omega \longrightarrow \mathbb{C}$$

a reproducing kernel for H if it satisfies

1. $K(\cdot, z) \in H$ *for all* $w \in \Omega$,
2. $\langle f, K(\cdot, z)\rangle_H = f(z)$ *for all* $f \in H, z \in \Omega$.

Obviously, each functional Hilbert space possesses a unique reproducing kernel. One can easily see that the identity

$$K(w, z) = \delta_w \delta_z^*$$

holds. Furthermore, one can show that

$$H = \vee\{K(\cdot, z); z \in \Omega\}.$$

If H has no common zeros, or equivalently $K(\cdot, z) \neq 0$ for all $z \in \Omega$, we define the normalized kernel function by

$$k(w, z) = \frac{K(w, z)}{\|K(\cdot, z)\|} = \frac{K(w, z)}{\sqrt{K(z, z)}} \text{ for } z, w \in \Omega.$$

One of the main tools we will use in this thesis is the Berezin transform of an operator in $L(H)$.

Definition 1.3. *Let Ω be an arbitrary set and $H \subset \mathbb{C}^\Omega$ a functional Hilbert space such that H has no common zeros. For $z \in \Omega$, we abbreviate*

$$k_z = k(\cdot, z)$$

and define the Berezin transform of an operator $X \in L(H)$ as

$$\Gamma(X) : \Omega \to \mathbb{C}, z \mapsto \langle X k_z, k_z \rangle.$$

One can easily deduce the following basic properties of the Berezin transformation.

Lemma 1.4. *Let Ω be an arbitrary set and $H \subset \mathbb{C}^\Omega$ a functional Hilbert space that has no common zeros. Furthermore, let $B(\Omega, \mathbb{C})$ denote the bounded functions from Ω into $\mathbb{C}$. The Berezin transformation*

$$\Gamma : L(H) \longrightarrow B(\Omega, \mathbb{C}), X \mapsto \Gamma(X)$$

is well defined, linear, contractive, and respects the involutions.

For our results, the behaviour of $\Gamma(T)(z)$ for $z \longrightarrow \partial\Omega$ will be of particular importance. We introduce some useful notations in the next definition.

Definition 1.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and let (F, t) be a topological space. In addition, let $f : \Omega \to (F, t)$ be a map and $x \in F$. We say*

$$\lim_{z \to \partial\Omega} f(z) = x$$

if for every neighbourhood U of x, there exists $\delta > 0$ such that

$$f(z) \in U \text{ for all } z \in \Omega \text{ with } \operatorname{dist}(z, \partial\Omega) < \delta.$$

1.2 Operators on Hilbert spaces

Besides the basic results on the Berezin transform, we will in this place also mention some general results about operators on Hilbert spaces that will be useful later on. We start with defining contractions of class $[C_{\cdot 0}]$ on a Hilbert space.

Definition 1.6. *Let H be a Hilbert space. A contraction $T \in L(H)$ is said to be a contraction of the class $[C_{\cdot 0}]$ if*

$$\lim_{k \to \infty} T^{*k} = 0 \text{ in } (L(H), \tau_{SOT}).$$

Contractions of class $[C_{\cdot 0}]$ have an additional property, which is introduced in the next definition.

Definition 1.7. *We call a contraction T completely non-unitary if there is no non-zero reducing subspace for T on which its restriction is unitary.*

Contractions of class $[C_{.0}]$ cannot have a non-zero unitary part, as the following lemma shows.

Lemma 1.8. *A contraction T on a Hilbert space H of the class $[C_{.0}]$ is completely non-unitary.*

Proof. Let $T \in L(H)$ be a contraction with

$$SOT - \lim_{k\to\infty} T^{*k} = 0.$$

Assume there is an orthogonal decomposition

$$H = H_U \oplus H_N$$

such that $H_U \neq \{0\}$ and

$$T = U \oplus N \in L\left(H_U \oplus H_N\right),$$

where U is an unitary operator. Let us consider the effect of the operator T^{*k} on an element $f \in H_U$ with $f \neq 0$. Since U is an unitary operator, U^{*k} is unitary as well and we have

$$\lim_{k\to\infty} \|T^{*k} f\| = \lim_{k\to\infty} \|U^{*k} f\| = \lim_{k\to\infty} \|f\| = \|f\| \neq 0.$$

This is a contradiction to

$$SOT - \lim_{k\to\infty} T^{*k} = 0$$

and hence the assumption was wrong, meaning that T is a completely non-unitary contraction. □

The next lemma deals with certain compact operators on a Hilbert space.

Lemma 1.9. *Let H, K be Hilbert spaces and $S \in L(H, K)$ a bounded operator. Then S is compact if and only if the operator S^*S in $L(H)$ is compact.*

Proof. Suppose that the operator S^*S is compact. Then every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in H has a subsequence $(x_{n_k})_k \in \mathbb{N}$ such that the sequence $(S^*Sx_{n_k})_{n\in\mathbb{N}}$ is convergent in H. Then we calculate for $k, l \in \mathbb{N}$ with the Cauchy Schwarz inequality

$$\begin{aligned}
\|Sx_{n_k} - Sx_{n_l}\|^2 &= |\langle S^*S(x_{n_k} - x_{n_l}), (x_{n_k} - x_{n_l})\rangle| \\
&\leq \|S^*S(x_{n_k} - x_{n_l})\| \|(x_{n_k} - x_{n_l})\| \\
&\leq \|S^*S(x_{n_k} - x_{n_l})\| (\|(x_{n_k}\| + \|x_{n_l})\|)
\end{aligned}$$

The second term is bounded, since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence. $(S^*Sx_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. Thus $(Sx_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent, since K is complete. Hence S is compact. The converse implication is clear. □

A class of of operators on Hilbert spaces that will play an important role later on are the multiplication operators.

Definition 1.10. *Let $\Omega \subset \mathbb{C}^n$ be an arbitrary set. Consider two functional Hilbert spaces $H_1, H_2 \subset \mathbb{C}^\Omega$. The elements of*

$$\mathcal{M}(H_1, H_2) = \{\varphi : \Omega \to \mathbb{C}; \varphi H_1 \subset H_2\}$$

are called multipliers from H_1 to H_2.
In the case $H_1 = H_2 = H$, we write $\mathcal{M}(H)$ instead of $\mathcal{M}(H, H)$.
For $f : \Omega \to \mathbb{C}$, the map

$$(\varphi f) : \Omega \to \mathbb{C}$$

is defined by

$$(\varphi f)(z) = \varphi(z)f(z) \text{ for all } z \in \Omega.$$

For a multiplier $\varphi \in \mathcal{M}(H_1, H_2)$, we call the operator

$$M_\varphi : H_1 \to H_2, f \mapsto \varphi f$$

the multiplication operator with symbol φ.

Remark 1.11. *By the closed graph theorem, multiplication operators are continuous. For $\Omega \subset \mathbb{C}^n$ and two functional Hilbert spaces $H_1, H_2 \subset \mathbb{C}^\Omega$ with reproducing kernels K_1, K_2, one easily calculates that*

$$M_\varphi^* K_2(\cdot, z) = \overline{\varphi(z)} K_1(\cdot, z) \text{ for all } \varphi \in \mathcal{M}(H_1, H_2), z \in \Omega. (*)$$

If $H_1 = H_2 = H$ and H has no common zeros, then

$$K(z, z) > 0 \text{ for all } z \in \Omega$$

and equation $(*)$ implies

$$\|\varphi\|_{\infty,\Omega} \leq \|M_\varphi\| \text{ for all } \varphi \in \mathcal{M}(H),$$

since

$$\begin{aligned} |\varphi(z)|^2 K(z, z) &= \left\langle \overline{\varphi(z)} K(\cdot, z), \overline{\varphi(z)} K(\cdot, z) \right\rangle \\ &\leq \|M_\varphi^*\|^2 \|K(\cdot, z)\|^2 \\ &= \|M_\varphi^*\|^2 K(z, z) \text{ for all } z \in \Omega. \end{aligned}$$

For an arbitrary measure μ on Ω and $\varphi \in L^\infty(\Omega, \mu)$, the operator

$$M_\varphi : L^2(\Omega, \mu) \to L^2(\Omega, \mu), [f] \mapsto [\varphi f]$$

is well defined, linear and continuous, since

$$\|\varphi f\|_{L^2(\Omega,\mu)} = \int_\Omega |\varphi f|^2 d\mu \leq \|\varphi\|^2_{L^\infty(\Omega,\mu)} \|f\|^2_{L^2(\Omega,\mu)}.$$

Hence

$$\|M_\varphi\| \leq \|\varphi\|_{L^\infty(\Omega,\mu)}.$$

In addition, a short calculation yields

$$M_\varphi^* = M_{\overline{\varphi}}.$$

2 Compact operators in the Toeplitz algebra

2.1 The generalized Bergman spaces

Our main theorem extends two previous results on compact Toeplitz operators. The first was formulated by Le in [Le09] and characterizes the compact Toeplitz operators with continuous symbols on Bergman spaces on the unit ball in $\mathbb{C}^n$ with respect to suitable rotation invariant measures. The second was obtained by Čučković and Şahutoğlu in [ČŞ13], where the compact operators in the Toeplitz algebra in the setting of Bergman spaces on pseudoregular domains in $\mathbb{C}^n$ with respect to the Lebesgue measure are described. We consider a class of functional Hilbert spaces that contains both of the above classes as examples, namely the generalized Bergman spaces. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set, $\mu \in M^+\left(\overline{\Omega}\right)$ a positive finite Borel measure and $A \subset C(\overline{\Omega})$ a closed subalgebra with $\mathbb{C}[z]|_{\overline{\Omega}} \subset A$. We define

$$H_A^2(\mu) = \overline{A}^{L^2(\overline{\Omega},\mu)},$$

which is, together with the restricted $L^2(\overline{\Omega},\mu)$-inner product, a Hilbert space. Suppose that for every $z \in \Omega$,

$$\left(\{[f]; f \in A\}, \|\cdot\|_{L^2(\overline{\Omega},\mu)}\right), [f] \mapsto f(z)$$

is a well defined continuous linear map, and denote by

$$\delta_z : H_A^2(\mu) \to \mathbb{C}$$

its continuous extension. Furthermore, let the map

$$\rho : H_A^2(\mu) \to \mathbb{C}^\Omega,\ \rho([f|])(z) = \delta_z([f])$$

be injective. In this case, the space $\mathcal{H}_A^2(\mu) = \mathrm{Im}(\rho) \subset \mathbb{C}^\Omega$ together with the inner product

$$\langle \rho(f), \rho(g)\rangle = \langle f, g\rangle_{L^2(\overline{\Omega},\mu)}$$

becomes a functional Hilbert space. Later on, we will be especially interested in spaces of the form $\mathcal{H}_A^2(\mu)$ that consist only of holomorphic functions. The next lemma shows when this is the case.

Lemma 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and $\mathcal{H}^2_A(\mu)$ a functional Hilbert space as above. Let*

$$K : \Omega \times \Omega \to \mathbb{C}$$

be the unique reproducing kernel function for $\mathcal{H}^2_A(\mu)$. We denote by Ω^* the set

$$\Omega^* = \{\overline{z}; z \in \Omega\}.$$

Then the following statements are equivalent:

(i) $A|_\Omega \subset \mathcal{O}(\Omega)$ *and* $\sup_{z\in K}\|\delta_z\| < \infty$ *for all compact sets* $K \subset \Omega$,

(ii) $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$,

(iii) *The map* $\Omega \times \Omega^*, (z,w) \mapsto K(z, \overline{w})$ *is holomorphic,*

(iv) *The map* $\delta : \Omega \to L(\mathcal{H}^2_A(\mu), \mathbb{C}), z \mapsto \delta_z$ *is holomorphic.*

Proof. $(i) \Rightarrow (ii)$. Assume that $A|_\Omega \subset \mathcal{O}(\Omega)$ and that

$$\sup_{z\in K}\|\delta_z\| < \infty \text{ for all compact subsets } K \subset \Omega.$$

For a function $f \in \mathcal{H}^2_A(\mu)$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in A such that

$$([f_n]) \overset{n\to\infty}{\longrightarrow} \rho^{-1}(f) \text{ in } H^2_A(\mu).$$

Furthermore, for every compact subset $K \subset \Omega$,

$$\begin{aligned}\|\rho([f_n])(z) - f(z)\| &= \|\delta_z(\rho([f_n]) - f)\| \\ &\leq \sup_{z\in K}\|\delta_z\|\|\rho([f_n]) - f\| \\ &= \sup_{z\in K}\|\delta_z\|\|[f_n] - \rho^{-1}(f)\|,\end{aligned}$$

and since $\sup_{z\in K}\|\delta_z\| < \infty$, the sequence $\rho([f_n])$ converges to f uniformly on compact subsets. We have

$$\rho([f_n])(z) = f_n(z) \text{ for all } z \in \Omega.$$

Thus the sequence $(f_n)_{n\in\mathbb{N}}$ also converges to f uniformly on compact subsets of Ω. Since f_n is holomorphic on Ω for all $n \in \mathbb{N}$, f is holomorphic on Ω.

$(ii) \Rightarrow (iii)$. Suppose now that every function in $\mathcal{H}^2_A(\mu)$ is holomorphic. For fixed $w \in \Omega^*$, the function $K(\cdot, \overline{w})$ is an element of $\mathcal{H}^2_A(\mu)$, and hence holomorphic.

Furthermore, for fixed $z \in \Omega$, this implies that the map

$$\Omega \to \mathbb{C}, w \mapsto K(z,w) = \overline{K(w,z)}$$

is antiholomorphic and hence that

$$\Omega^* \to \mathbb{C}, w \mapsto K(z,\overline{w})$$

is holomorphic. Hartog's theorem then implies that the map

$$\Omega \times \Omega^* \to \mathbb{C}, (z,w) \mapsto K(z,\overline{w})$$

is holomorphic.

$(iii) \Rightarrow (i)$. For $f \in \mathcal{H}^2_A(\mu)$ and $z \in \Omega$, we have

$$|f(z)| = |\langle f, K(\cdot,z)\rangle| \leq \|f\|\|K(\cdot,z)\| = \|f\|K(z,z)^{\frac{1}{2}}.$$

Hence the continuity of K implies that convergent sequences in $\mathcal{H}^2_A(\mu)$ converge uniformly on all compact subsets of Ω. Since by (iii) all functions of the form $K(\cdot,w)$, $w \in \Omega$, are holomorphic and since the finite linear combinations of these functions are dense in $\mathcal{H}^2_A(\mu)$, it follows that $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$. Since $A|_\Omega \subset \mathcal{H}^2_A(\mu)$, we also obtain (i).

$(ii) \Leftrightarrow (iv)$. It is well known that the operator-valued map

$$\delta : \Omega \to L(\mathcal{H}^2_A(\mu), \mathbb{C}),\ z \mapsto \delta_z$$

is holomorphic if and only if all the functions

$$\Omega \to \mathbb{C},\ z \mapsto \delta_z(f) = f(z) \quad (f \in \mathcal{H}^2_A(\mu))$$

are holomorphic. Thus (ii) and (iv) are equivalent.

□

We now introduce Toeplitz operators on the spaces of the form $\mathcal{H}^2_A(\mu)$.

Definition 2.2. *For $\varphi \in L^\infty(\overline{\Omega},\mu)$, we call the operator*

$$T_\varphi = \rho\left(P_{H^2_A(\mu)} M_\varphi\right)\rho^{-1} \in L\left(\mathcal{H}^2_A(\mu)\right)$$

the Toeplitz operator with symbol φ. Note that

$$\|T_\varphi\| = \|\rho(P_{H^2_A(\mu)} M_\varphi)\rho^{-1}\| \leq \|\varphi\|_{L^\infty(\overline{\Omega},\mu)}.$$

The C^*-algebra

$$\mathcal{T}_A = C^*\left(\left\{T_\varphi; \varphi \in C\left(\overline{\Omega}\right)\right\}\right)$$

is called the Toeplitz algebra on $\mathcal{H}^2_A(\mu)$.

The following lemma shows that we already know some Toeplitz operators on $\mathcal{H}_A^2(\mu)$.

Lemma 2.3. *We have* $A|_\Omega \subset \mathcal{M}(\mathcal{H}_A^2(\mu))$ *and* $M_{(f|\Omega)} = T_f$ *for all* $f \in A$.

Proof. Consider a function $f \in A$. The operator

$$M_f : L^2(\overline{\Omega}, \mu) \to L^2(\overline{\Omega}, \mu), [g] \mapsto [fg]$$

is continuous and, since $fA \subset A$, leaves $H_A^2(\mu)$ invariant. It follows that the operator

$$M_f : H_A^2(\mu) \to H_A^2(\mu)$$

is well defined, linear, and continuous. Consider now the continuous operator

$$\rho M_f \rho^{-1} \in L(\mathcal{H}_A^2(\mu))$$

and $[g] \in H_A^2(\mu)$. We choose a sequence $(g_n)_{n\in\mathbb{N}}$ in A such that $([g_n])_{n\in\mathbb{N}}$ converges to $[g]$ and calculate

$$\begin{aligned} \rho M_f \rho^{-1}(\rho([g])(z) &= \lim_{n\to\infty} \rho([fg_n])(z) \\ &= \lim_{n\to\infty} f(z)g_n(z) \\ &= f(z) \lim_{n\to\infty} \rho([g_n])(z) \\ &= f(z)\rho([g])(z) \text{ for all } z \in \Omega. \end{aligned}$$

Hence $f|_\Omega \in \mathcal{M}(\mathcal{H}_A^2(\mu))$ and

$$M_{(f|\Omega)} = \rho M_f \rho^{-1} = \rho P_{H_A^2(\mu)} M_f|_{H_A^2(\mu)} \rho^{-1} = T_f.$$

□

In the next lemma, we obtain yet another useful characterization of $\mathcal{T}_A$.

Theorem 2.4. *For the* C^**-algebra* $\mathcal{T}_A$ *introduced in Definiton 2.2, we have*

$$\mathcal{T}_A = C^*(T_z),$$

where $C^*(T_z)$ *denotes the unital* C^**-algebra generated by* $T_z = (T_{z_1}, \ldots, T_{z_n})$.

Proof. Consider the set

$$S = \{\varphi \in C(\overline{\Omega}); T_\varphi \in C^*(T_z)\} \subset C(\overline{\Omega}),$$

which contains the constant function 1 and the coordinate functions $z_1, \ldots, z_n$.

This set is a closed linear subspace of $C(\overline{\Omega})$, since the map

$$C(\overline{\Omega}) \to \mathcal{T}_A, \varphi \mapsto T_\varphi$$

is linear and continuous. Let now $\varphi \in C(\overline{\Omega})$ and $f \in A$ with $\varphi, f \in S$. In the proof of Lemma 2.3 it is shown that the operator

$$M_f : H_A^2(\mu) \to H_A^2(\mu), [g] \mapsto [fg]$$

is well defined, continuous and linear. Thus, since

$$\begin{aligned} T_{\varphi f} &= \rho P_{H_A^2(\mu)} M_{\varphi f} \rho^{-1} \\ &= \rho P_{H_A^2(\mu)} M_\varphi M_f \rho^{-1} \\ &= \rho P_{H_A^2(\mu)} M_\varphi P_{H_A^2(\mu)} M_f \rho^{-1} \\ &= \rho P_{H_A^2(\mu)} M_\varphi \rho^{-1} \rho P_{H_A^2(\mu)} M_f \rho^{-1} \\ &= T_\varphi T_f, \end{aligned}$$

the function φf is also an element of S.
Furthemore, for $\varphi \in S$, we have

$$T_{\overline{\varphi}} = T_\varphi^* \in C^*(T_z),$$

so S contains $\overline{\varphi}$.
Hence, since the coordinate functions lie in A, the set $S \subset C(\overline{\Omega})$ is a closed subspace containing $C[z, \overline{z}]|_{\overline{\Omega}}$. By the Stone-Weierstrass theorem, it follows that $S = C(\overline{\Omega})$ and $\mathcal{T}_A = C^*(T_z)$.

□

One of our main objectives in this thesis is to show that the compactness of an operator in the Toeplitz algebra depends on the behaviour of its Berezin transform when approaching boundary points. Of particular interest are the so called peak points for the algebra A which are defined in the following way.

Definition 2.5. *A point $z_0 \in \partial\Omega$ is called a peak point for A if there exists a function $f \in A$ satisfying*

$$f(z) = 1 > |f(z)| \text{ for all } z \in \overline{\Omega} \setminus \{z_0\}.$$

We write $\partial_p A$ for the set of all peak points for A in $\partial\Omega$.

2.2 Characterization of compact Toeplitz operators

Our goal in the following sections is to characterise the compact operators in the Toeplitz algebra. As a first step, we look at the generators of the Toeplitz algebra. The next theorem contains a sufficient condition for such a Toeplitz operator to be compact in case that all functions in $\mathcal{H}_A^2(\mu)$ are holomorphic. This result will play an important role in the proof of a more general theorem for operators in the Toeplitz algebra later on.

Theorem 2.6. *Let T_φ be a Toeplitz operator with symbol $\varphi \in C\left(\overline{\Omega}\right)$ on a functional Hilbert space $\mathcal{H}^2_A(\mu)$ constructed as in Section 2.1. In addition, suppose that*

$$\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega).$$

If $\varphi|_{\partial\Omega} = 0$, then T_φ is compact.

Proof. We begin by showing that the assertion holds for every Toeplitz operator T_ξ with a symbol $\xi \in C\left(\overline{\Omega}\right)$ that vanishes not only on the boundary itself, but on $U \cap \overline{\Omega}$ for some open neighbourhood $U \supset \partial\Omega$. Given a bounded sequence$(f_k)_{k\in\mathbb{N}}$ in $\mathcal{H}^2_A(\mu)$, we will show that the sequence $(T_\xi f_k)_{k\in\mathbb{N}}$ has a converging subsequence.
By a theorem of Eberlein and Smulian (see Theorem V.6.1 in [DS58]), $(f_k)_{k\in\mathbb{N}}$ possesses a weakly convergent subsequence $\left(f_{k_j}\right)_{j\in\mathbb{N}}$. Let $f \in \mathcal{H}^2_A(\mu)$ denote the weak limit of this subsequence. By Lemma 2.1, the map

$$\delta : \Omega \to L\left(\mathcal{H}^2_A(\mu), \mathbb{C}\right), z \mapsto \delta_z$$

is continuous, because all elements of $\mathcal{H}^2_A(\mu)$ are holomorphic functions. In consequence, the set

$$\{\delta_z; z \in K\} \subset \mathcal{H}^2_A(\mu)$$

is compact for every compact set $K \subset \Omega$. The sequence $\left(\langle\cdot, f_{k_j}\rangle\right)_{j\in\mathbb{N}}$ of continuous linear forms $\langle\cdot, f_{k_j}\rangle \in \mathcal{H}^2_A(\mu)'$converges pointwise, hence uniformly on all compact subsets of $\mathcal{H}^2_A(\mu)$, to the functional $\langle\cdot, f\rangle$. In particular, the sequence $(f_{k_j})_{j\in\mathbb{N}}$ converges uniformly on compact subsets of Ω to f. Since $\rho(A) \subset \mathcal{H}^2_A(\mu)$ is dense, there is a sequence $(F_j)_{j\in\mathbb{N}}$ in A such that

$$\|\rho(F_j) - f_{k_j}\|_{\mathcal{H}^2_A(\mu)} \overset{j\to\infty}{\longrightarrow} 0.$$

This implies pointwise convergence on $\mathcal{H}^2_A(\mu)$ and by the same argument as above, also uniform convergence on compact subsets of Ω. Therefore

$$F_j|_\Omega = \rho(F_j) \overset{j\to\infty}{\longrightarrow} f$$

uniformly on compact subsets of Ω. The hypothesis that $\xi|_{U\cap\overline{\Omega}} = 0$ for some open neighbourhood $U \supset \partial\Omega$ implies that $\operatorname{supp}(\xi)$ is a compact subset of Ω. We then have

$$\xi F_j \overset{j\to\infty}{\longrightarrow} h$$

uniformly on $\overline{\Omega}$, where the function $h : \overline{\Omega} \to \mathbb{C}$ is defined as

$$h(z) = \begin{cases} \xi(z) f(z), & z \in \operatorname{supp}(\xi) \\ 0, & \text{else.} \end{cases}$$

It follows that

$$T_\xi\left(f_{k_j}-\rho\left(F_j\right)\right)=\rho\left(P_{H^2_A(\mu)}M_\xi\right)\rho^{-1}\left(f_{k_j}-\rho\left(F_j\right)\right)\overset{k\to\infty}{\longrightarrow}0\text{ in }\mathcal{H}^2_A\left(\mu\right)$$

and that

$$\begin{aligned}T_\xi\left(\rho\left(F_j\right)\right)&=\rho\left(P_{H^2_A(\mu)}M_\xi\right)\rho^{-1}\left(\rho F_j\right)\\&=\rho\left(P_{H^2_A(\mu)}M_\xi\right)\left(F_j\right)\overset{k\to\infty}{\longrightarrow}\rho P_{H^2_A(\mu)}\left(h\right)\end{aligned}$$

as ρ and $P_{H^2_A(\mu)}$ are continuous. Together, this yields

$$\begin{aligned}\lim_{k\to\infty}T_\xi\left(f_{k_j}\right)&=\lim_{k\to\infty}T_\xi\left(\rho\left(F_j\right)\right)+\lim_{k\to\infty}T_\xi\left(f_{k_j}-\rho\left(F_j\right)\right)\\&=\rho P_{H^2_A(\mu)}\left(h\right)\in\mathcal{H}^2_A\left(\mu\right),\end{aligned}$$

so T_ξ is a compact operator.
Consider now $\varphi\in C(\overline{\Omega})$ with $\varphi|_{\partial\Omega}=0$. For $\epsilon>0$, there exists an open neighbourhood $U\supset\partial\Omega$ in $\mathbb{C}^n$ with

$$|\varphi(z)|\leq\epsilon\text{ for all }z\in U\cap\overline{\Omega}.$$

We choose an open neighbourhood $V\supset\partial\Omega$ with $V\subset\overline{V}\subset U$ and a cut-off function $\theta\in C(\overline{\Omega})$ such that $0\leq\theta\leq 1$ on Ω, $\theta|_{\overline{\Omega}\cap(\mathbb{C}^n\setminus U)}=1$, and $\theta|_{V\cap\overline{\Omega}}=0$. The operator $T_{\theta\varphi}$ is then compact, since $\theta\varphi\in C(\overline{\Omega})$ vanishes on $V\cap\overline{\Omega}$. We then have

$$\begin{aligned}\|T_{\theta\varphi}-T_\varphi\|&\leq\|(1-\theta)\varphi\|_{\infty,\overline{\Omega}}\\&\leq\|(1-\theta)\varphi\|_{\infty,U\cap\overline{\Omega}}\\&\leq\epsilon.\end{aligned}$$

As ϵ was arbitrary, the operator T_φ can be approximated in the operator norm by compact operators and is hence compact. □

2.3 Compact operators in the Toeplitz algebra

In Satz 6.22 in [Kre11], it was proven that for a Toeplitz operator $T_\varphi\in L(\mathcal{H}^2_A\left(\mu\right))$ with $\varphi\in C(\overline{\Omega})$, we have

$$\varphi(z_0)=\lim_{z\to z_0}\Gamma(T_\varphi)(z)\text{ for all }z_0\in\partial_p A.$$

If we require the peak points to be dense in $\partial\Omega$, then in Theorem 2.6, the condition $\varphi|_{\partial\Omega}=0$ follows from the condition that

$$\lim_{z\to\partial\Omega}\Gamma(T_\varphi)(z)=0.$$

The last condition also makes sense in the case of an arbitrary operator in the Toeplitz algebra. It turns out, that in certain cases, this is indeed the condition that characterizes the compact operators in the Toeplitz algebra. Before we are able to prove this result, we need to make some preparations. Let us first define what it means for an operator tuple to be essentially normal.

Definition 2.7. *Let H be a Hilbert space. We call a commuting tuple*

$$T = (T_1, \cdots, T_n) \in L(H)^n$$

essentially normal if

$$T_j^* T_j - T_j T_j^* \text{ compact for all } j \in 1, \cdots, n.$$

Another important tool will be the class of Hankel operators.

Definition 2.8. *Let $H_A^2(\mu)$ be as in Section 2.1 and let*

$$P_{H_A^2(\mu)} : L^2(\overline{\Omega}, \mu) \to H_A^2(\mu)$$

be the projection onto $H_A^2(\mu)$. We define the Hankel operator with symbol $\varphi \in L^\infty(\overline{\Omega}, \mu)$ as

$$H_\varphi : H_A^2(\mu) \to L^2(\overline{\Omega}, \mu), \;\; f \mapsto (I - P_{H_A^2(\mu)}) M_\varphi f.$$

We want to use Hankel operators to show the compactness of operators of the form

$$T_{\varphi\xi} - T_\varphi T_\xi \in L(\mathcal{H}_A^2(\mu)),$$

where $\varphi,\ \xi \in C(\overline{\Omega})$ are continuous symbols. The main idea is contained in the next lemma.

Lemma 2.9. *Let $\varphi, \xi \in L^\infty(\overline{\Omega}, \mu)$ be functions with induced Toeplitz operators $T_\varphi, T_\xi \in L(\mathcal{H}_A^2(\mu))$. Then we have*

- $H_{\overline{\varphi}}^* H_\xi = \rho^{-1}(T_{\varphi\xi} - T_\varphi T_\xi)\rho.$
- $H_{\varphi\xi} = H_\varphi \rho^{-1} T_\xi \rho + (I - P_{H_A^2(\mu)}) M_\varphi H_\xi.$

Proof. Let $g \in H_A^2(\mu)$ and $f \in L^2(\overline{\Omega}, \mu)$ be arbitrary functions. We calculate

$$\begin{aligned}
\langle H_{\overline{\varphi}}^* f, g \rangle_{H_A^2(\mu)} &= \langle f, H_{\overline{\varphi}} g \rangle_{L^2(\overline{\Omega},\mu)} \\
&= \langle f, (I - P_{H_A^2(\mu)}) M_{\overline{\varphi}} g \rangle_{L^2(\overline{\Omega},\mu)} \\
&= \langle f, M_{\overline{\varphi}} g \rangle_{L^2(\overline{\Omega},\mu)} - \langle f, P_{H_A^2(\mu)} M_{\overline{\varphi}} g \rangle_{L^2(\overline{\Omega},\mu)} \\
&= \langle M_\varphi f, g \rangle_{L^2(\overline{\Omega},\mu)} - \langle M_\varphi P_{H_A^2(\mu)} f, g \rangle_{L^2(\overline{\Omega},\mu)} \\
&= \langle P_{H_A^2(\mu)} M_\varphi f, g \rangle_{H_A^2(\mu)} - \langle P_{H_A^2(\mu)} M_\varphi P_{H_A^2(\mu)} f, g \rangle_{H_A^2(\mu)}
\end{aligned}$$

As f and g were arbitrary, we obtain

$$H_{\overline{\varphi}}^* = P_{H_A^2(\mu)} M_\varphi - P_{H_A^2(\mu)} M_\varphi P_{H_A^2(\mu)}.$$

It follows that

$$\begin{aligned}H_{\overline{\varphi}}^*H_\xi &= P_{H_A^2(\mu)}M_\varphi(I-P_{H_A^2(\mu)})M_\xi|_{H_A^2(\mu)}\\ &= P_{H_A^2(\mu)}M_{\varphi\xi}|_{H_A^2(\mu)} - P_{H_A^2(\mu)}M_\varphi|_{H_A^2(\mu)}P_{H_A^2(\mu)}M_\xi|_{H_A^2(\mu)}\\ &= \rho^{-1}T_{\varphi\xi}\rho - \rho^{-1}T_\varphi\rho\rho^{-1}T_\xi\rho\\ &= \rho^{-1}(T_{\varphi\xi}-T_\varphi T_\xi)\rho.\end{aligned}$$

The calculation

$$\begin{aligned}H_{\varphi\xi} &= (I-P_{H_A^2(\mu)})M_{\varphi\xi}|_{H_A^2(\mu)}\\ &= (I-P_{H_A^2(\mu)})M_\varphi M_\xi|_{H_A^2(\mu)}\\ &= (I-P_{H_A^2(\mu)})M_\varphi(P_{H_A^2(\mu)}M_\xi|_{H_A^2(\mu)} + (I-P_{H_A^2(\mu)})M_\xi|_{H_A^2(\mu)})\\ &= (I-P_{H_A^2(\mu)})M_\varphi|_{H_A^2(\mu)}P_{H_A^2(\mu)}M_\xi|_{H_A^2(\mu)} + (I-P_{H_A^2(\mu)})M_\varphi(I-P_{H_A^2(\mu)})M_\xi|_{H_A^2(\mu)})\\ &= H_\varphi\rho^{-1}T_\xi\rho + (I-P_{H_A^2(\mu)})M_\varphi H_\xi\end{aligned}$$

yields the second equality. □

Remark 2.10. *The identity*

$$H_\xi^*H_\xi = \rho^{-1}(T_{|\xi|^2} - T_\xi^*T_\xi)\rho$$

together with Theorem 2.6 and Lemma 1.9 shows that on each analytic functional Hilbert space $\mathcal{H}_A^2(\mu) \subset \mathcal{O}(\Omega)$ every Hankel operator with continuous symbol $\varphi \in C(\overline{\Omega})$ such that

$$\varphi|_{\partial\Omega} = 0$$

is compact.

The following lemmata show how the essential normality of the multiplication tuple $T_z \in L(H_A^2(\mu))^n$ will be used.

Lemma 2.11. *The set*

$$B = \{\varphi \in C(\overline{\Omega}); H_\varphi \text{ compact }\}$$

is a closed subalgebra of $C(\overline{\Omega})$ that contains A.

Proof. It follows immediately from the second part of Lemma 2.9 that B is an algebra. Since for every $\varphi \in L^\infty(\overline{\Omega},\mu)$

$$\|H_\varphi\| = \|(I-P_{H_A^2(\mu)})M_\varphi|_{H_A^2(\mu)}\| \leq \|M_\varphi\| \leq \|\varphi\|_{L^\infty(\overline{\Omega},\mu)},$$

the map

$$C(\overline{\Omega}) \longrightarrow L(H_A^2(\mu), L^2(\overline{\Omega},\mu)),\ \varphi \mapsto H_\varphi$$

is continuous, so B is closed as an inverse image of a closed set. By the proof of Lemma 2.3, for $f \in A$, $g \in H_A^2(\mu)$, we have $M_f g \in H_A^2(\mu)$. We conclude

$$H_f g = (I-P_{H_A^2(\mu)})M_f g = 0,$$

which in particular means that H_f is compact for all $f \in A$, so $A \subset B$. □

Remark 2.12. *Note that with the same proof, one can also show that*

$$\{\varphi \in L^\infty(\overline{\Omega}, \mu); H_f \text{ compact } \} \subset L^\infty(\overline{\Omega}, \mu)$$

is a closed subalgebra.

Compactness of the Hankel operators with continuous symbols is equivalent to the essential normality of T_z, as one can see from the next lemma.

Lemma 2.13. *The following conditions are equivalent:*

(i) $T_z = (T_{z_1}, \cdots, T_{z_n}) \in L(\mathcal{H}^2_A(\mu))^n$ *is essentially normal,*

(ii) $\overline{z_1}, \ldots \overline{z_n} \in B$,

(iii) $B = C(\overline{\Omega})$,

(iv) $T_{\varphi\xi} - T_\varphi T_\xi \in L(\mathcal{H}^2_A(\mu))$ *is compact for all* $\varphi, \xi \in C(\overline{\Omega})$.

Proof. Since the coordinate functions z_j are elements of A, we have for $j \in \{1, \ldots, n\}$

$$T^*_{z_j} T_{z_j} - T_{z_j} T^*_{z_j} = T_{\overline{z_j} z_j} - T_{z_j} T_{\overline{z_j}}.$$

The first part of Lemma 2.9 then yields that

$$T^*_{z_j} T_{z_j} - T_{z_j} T^*_{z_j} = \rho H^*_{\overline{z_j}} H_{\overline{z_j}} \rho^{-1}.$$

Hence, the essential normality of T_z is equivalent to the compactness of the operators

$$H^*_{\overline{z_j}} H_{\overline{z_j}} \text{ for } 1 \leq j \leq n.$$

By Lemma 1.9, these operators are compact if and only if the Hankel operators

$$H_{\overline{z_j}} \text{ for } j \in \{1, \ldots, n\}$$

are compact.
Assume now that $\overline{z_1}, \ldots, \overline{z_j} \in B$. By Lemma 2.11, the set B is then a closed subalgebra containing $\mathbb{C}[z, \overline{z}]$. The Stone-Weierstrass theorem then yields $B = C(\overline{\Omega})$.

Let now on the other hand all Hankel operators with continuous symbol on $\overline{\Omega}$ be compact. Then by Lemma 2.9, the operators

$$T_{\varphi\xi} - T_\varphi T_\xi$$

are compact for all $\varphi, \xi \in C(\overline{\Omega})$.
In order to show the implication $(iv) \Rightarrow (i)$, suppose that $T_\varphi T_\xi - T_{\varphi\xi} \in L(\mathcal{H}^2_A(\mu))$ is compact for all $f, g \in C(\overline{\Omega})$. This implies the essential normality of T_z, since for $j \in \{1, \ldots, n\}$

$$\begin{aligned} T^*_{z_j} T_{z_j} - T_{z_j} T^*_{z_j} &= T_{\overline{z_j}} T_{z_j} - T_{z_j} T_{\overline{z_j}} \\ &= (T_{\overline{z_j}} T_{z_j} - T_{\overline{z_j} z_j}) + (T_{z_j \overline{z_j}} - T_{z_j} T_{\overline{z_j}}), \end{aligned}$$

which is compact as a sum of compact operators. □

Another tool that will be important later on is the essential norm of an operator.

Definition 2.14. *Let X be a Banach space and $S \in L(X)$ a bounded linear operator on X. We define the essential norm of S as*

$$\|S\|_e = \inf\{\|S - K\|; K \in L(X) \text{ compact }\}.$$

The next lemma yields some information about the essential norm of a Toeplitz operator with continuous symbol. However, we have to require that the generalized Bergman space $\mathcal{H}^2_A(\mu)$ consists only of holomorphic functions.

Lemma 2.15. *Let $\mathcal{H}^2_A(\mu)$ be a functional Hilbert space as in Section 2.1 which additionally satisfies $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$. Furthermore, let $T_\varphi \in L(\mathcal{H}^2_A(\mu))$ be the Toeplitz operator with symbol $\varphi \in C(\overline{\Omega})$. Then the inequality*

$$\|T_\varphi\|_e \leq \|\varphi\|_{\infty,\partial\Omega}$$

holds.

Proof. Since Ω is a bounded subset of $\mathbb{C}^n$, $\partial\Omega$ is compact. Let

$$\alpha > \|\varphi\|_{\infty,\partial\Omega} \geq 0$$

be arbitrary. Since φ is continuous on $\overline{\Omega}$, we can find an open neighbourhood $U \supset \partial\Omega$ with

$$|\varphi(w)| < \alpha \text{ for all } w \in U \cap \overline{\Omega}.$$

By Urysohn's Lemma, there exists a function $\psi \in C(\overline{\Omega})$ with

1. $0 \leq \psi(z) \leq 1$ for all $z \in \overline{\Omega}$,
2. $\psi = 1$ on $\overline{\Omega} \cap (\mathbb{C}^n \setminus U)$,
3. $\psi = 0$ on $\partial\Omega$.

By Theorem 2.6, the operator $S = T_{\varphi\psi}$ is compact. For the essential norm of T_φ, we then have

$$\|T_\varphi\|_e \leq \|T_\varphi - S\| \leq \sup(|(1 - \psi(z))\varphi(z)|; z \in \overline{\Omega}) \leq \alpha.$$

Since $\alpha > \|\varphi\|_{\infty,\partial\Omega}$ was arbitrary, it follows that

$$\|T_\varphi\| \leq \|\varphi\|_{\infty,\partial\Omega}.$$

□

These preparations enable us to formulate the main theorem. A version for Bergman spaces with respect to the Lebesgue measure on a certain class of bounded domains in $\mathbb{C}^n$ is proven by Čučković and Sahutolgu in [ČŞ13]. We will consider this situation as an example later on. Note that if

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0,$$

then the Berezin transform $\Gamma(K)(z)$ of every compact operator $K \in L(\mathcal{H}_A^2(\mu))$ converges to zero, as $z \longrightarrow \partial\Omega$. Indeed, for every sequence $(z_n)_{n\in\mathbb{N}}$ that converges to the boundary of Ω, the sequence $(k_{z_n})_{n\in\mathbb{N}}$ is weakly convergent to zero. Hence $(Kk_{z_n})_{n\in\mathbb{N}}$ converges to zero in norm and

$$\lim_{n\to\infty} |\Gamma(K)(z_n)| = |\langle Kk_{z_n}, k_{z_n}\rangle| \leq \|Kk_{z_n}\|\|k_{z_n}\| \leq \|Kk_{z_n}\| \overset{n\to\infty}{\longrightarrow} 0.$$

This yields one implication in the next theorem.

Theorem 2.16. *Let $\mathcal{H}_A^2(\mu)$ be a functional Hilbert space as in Section 2.1, which in addition satisfies $\mathcal{H}_A^2(\mu) \subset \mathcal{O}(\Omega)$. Assume that $T_z \in L(\mathcal{H}_A^2(\mu))^n$ is essentially normal and that*

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0.$$

Furthermore, let $\partial_p A \subset \partial\Omega$ be a dense subset. Then an operator $T \in \mathcal{T}_A$ is compact if and only if

$$\lim_{z\to\partial\Omega} \Gamma(T)(z) = 0.$$

Proof. Let $T \in \mathcal{T}_A = C^*(\{T_\varphi; \varphi \in C(\overline{\Omega})\})$ be an operator such that

$$\lim_{z\to\partial\Omega} \Gamma(T)(z) = 0.$$

Then T is the limit with respect to the operator norm of finite sums of operators of the form

$$T_{\varphi_1} \cdots T_{\varphi_m} \text{ with } m \in \mathbb{N} \text{ and } \varphi_1, \ldots, \varphi_m \in C(\overline{\Omega}).$$

By an elementary induction, it follows that any such product is of the form

$$T_{\varphi_1\cdots\varphi_m} + K$$

with a suitable compact operator $K \in L(\mathcal{H}_A^2(\mu))$. The case $m = 2$ follows immediately from Lemma 2.13. If the assertion has been shown for products of length m and if $\varphi_1, \ldots, \varphi_{m+1} \in C(\overline{\Omega})$, then using the induction hypothesis we find a compact operator $K \in L(\mathcal{H}_A^2(\mu))$ such that

$$T_{\varphi_1} \cdots T_{\varphi_{m+1}} = (T_{\varphi_1\cdots\varphi_m} + K)T_{\varphi_{m+1}}.$$

The case $m = 2$ then yields the existence of a compact operator $\tilde{K} \in L(\mathcal{H}^2_A(\mu))$ with

$$T_{\varphi_1} \cdots T_{\varphi_{m+1}} = T_{\varphi_1 \cdots \varphi_{m+1}} + \tilde{K} + K T_{\varphi_{m+1}}.$$

Hence, for every $\epsilon > 0$, there are a function $\xi_\epsilon \in C(\overline{\Omega})$ and a compact operator $K_\epsilon \in L(\mathcal{H}^2_A(\mu))$ with

$$\|T - T_{\xi_\epsilon} - K_\epsilon\| < \epsilon.$$

Since the Berezin transform is contractive, it follows that

$$|\Gamma(T) - \Gamma(T_{\xi_\epsilon}) - \Gamma(K_\epsilon)| < \epsilon$$

on Ω. By hypothesis

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0.$$

Hence

$$\lim_{z \to \partial\Omega} \Gamma(K_\epsilon)(z) = 0.$$

Since by hypothesis

$$\lim_{z \to \partial\Omega} \Gamma(T)(z) = 0,$$

we can choose an open neighbourhood $U \supset \partial\Omega$ such that

$$|\Gamma(T_{\xi_\epsilon})| < 2\epsilon$$

on $U \cap \Omega$. From Satz 6.22 in [Kre11] we know that

$$|\xi_\epsilon| \leq 2\epsilon \text{ on } \partial_p A.$$

Since by hypothesis $\partial_p A \subset \partial\Omega$ is dense, it follows that

$$\|\xi_\epsilon\|_{\infty, \partial\Omega} \leq 2\epsilon.$$

By Lemma 2.15, there is a compact operator $C_\epsilon \in \mathcal{K}(\mathcal{H}^2_A(\mu))$ with

$$\|T_{\xi_\epsilon} - C_\epsilon\| < 3\epsilon.$$

But then

$$\|T - C_\epsilon - K_\epsilon\| < 4\epsilon$$

and since ϵ is arbitrary, we conclude that T is compact.
The reverse implication is clear. $\square$

The question arises under which circumstances the requirements in the above theorem are satisfied. In the next two sections, we will consider some situations in which Theorem 2.16 is applicable.

2.4 A class of Bergman spaces on pseudoregular domains

The first class of spaces we want to look at are Bergman spaces on pseudogular open domains in $\mathbb{C}^n$. Before we can define pseudoregular open sets, we need to introduce the $\overline{\partial}$-Neumann operator. For more information about the $\overline{\partial}$-Neumann problem and pseudoregular sets, consider the appendix.

Definition 2.17. *By Theorem 2.9 (1) in [Str10], on a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the complex Laplacian*

$$\overline{\partial^*}\,\overline{\partial} + \overline{\partial}\,\overline{\partial^*}$$

has a bounded inverse N on the square-integrable (0,1)-forms on Ω. We call N the $\overline{\partial}$-Neumann operator of Ω.

A pseudoregular set is then defined as follows.

Definition 2.18. *We call a bounded open set $\Omega \subset \mathbb{C}^n$ pseudoregular if Ω is pseudoconvex with smooth boundary and if, in addition, the $\overline{\partial}$-Neumann operator N of Ω is compact.*

We can now define Bergman spaces on these sets.

Definition 2.19. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $\lambda \in M(\Omega)$ the restriction of the Lebesgue measure on $\mathbb{C}^n$ to Ω. The Bergman space on Ω is defined as*

$$L_a^2(\Omega, \lambda) = \left\{ f \in \mathcal{O}(\Omega); \|f\|_p = \left(\int_\Omega |f|^p d\lambda \right)^{\frac{1}{p}} < \infty \right\}.$$

One can show that $(L_a^2(\Omega, \lambda), \|\cdot\|_{L^2(\Omega,\lambda)})$ is a functional Hilbert space. It is well known that the Bergman space is a closed subspace of $L^2(\Omega, \lambda)$ (see Corollary 1.10 in [Ran86]). For Bergman spaces on pseudoregular domains, Čučković and Şahutoğlu proved a version of Theorem 2.16 in [ČŞ13] which we now obtain as a corollary. While working with Bergman spaces, the domain algebra plays a central role.

Definition 2.20. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Then we call*

$$A(\Omega) = \{f \in C(\overline{\Omega}); f|_\Omega \in \mathcal{O}(\Omega)\}$$

the domain algebra of Ω.

Since we would like to apply our results to the Bergman spaces defined above, we have to make sure they are of the right form.

Lemma 2.21. *Let $\Omega \subset \mathbb{C}^n$ a pseudoregular set and let $\mu \in M^+(\overline{\Omega})$ denote the trivial extension of the usual Lebesgue measure λ on Ω. The Bergman space $L^2_a(\Omega, \lambda)$ is a functional Hilbert space of the form $\mathcal{H}^2_A(\mu)$ as defined in Section 2.1 with $A = A(\Omega)$.*

Proof. Since Ω is pseudoregular, it is pseudoconvex with class C^∞ boundary, so by Theorem 20 in [BS99], the domain algebra is a dense subset of $L^2_a(\Omega, \lambda)$. We now need to show that for every $z \in \Omega$, the map

$$\delta_z^{A(\Omega)}: \ \{[f]; f \in A(\Omega)\} \to \mathbb{C}, [f] \mapsto f(z)$$

is a well defined continuous linear map. Let $z \in \Omega$ be arbitrary. Since $\Omega \subset \mathbb{C}^n$ is open, there is $r > 0$ such that the open polydisk $P_r(z)$ is entirely contained in Ω. For every $f \in A(\Omega)$, by 1.6 in [ABR01], we then have,

$$f(z) = \frac{1}{(\pi r^2)^n} \int_{P_r(z)} f(w) d\lambda(w).$$

This implies

$$\begin{aligned} |f(z)| &\leq \frac{1}{(\pi r^2)^n} \int_{P_r(z)} |f(w)| d\lambda(w) \\ &\leq K_1 \left(\int_{P_r(z)} 1 d\lambda(w) \int_{P_r(z)} |f(w)|^2 d\lambda(w) \right)^{\frac{1}{2}} \\ &\leq K_1 K_2 \left(\int_{\overline{\Omega}} |f(w)|^2 d\mu(w) \right)^{\frac{1}{2}} \\ &= K \|f\|_{L^2(\overline{\Omega}, \mu)}. \end{aligned}$$

Here

$$K_1 = \frac{1}{(\pi r^2)^n}, \ K_2 = \left(\int_{P_r(z)} 1 d\lambda(w) \right)^{\frac{1}{2}} \text{ and } K = K_1 K_2$$

are constants only depending on z. So for every $z \in \Omega$ the map $\delta_z^{A(\Omega)}$ is well-defined, linear and continuous and therefore has a continuous extension to $H^2_A(\mu) = \overline{A}^{L^2(\overline{\Omega}, \mu)}$ which we denote by δ_z. We then look at the linear map

$$\rho : H^2_A(\mu) \to \mathbb{C}^\Omega, \rho([f])(z) = \delta_z([f]) \text{ for all } z \in \Omega, [f] \in H^2_A(\mu).$$

Let $[f] \in H^2_A(\mu)$ with

$$\rho([f])(z) = 0 \text{ for all } z \in \Omega.$$

Since $A(\Omega)$ is a dense subset of $H^2_A(\mu)$, we can choose a sequence $(f_n)_{n \in \mathbb{N}}$ in $A(\Omega)$ with

$$[f_n] \xrightarrow{n \to \infty} [f] \text{ in } H^2_A(\mu).$$

We then have, by the continuity of δ_z,

$$\begin{aligned}\lim_{n\to\infty} f_n(z) &= \lim_{n\to\infty} \delta_z([f_n]) \\ &= \delta_z(\lim_{n\to\infty}[f_n]) \\ &= \delta_z([f]) \\ &= \rho([f])(z) \\ &= 0 \text{ for all } z \in \Omega.\end{aligned}$$

Since $[f_n]$ converges towards $[f]$ in $L^2(\overline{\Omega}, \mu)$, there is a μ-zero set $N \subset \overline{\Omega}$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that

$$f_{n_k}(z) \overset{k\to\infty}{\longrightarrow} f(z) \text{ for all } z \in \overline{\Omega} \setminus N.$$

We conclude, that

$$f(z) = 0 \text{ on } \Omega \cap N^c.$$

Since μ is the trivial extension of λ to $\overline{\Omega}$, the boundary of Ω is a μ-null set, so we have

$$[f] = 0.$$

Hence, the map ρ is injective. By construction $\rho(H_A^2(\mu))$ consists precisely of all functions $f : \Omega \to \mathbb{C}$ for which there is an $L^2(\overline{\Omega}, \mu)$-Cauchy sequence $(f_k)_{k\in\mathbb{N}}$ in A such that

$$f(z) = \lim_{k\to\infty} f_k(z) \text{ for all } z \in \Omega.$$

Since μ is the trivial extension of the Lebesgue measure on Ω and since

$$A|_\Omega \subset L_a^2(\Omega, \lambda)$$

is dense, it follows that

$$\rho(H_A^2(\mu)) = L_a^2(\Omega, \lambda).$$

□

Note that, for $f \in C(\overline{\Omega})$, the Toeplitz operators on $L_a^2(\Omega, \lambda)$

$$T_f = \rho(P|_{H_A^2(\mu)} M_f)\rho^{-1} \in L(L_a^2(\Omega, \lambda))$$

with respect to Definition 2.2 coincide withe the usual Toeplitz operators defined by

$$P_{L_a^2(\Omega,\lambda)}(M_{f|_\Omega})|_{L_a^2(\Omega,\lambda)}.$$

Theorem 1 in [ČŞ13] then follows as a corollary from Theorem 2.16.

Corollary 2.22. *Let Ω be a bounded pseudoregular domain in $\mathbb{C}^n$. Let $\mathcal{T}_{A(\Omega)}$ denote the Toeplitz algebra on the Bergman space on Ω with respect to the Lebesgue measure. Then $T \in \mathcal{T}_{A(\Omega)}$ is compact if and only if*

$$\lim_{z \to \partial\Omega} \Gamma(T)(z) = 0$$

Proof. By the definition of pseudoregularity, the $\overline{\partial}$-Neumann operator is compact on Ω. Thus, Proposition 4.1 and Proposition 4.2 in [Str10] imply that all Hankel operators on $L^2_a(\Omega, \lambda)$ with continuous symbols are compact. By Lemma 2.13, this is equivalent to the condition that the multiplication tuple $T_z \in L(L^2_a(\Omega, \lambda))^n$ is essentially normal. Since $\Omega \subset \mathbb{C}^n$ is a bounded domain with C^1-boundary, Theorem 3.3.4 and Theorem 3.3.6 in [JP00] yield that

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0.$$

By construction of $L^2_a(\Omega, \lambda)$, every function in $L^2_a(\Omega, \lambda)$ is holomorphic. Moreover, a remark from Čučković and Şahutoğlu in [ČŞ13] states that the strictly pseudoconvex points form a dense subset of $\partial\Omega$. Since by Theorem 2.3 in [Noe08], on bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary, every strictly pseudoconvex point is a peak point for the domain algebra, the set $\partial_p A(\Omega)$ is a dense subset of $\partial\Omega$. Hence the statements follow from Theorem 2.16. □

2.5 Generalized Bergman spaces on convex sets

We consider again arbitrary functional Hilbert spaces of the form $\mathcal{H}^2_A(\mu)$ as defined in Section 2.1. One of the central requirements in Theorem 2.16 is that for the normalized kernel function of $\mathcal{H}^2_A(\mu)$

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0.$$

In general, it is not obvious whether this condition is satisfied. However, we are able to show that it always holds for generalized Bergman spaces on convex domains in $\mathbb{C}^n$. The proof consists of several steps. We start with showing that, for convex Ω, the space $\mathcal{H}^2_A(\mu)$ has the l^∞-interpolation property, as defined in [DE12].

Definition 2.23. *A functional Hilbert space H on an open set $\Omega \subset \mathbb{C}^n$ possesses the l^∞-interpolation property if, for every sequence $(z_k)_{k\in\mathbb{N}}$ in Ω converging to a boundary point $z_0 \in \partial\Omega$, there exists a subsequence $(w_k)_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ in Ω with*

$$\{(h(w_k))_{k\in\mathbb{N}}; h \in \mathcal{M}(H)\} = l^\infty$$

In order to show that generalized Bergman spaces on convex sets possess this property, the next lemma will be helpful. It can also be found in [DE12] and it states that, for every boundary point Ω, there exists a function in A which is almost a peak function.

Lemma 2.24. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open convex set. For every $z_0 \in \partial\Omega$, there is a function $h \in A$ with $h(\Omega) \subset \mathbb{D}$ and $h(z_0) = 1$.*

Proof. Since Ω is convex and we have $z_0 \in \partial\Omega$, the separation theorems yield a homogenous polynomial $p(z) = \sum_{i=1}^{n} a_i z_i$ of degree one, satisfying

$$\operatorname{Re}(p(z)) < \operatorname{Re}(p(z_0)) \text{ for all } z \in \Omega.$$

The algebra A contains the restrictions of the polynomials, hence we have $p|_{\overline{\Omega}} \in A$. Then

$$e^p = \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

is also an element of A, since A is a Banach algebra. Therefore e^p is a multiplier on $\mathcal{H}_A^2(\mu)$. The real exponential function is strictly increasing, hence we have

$$\left|e^{p(z)}\right| = e^{\operatorname{Re}(p(z))} < e^{\operatorname{Re}(p(z_0))} = \left|e^{p(z_0)}\right|$$

for all $z \in \Omega$. The function $h : \overline{\Omega} \to \mathbb{C}$ with

$$h(z) = \frac{e^{p(z)}}{e^{p(z_0)}} \text{ for all } z \in \overline{\Omega}$$

then possesses the required properties. $\square$

By Lemma 2.3 and Remark 1.11, the restriction of the function h to Ω is then a multiplier on $\mathcal{H}_A^2(\mu)$ with multiplier norm

$$\|h|_\Omega\|_{\mathcal{M}} = \|h\|_{\infty,\Omega} = 1.$$

The following lemma will allow us to show that the contraction $M_{h|_\Omega}$ possesses a w^*-continuous $H^\infty(\mathbb{D})$-functional calculus.

Lemma 2.25. *The operator $T = M_{h|_\Omega} \in L(\mathcal{H}_A^2(\mu))$ is a contraction of class $[C_{.0}]$.*

Proof. It is to show that

$$SOT - \lim_{k\to\infty} T^{*k} = 0,$$

which means that for every $f \in \mathcal{H}_A^2(\mu)$,

$$\lim_{k\to\infty} \|T^{*k} f\| = 0.$$

The space $\mathcal{H}_A^2(\mu)$ is a scalar-valued functional Hilbert space, hence it possesses a reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$ for which

$$\vee(\{K(\cdot, z) ; z \in \Omega\}) = \mathcal{H}_A^2(\mu).$$

For a function $f = K(\cdot, z)$ with $z \in \Omega$, we can calculate

$$\begin{aligned}\|T^{*k} f\| &= \|M_{h|_\Omega}^{*k} K(\cdot, z)\| \\ &= \|\overline{h(z)}^k K(\cdot, z)\| \\ &= |h(z)|^k \|f\| \overset{k\to\infty}{\longrightarrow} 0,\end{aligned}$$

for all $z \in \Omega$, since $|h(z)| < 1$ and for every $g \in \mathcal{H}_A^2(\mu)$

$$\begin{aligned}\langle M_{h|_\Omega}^{*k} f, g\rangle &= \langle f, M_{h|_\Omega}^k g\rangle \\ &= \langle f, h|_\Omega^k g\rangle \\ &= \overline{\langle h|_\Omega^k g, f\rangle} \\ &= \overline{\langle h|_\Omega^k g, K(\cdot, z)\rangle} \\ &= \overline{h|_\Omega^k(z) g(z)} \\ &= \langle \overline{h|_\Omega^k(z)} f, g\rangle\end{aligned}$$

With help of the triangle inequality, we also get

$$\lim_{k\to\infty} \|T^{*k} f\| = 0 \text{ for all } f \in \operatorname{span}(\{\mathrm{K}(\cdot, \mathrm{z}); \mathrm{z} \in \Omega\}).$$

The sequence of operators $\left(T^{*k}\right)_{k\in\mathbb{N}}$ is norm-bounded, since T is a contraction. The latter and the convergence of the sequence on a dense subset of $\mathcal{H}_A^2(\mu)$ prove the SOT-convergence. □

Concluding from Lemma 1.8 that T is a completely non-unitary contraction, Corollary 14.1.14 in [DAE+03] yields a unique w^*-continuous algebra homomorphism

$$\Phi : H^\infty(\mathbb{D}) \to L\left(\mathcal{H}_A^2(\mu)\right)$$

with $\|\Phi\| = 1$, $\Phi(1) = 1_{\mathcal{H}_A^2(\mu)}$ and $\Phi(z) = T$. The next lemma specializes how the homomorphism Φ looks like in this case.

Lemma 2.26. *In the situation above, the algebra homomorphism Φ acts as*

$$\Phi(f) = M_{f\circ(h|_\Omega)} \textit{ for all } f \in H^\infty(\mathbb{D}).$$

Proof. The map Φ is an algebra homomorphism with $\Phi(z) = T$, hence for a polynomial p, it acts as

$$\Phi(p) = p(T) = p\left(M_{h|_\Omega}\right) = M_{p\circ(h|_\Omega)}.$$

Let now $f \in H^\infty(\mathbb{D})$ be arbitrary. According to Exercise II.4 in [Gar81] and Lemma 14.1.6 in [DAE+03], the polynomials are w^*-sequentially dense in $H^\infty(\mathbb{D})$.

Thus there exists a sequence $(p_k)_k$ of polynomials converging to f with respect to the w^*-topology. Since Φ is w^*-continuous,

$$\Phi(p_k) \overset{k\to\infty}{\longrightarrow} \Phi(f) \text{ in } \left(L\left(\mathcal{H}_A^2(\mu)\right), \tau_{w*}\right).$$

This implies convergence in the weak operator topology and since, according to Lemma 14.1.6 in [DAE+03], pointwise convergence in $H^\infty(\mathbb{D})$ follows from w^*-convergence, it follows that

$$\begin{aligned}
(\Phi(f)u)(z) &= \langle \Phi(f)u, K(\cdot,z)\rangle \\
&= \lim_{k\to\infty} \langle \Phi(p_k)u, K(\cdot,z)\rangle \\
&= \lim_{k\to\infty} \langle M_{p_k\circ(h|_\Omega)}u, K(\cdot,z)\rangle \\
&= \lim_{k\to\infty} (p_k\circ(h|_\Omega))(z)\,u(z) \\
&= (f\circ h)(z)\,u(z) \text{ for all } u\in\mathcal{H}_A^2(\mu) \text{ and } z\in\Omega.
\end{aligned}$$

□

Corollary 2.27. *Let the function h be as in the preceeding lemma. Then for every $f\in H^\infty(\mathbb{D})$,*

$$f\circ(h|_\Omega)\in\mathcal{M}(\mathcal{H}_A^2(\mu)).$$

The next lemma, originally stated in [DE12] even for functional Banach spaces, indicates why these results are important.

Lemma 2.28. *Let H be a functional Hilbert space on a bounded open set $\Omega\subset\mathbb{C}^n$ with $1\in H$. If for every boundary point $z_0\in\partial\Omega$, there exists a multiplier h of H such that $h(\Omega)\subset\mathbb{D}$,*

$$\lim_{z\to z_0} h(z) = 1$$

and

$$\{f\circ h; f\in H^\infty(\mathbb{D})\}\subset\mathcal{M}(H),$$

then H has the l^∞-interpolation property.

Proof. In the proof we will use the well-known fact that the classical Hardy space on the unit disk possesses the l^∞-interpolation property (see [Hof62], p.204). If we consider a sequence $(z_k)_{k\in\mathbb{N}}$ in Ω with

$$z_k \overset{k\to\infty}{\longrightarrow} z_0$$

for a point z_0 in $\partial\Omega$, then for a multiplier $h\in\mathcal{M}(H)$ as above, the sequence $(h(z_k))_{k\in\mathbb{N}}$ is a sequence in the open unit disk $\mathbb{D}\subset\mathbb{C}$ with

$$\lim_{k\to\infty} h(z_k) = 1.$$

Since $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$ and the classical Hardy space possesses the l^∞-interpolation property, we can find a subsequence $(w_k)_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ with

$$\{(f(h(w_k)))_{k\in\mathbb{N}}; f \in H^\infty(\mathbb{D})\} = l^\infty.$$

We required that $f \circ h \in \mathcal{M}(H)$ for all $f \in H^\infty(\mathbb{D})$. So we obtain

$$\{(g(w_k))_{k\in\mathbb{N}}; g \in \mathcal{M}(H)\} \supset l^\infty.$$

In addition, since by Remark 1.11 every multiplier $g \in \mathcal{M}(H)$ is bounded on Ω, it follows

$$\{(g(w_k))_{k\in\mathbb{N}}; g \in \mathcal{M}(H)\} = l^\infty.$$

□

Theorem 2.29. *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex open set and $\mathcal{H}^2_A(\mu)$ a functional Hilbert space on Ω as in Section 2.1. Then the space $\mathcal{H}^2_A(\mu)$ possesses the l^∞-interpolation property.*

Proof. Lemma 2.24 and Corollary 2.27 allow us to apply Lemma 2.28 from which the assertion follows. □

Lemma 2.30. *Let H be a functional Hilbert space on a bounded open set $\Omega \subset \mathbb{C}^n$ with reproducing kernel $K : \Omega \times \Omega \longrightarrow \mathbb{C}$ such that H has the l^∞- property and $1 \in H$. Then we have*

$$w - \lim_{z\to\partial\Omega} k(\cdot, z) = 0.$$

Proof. Assume that there were a zero-neighbourhood $U \subset H$ with respect to the weak topology and a sequence $(z_j)_{j\in\mathbb{N}}$ such that

$$\operatorname{dist}(z_j, \partial\Omega) \overset{j\to\infty}{\longrightarrow} 0,$$

and

$$k(\cdot, z_j) \notin U \text{ for all } j.$$

Since the space H possesses the l^∞-interpolation property, there is a subsequence $(z_{j_k})_{k\in\mathbb{N}}$ of $(z_j)_{j\in\mathbb{N}}$ such that

$$\{(h(z_{j_k}))_{k\in\mathbb{N}}; h \in \mathcal{M}(H)\} = l^\infty.$$

By the proof of Lemma 3.2 in [DE12], we then conclude that

$$\{(\langle f, k(\cdot, z_{j_k})\rangle)_{k\in\mathbb{N}}; f \in H\} = l^2.$$

We infer that $(k(\cdot, z_{j_k}))_{k\in\mathbb{N}}$ is a weak zero sequence, which is a contradiction to our assumption. Hence the assumption was wrong and

$$w - \lim_{z\to\partial\Omega} k(\cdot, z) = 0.$$

□

Hence we conclude that for a bounded convex open set $\Omega \subset \mathbb{C}^n$, the normalized kernel functions converge to zero for $z \longrightarrow \partial\Omega$.

Lemma 2.31. *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex open set and let $\mathcal{H}_A^2(\mu)$ be a functional Hilbert space on Ω as defined in Section 2.1. Then*

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0.$$

Proof. By Theorem 2.29, $\mathcal{H}_A^2(\mu)$ possesses the l^∞-interpolation property. The result then follows from Lemma 2.30. □

We can now formulate a corollary of Theorem 2.16 for functional Hilbert spaces of the form $\mathcal{H}_A^2(\mu)$ as in Section 2.1 on bounded convex open sets $\Omega \subset \mathbb{C}^n$.

Corollary 2.32. *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex open set and let $\mathcal{H}_A^2(\mu)$ be a functional Hilbert space on Ω as defined in Section 2.1, which in addition satisfies $\mathcal{H}_A^2(\mu) \subset \mathcal{O}(\Omega)$. Assume that $T_z \in L(\mathcal{H}_A^2(\mu))^n$ is essentially normal and that $\partial_p A \subset \partial\Omega$ is a dense subset. Then an operator $T \in \mathcal{T}_A$ is compact if and only if*

$$\lim_{z \to \partial\Omega} \Gamma(T)(z) = 0.$$

On the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$, it turns out that we can apply Theorem 2.16 not only to the Bergman space with respect to the Lebesgue measure, but also to a class of Bergman spaces formed with respect to a larger class of measures. In his paper "Compact Toeplitz operators with continuous symbols" ([Le09]), Trieu Le considers a normalized regular positive Borel measure μ on the unit interval in $\mathbb{R}$ with $1 \in \operatorname{supp}(\mu)$ and the unique rotation invariant probability measure σ on the unit sphere. Let $\tilde{\nu} = \mu \times \sigma$ denote their product measure on $[0,1] \times \partial\mathbb{B}_n$. The measure $\tilde{\nu}$ is again a positive Borel measure. The map

$$\lambda_\mu : C\left(\overline{\mathbb{B}}_n\right) \longrightarrow \mathbb{C}, f \mapsto \int_{[0,1]\times\partial\mathbb{B}_n} f(r\xi)\, d(\mu \times \sigma)(r, \xi)$$

is a positive linear functional with

$$\|\lambda_\mu\| = \lambda_\mu(1) = 1.$$

By the Riesz representation theorem , there is a unique positive regular Borel measure $\nu = \nu_\mu$ on $\overline{\mathbb{B}}_n$ such that

$$\int_{\overline{\mathbb{B}}_n} f d\nu = \lambda_\mu(f)$$

is valid for all $f \in C\left(\overline{\mathbb{B}}_n\right)$. We then look at the Bergman space defined as

$$L_a^2(\overline{\mathbb{B}}_n, \nu) = H^2_{A(\mathbb{B}_n)}(\nu) = \overline{A(\mathbb{B}_n)}^{L^2(\overline{\mathbb{B}}_n, \nu)}.$$

By Propositions 1.4.8 and 1.4.9 in [Rud80], for multiindices $\alpha, \beta \in \mathbb{N}^n$, we can calculate

$$\begin{aligned}\int_{\overline{\mathbb{B}}_n} z^\alpha \bar{z}^\beta d\nu\left(z\right) &= \int_{[0,1]\times\partial\mathbb{B}_n} r^{|\alpha|+|\beta|}\xi^\alpha\bar{\xi}^\beta d\left(\mu\times\sigma\right)\left(r,\xi\right)\\ &= \left(\int_{[0,1]} r^{|\alpha|+|\beta|}d\mu\right)\left(\int_{\partial\mathbb{B}_n}\xi^\alpha\bar{\xi}^\beta d\sigma\left(\xi\right)\right)\\ &= \begin{cases} 0, & \alpha\neq\beta\\ \dfrac{(n-1)!\alpha!}{(n+|\alpha|-1)!}\displaystyle\int_0^1 r^{2|\alpha|}d\mu, & \alpha=\beta\end{cases}.\end{aligned}$$

We define for $k \in \mathbb{N}$

$$a_k = \int_{[0,1]} r^{2k} d\mu\left(>0\right).$$

Hence, we obtain an orthonormal basis of $L_a^2(\overline{\mathbb{B}}_n, \nu)$ consisting of the functions

$$e_\alpha = \left(\frac{(n+|\alpha|-1)!}{(n-1)!\alpha! a_{|\alpha|}}\right)^{\frac{1}{2}} z^\alpha \left(\alpha\in\mathbb{N}^n\right).$$

The orthonormality follows from the calculation above. To see that $(e_\alpha)_{\alpha\in\mathbb{N}^n}$ is complete, not that the closed linear span of

$$\{e_\alpha; \alpha\in\mathbb{N}^n\}$$

is a closed subset of $L^2(\overline{\mathbb{B}}_n, \nu)$ that contains all polynomials and therefore also the algebra

$$A\left(\mathbb{B}_n\right) = \overline{\mathbb{C}[z]|_{\overline{\mathbb{B}}_n}}^{C(\overline{\mathbb{B}}_n)}.$$

We begin by stating a rather technical lemma that will be helpful later on.

Lemma 2.33. *For the orthonormal basis $(e_\alpha)_{\alpha\in\mathbb{N}^n}$ defined above, the sum*

$$\sum_{\alpha\in\mathbb{N}^n} |e_\alpha(z)|^2$$

is convergent for all $z \in \mathbb{B}_n$.

Proof. For $z \in \mathbb{B}_n$, we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^n}|e_\alpha(z)|^2 &= \sum_{\alpha\in\mathbb{N}^n}\frac{(n+|\alpha|-1)!}{(n-1)!a_{|\alpha|}}\frac{|z^\alpha|^2}{\alpha!}\\ &= \sum_{k=0}^\infty\frac{(n+k-1)!}{(n-1)!k!a_k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}|z^\alpha|^2\\ &= \sum_{k=0}^\infty\frac{(n+k-1)!}{(n-1)!k!a_k}\left(|z|^2\right)^k.\end{aligned}$$

We show in two steps that the last series is convergent. First we prove that

$$\lim_{k\to\infty} a_k^{\frac{1}{k}} = 1.$$

Note that, for $0 < \epsilon < 1$ and $k \geq 1$,

$$\begin{aligned}\left((1-\epsilon)^{2k}\,\mu\left([1-\epsilon,1]\right)\right)^{\frac{1}{k}} &\leq \left(\int_{[1-\epsilon,1]} r^{2k}d\mu\right)^{\frac{1}{k}}\\ &\leq \left(\int_{[0,1]} r^{2k}d\mu\right)^{\frac{1}{k}}\\ &\leq \|r^{2k}\|_{L^\infty(\mu)}^{\frac{1}{k}}.\end{aligned}$$

Since $1 \in \operatorname{supp}(\mu)$,

$$\begin{aligned}(1-\epsilon)^2 &\leq \varliminf_{k\to\infty}\,(a_k)^{\frac{1}{k}}\\ &\leq \varlimsup_{k\to\infty}(a_k)^{\frac{1}{k}}\\ &\leq \lim_{k\to\infty}\,\|r^{2k}\|_{L^\infty(\mu)}^{\frac{1}{k}}\\ &= \|r^2\|_{L^\infty(\mu)}\\ &\leq 1.\end{aligned}$$

Thus it follows that

$$\lim_{k\to\infty} a_k^{\frac{1}{k}} = 1.$$

Since for $k \geq n-1$,

$$\begin{aligned}\left(k^{\frac{1}{k}}\right)^{n-1}\left(\frac{1}{a_k}\right)^{\frac{1}{k}} &\leq \left(\frac{(k+1)\cdots(k+(n-1))}{a_k}\right)^{\frac{1}{k}}\\ &= \left(\frac{(n+k-1)!}{k!a_k}\right)^{\frac{1}{k}}\\ &\leq \left((2k)^{(n-1)}\right)^{\frac{1}{k}}\left(\frac{1}{a_k}\right)^{\frac{1}{k}},\end{aligned}$$

it follows that the power series

$$\sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!k!a_k}z^k$$

has radius of convergence $R = 1$. $\square$

In order to apply Theorem 2.16 to the situation above, we have to show that the Bergman spaces on the unit ball, with respect to measures ν defined as above, are functional Hilbert spaces of the form $\mathcal{H}^2_A(\mu)$.

Lemma 2.34. *Consider the Bergman space $L^2_a(\overline{\mathbb{B}}_n, \nu)$ on the closed unit ball $\overline{\mathbb{B}}_n \subset \mathbb{C}^n$ with respect to a Borel measure ν defined as above and let $A(\mathbb{B}_n)$ be the domain algebra on the unit ball in $\mathbb{C}^n$. Then the point evaluations*

$$\{[f]; f \in A(\mathbb{B}_n)\} \to \mathbb{C}, [f] \mapsto f(z) \ (z \in \mathbb{B}_n)$$

possess continuous extensions

$$\delta_z : L^2_a(\overline{\mathbb{B}}_n, \nu) \to \mathbb{C}$$

and the induced map

$$\rho : L^2_a(\overline{\mathbb{B}}_n, \nu) \to \mathbb{C}^{\mathbb{B}_n}, \rho([f])(z) = \delta_z([f]) \text{ for all } z \in \mathbb{B}_n$$

is injective with

$$\mathrm{Im}(\rho) \subset \mathcal{O}(\Omega).$$

Hence $\mathrm{Im}(\rho)$ *is a functional Hilbert space of the form $\mathcal{H}^2_A(\mu)$ as in Section 2.1 which in addition satisfies $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$.*

Proof. Let us first consider a polynomial $p \in \mathbb{C}[z]$. We can write p as

$$p(z) = \sum_{|\alpha| \leq N} c_\alpha e_\alpha(z) \text{ for } z \in \mathbb{B}_n.$$

Applying the Cauchy-Schwarz inequality leads to

$$\begin{aligned} |p(z)| &\leq \left(\sum_{|\alpha| \in \mathbb{N}^n} |e_\alpha(z)|^2 \right)^{\frac{1}{2}} \left(\sum_{|\alpha| \leq N} |c_\alpha|^2 \right)^{\frac{1}{2}} \\ &= C_z \|p\|_{L^2(\overline{\mathbb{B}}_n, \nu)}, \end{aligned}$$

where

$$C_z = \left(\sum_{\alpha \in \mathbb{N}^n} |e_\alpha(z)|^2 \right)^{\frac{1}{2}}.$$

Consider a function $f \in A(\mathbb{B}_n)$. Since $A(\mathbb{B}_n)$ is the closure of the set of holomorphic polynomials in $C(\overline{\mathbb{B}}_n)$, we can find a sequence $(p_k)_{k\in\mathbb{N}}$ in $\mathbb{C}[z]$ with

$$p_k \overset{k\to\infty}{\longrightarrow} f$$

uniformly on $\overline{\mathbb{B}}_n$. This implies $\|f - p_k\| \overset{k\to\infty}{\longrightarrow} 0$ in $L^2(\overline{\mathbb{B}}_n, \nu)$ and therefore

$$\begin{aligned} |f(z)| &= \lim_{k\to\infty} |p_k(z)| \\ &\le C_z \lim_{k\to\infty} \|p_k\|_{L^2(\overline{\mathbb{B}}_n,\nu)} \\ &= C_z \|f\|_{L^2\left(\overline{\mathbb{B}}_n,\nu\right)} \text{ for all } z \in \mathbb{B}_n. \end{aligned}$$

Hence

$$\left(\{[f] \in L^2(\overline{\mathbb{B}}_n, \nu); f \in A(\mathbb{B}_n)\}, \|\cdot\|_{L^2(\mathbb{B}_n,\nu)}\right) \longrightarrow \mathbb{C}, [f] \mapsto f(z)$$

is a well defined continuous linear map and has a unique continuous linear extension

$$\delta_z : L^2_a(\overline{\mathbb{B}}_n, \nu) \longrightarrow \mathbb{C}.$$

If

$$f = \sum_{\alpha\in\mathbb{N}} c_\alpha e_\alpha \in L^2_a(\overline{\mathbb{B}}_n, \nu)$$

satisfies

$$\delta_z(f) = 0$$

for all $z \in \mathbb{B}_n$, it follows that

$$\begin{aligned} \sum_{\alpha\in\mathbb{N}^n} c_\alpha e_\alpha(z) &= \sum_{\alpha\in\mathbb{N}^n} c_\alpha \delta_z(e_\alpha) \\ &= \delta_z\left(\sum_{\alpha\in\mathbb{N}^n} c_\alpha e_\alpha\right) \\ &= 0 \text{ for all } z \in \mathbb{B}_n. \end{aligned}$$

The series above can be seen as a power series

$$\sum_{\alpha\in\mathbb{N}^n} \tilde{c}_\alpha z^\alpha,$$

as $e_\alpha = k_\alpha z^\alpha$ for some $k_\alpha > 0$. Since this power series is converging to zero on the whole unit ball, it follows that all the coefficients are zero. So $\tilde{c}_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$ and hence $c_\alpha = \frac{\tilde{c}_\alpha}{k_\alpha} = 0$, which leads to $f = 0$. Altogether, this means that the map

$$\rho : L^2_a(\overline{\mathbb{B}}_n, \nu) \longrightarrow \mathbb{C}^{\mathbb{B}_n},\ \rho(f)(z) = \delta_z(f)$$

is well defined, linear and injective. For $f \in L^2_a(\overline{\mathbb{B}}_n, \nu)$, $\rho(f) \in \mathcal{O}(\mathbb{B}_n)$ is the holomorphic function with power series expansion

$$\rho(f)(z) = \sum_{\alpha\in\mathbb{N}^n} \langle f, e_\alpha\rangle_{L^2(\overline{\mathbb{B}}_n,\nu)} e_\alpha(z)\ (z \in \mathbb{B}_n).$$

. $\square$

We denote by

$$\mathcal{H}^2_{A(\mathbb{B}_n)}(\nu) = \rho L^2_a(\overline{\mathbb{B}}_n, \nu) \subset \mathcal{O}(\mathbb{B}_n)$$

the functional Hilbert space associated with

$$L^2_a(\overline{\mathbb{B}}_n, \nu) = H^2_{A(\mathbb{B}_n)}(\nu)$$

as explained in Section 2.1. In a next step, we show that the multiplication tuple with the coordinate functions on such a space is essentially normal.

Lemma 2.35. *Consider the Bergman space $L^2_a(\overline{\mathbb{B}}_n, \nu)$ on the closed unit ball $\overline{\mathbb{B}}_n \subset \mathbb{C}^n$ with respect to a Borel measure ν defined as above. Then the multiplication tuple*

$$T_z = (T_{z_1}, \cdots, T_{z_n}) \in L(\mathcal{H}^2_{A(\mathbb{B}_n)}(\nu))^n$$

is essentially normal.

Proof. This lemma can be found in [ČC99], but for the sake of completeness we include a proof. We have to show that the operator $T_k = T^*_{z_k} T_{z_k} - T_{z_k} T^*_{z_k}$ is compact for every $1 \le k \le n$. We will only consider the case $k = 1$, as the other cases can be treated analogously. The strategy is to approximate the operator T_1 by a sequence of finite rank operators. Therefore, we define operators S_m, $m \in \mathbb{N}$, by

$$S_m(e_\alpha) = \begin{cases} (T^*_{z_1} T_{z_1} - T_{z_1} T^*_{z_1})(e_\alpha), \; \alpha \in \mathbb{N}^n \text{ with } |\alpha| \le m, \\ 0 \qquad\qquad\qquad\qquad\qquad\quad , \text{ otherwise.} \end{cases}$$

With the definition $w_\alpha = \frac{\|z^{\alpha+e_1}\|}{\|z^\alpha\|}$, where $e_1 \in \mathbb{N}^n$ with $e_{11} = 1$ and $e_{1i} = 0$ for $i \in \{2, \dots, m\}$, we obtain that

$$T_{z_1}(e_\alpha) = z_1(\frac{z^\alpha}{\|z^\alpha\|}) = \frac{z^{\alpha+e_1}}{\|z^\alpha\|} = w_\alpha \frac{z^{\alpha+e_1}}{\|z^{\alpha+e_1}\|}.$$

Furthermore, we calculate

$$\begin{aligned} T^*_{z_1}(e_\alpha) &= \sum_{\beta \in \mathbb{N}^n} \left\langle T^*_{z_1}\left(\frac{z^\alpha}{\|z^\alpha\|}\right), \frac{z^\beta}{\|z^\beta\|} \right\rangle e_\beta \\ &= \sum_{\beta \in \mathbb{N}^n} \left\langle \frac{z^\alpha}{\|z^\alpha\|}, T_{z_1}\left(\frac{z^\beta}{\|z^\beta\|}\right) \right\rangle e_\beta \\ &= \sum_{\beta \in \mathbb{N}^n} \left\langle \frac{z^\alpha}{\|z^\alpha\|}, w_\beta \frac{z^{\beta+e_1}}{\|z^{\beta+e_1}\|} \right\rangle e_\beta \\ &= w_{\alpha-e_1} e_{\alpha-e_1}, \end{aligned}$$

where we read the last term as zero in the case $\alpha_1 = 0$.

This means

$$T_1(e_\alpha) = (w_\alpha^2 - w_{\alpha-e_1}^2)\frac{z_\alpha}{\|z^\alpha\|}.$$

The weights w_α are given by

$$\begin{aligned} w_\alpha^2 &= (\frac{\|z^{\alpha+e_1}\|}{\|z^\alpha\|})^2 \\ &= \frac{\frac{(n-1)!(\alpha+e_1)!}{(n+|\alpha+e_1|-1)!}\int_0^1 r^{2|\alpha+e_1|}d\mu}{\frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}\int_0^1 r^{2|\alpha|}d\mu} \\ &= \frac{(\alpha_1+1)\int_0^1 r^{2|\alpha|+2}d\mu(r)}{(n+|\alpha|)\int_0^1 r^{2|\alpha|}d\mu(r)}. \end{aligned}$$

As an application of Hölder's inequality, we obtain that

$$\begin{aligned} \left(\int_0^1 r^{2|\alpha|}d\mu(r)\right)^2 &= \left(\int_0^1 r^{|\alpha|+1}r^{|\alpha|-1}d\mu(r)\right)^2 \\ &\leq \left(\int_0^1 r^{2|\alpha|+2}d\mu(r)\right)\left(\int_0^1 r^{2|\alpha|-2}d\mu(r)\right) \end{aligned}$$

for all $\alpha \in \mathbb{N}^n \setminus \{0\}$, or equivalently, that

$$\frac{\int_0^1 r^{2|\alpha|}d\mu(r)}{\int_0^1 r^{2|\alpha|-2}d\mu(r)} \leq \frac{\int_0^1 r^{2|\alpha|+2}d\mu(r)}{\int_0^1 r^{2|\alpha|}d\mu(r)}.$$

In consequence, the function $\rho : \mathbb{N} \to [0,1]$ defined by

$$\rho(k) = \frac{\left(\int_0^1 r^{2k+2}d\mu(r)\right)}{\left(\int_0^1 r^{2k}d\mu(r)\right)}$$

is increasing. Thus, for $k \to \infty$, it converges to some $\rho_0 \in [0,1]$. We look at the difference

$$\Delta(\alpha) = w_\alpha^2 - w_{\alpha-e_1}^2 = \frac{\alpha_1+1}{n+|\alpha|}\rho(|\alpha|) - \frac{\alpha_1}{n+|\alpha|-1}\rho(|\alpha|-1)$$

and calculate, as ρ is an increasing function,

$$\begin{aligned} \Delta(\alpha) &\geq \frac{\alpha_1+1}{n+|\alpha|}\rho(|\alpha|-1) - \frac{\alpha_1}{n+|\alpha|-1}\rho(|\alpha|-1) \\ &= \left(\frac{(\alpha_1+1)(n+|\alpha|-1)-\alpha_1(n+|\alpha|)}{(n+|\alpha|)(n+|\alpha|-1)}\right)\rho(|\alpha|-1) \\ &= \left(\frac{n+|\alpha|-\alpha_1-1}{|\alpha|^2+(2n-1)|\alpha|+n^2-n}\right)\rho(|\alpha|-1). \end{aligned}$$

The last term converges to zero as $|\alpha| \longrightarrow \infty$, since

$$\begin{aligned}0 = \lim_{|\alpha|\to\infty} \frac{n-1}{|\alpha|^2 + (2n-1)|\alpha| + n^2 - n} \\ \leq \lim_{|\alpha|\to\infty} \frac{n + |\alpha| - \alpha_1 - 1}{|\alpha|^2 + (2n-1)|\alpha| + n^2 - n} \\ \leq \lim_{|\alpha|\to\infty} \frac{n + |\alpha|}{|\alpha|^2 + (2n-1)|\alpha| + n^2 - n} = 0.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\Delta(\alpha) &\leq \frac{\alpha_1 + 1}{n + |\alpha|}\rho(|\alpha|) - \frac{\alpha_1}{n + |\alpha|}\rho(|\alpha| - 1) \\ &= \left(\frac{\alpha_1}{n + |\alpha|}\right)(\rho(|\alpha|) - \rho(|\alpha| - 1)) + \frac{1}{n + |\alpha|}\rho(|\alpha|) \\ &\leq (\rho(|\alpha|) - \rho(|\alpha| - 1)) + \frac{1}{n + |\alpha|}\rho(|\alpha|),\end{aligned}$$

which also converges to zero as $|\alpha| \to \infty$. Thus

$$\lim_{|\alpha|\to\infty} \Delta(\alpha) = 0.$$

Hence the operator T_1 is compact as a limit of finite rank operators. A similar argument yields the compactness of the operators

$$T_k, k \in \{2, \dots, n\}.$$

Hence $T_z \in L(\mathcal{H}^2_{A(\mathbb{B}_n)}(\nu))^n$ is essentially normal. □

Therefore, we can formulate the following corollary.

Corollary 2.36. *Let $L^2_a(\overline{\mathbb{B}}_n, \nu)$ be a Bergman space on the closed unit ball $\overline{\mathbb{B}}_n \subset \mathbb{C}^n$ with respect to a Borel measure ν defined as above. Then an operator $T \in \mathcal{T}_{A(\mathbb{B}_n)}$ is compact if and only if*

$$\lim_{|z|\to 1} \Gamma(T)(z) = 0.$$

Proof. Note that since the unit ball in $\mathbb{C}^n$ is strictly pseudoconvex, Theorem 2.3 in [Noe08] implies that every point in its boundary is a peak point for the ball algebra $A(\mathbb{B}_n)$. By Lemma 2.34, Lemma 2.35, and the convexity of $\mathbb{B}_n$, the requirements of Corollary 2.32 are satisfied. This yields the result. □

As a special case, we obtain Theorem 1.1 in [Le09].

Corollary 2.37. *Let $L^2_a(\overline{\mathbb{B}}_n, \nu)$ be a Bergman space on the closed unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ with respect to a Borel measure ν defined as above. Then a Toeplitz operator $T_\varphi \in L(\mathcal{H}^2_{A(\mathbb{B}_n)}(\nu))$ with continuous symbol $\varphi \in C(\overline{\mathbb{B}}_n)$ is compact if and only if*

$$\varphi|_{\partial\mathbb{B}_n} = 0.$$

Proof. As $\mathbb{B}_n$ is convex, Lemma 2.29 and Lemma 2.30 lead to

$$w - \lim_{|z|\to 1} k(\cdot, z) = 0.$$

Thus we can apply Satz 6.18 from [Kre11] and obtain

$$\lim_{z\to z_0} \Gamma(T_\varphi)(z) = \varphi(z_0)$$

for all peak points $z_0 \in \partial\mathbb{B}_n$. Since every point in the boundary of the unit ball is a peak point for the ball algebra, we have

$$\lim_{z\to z_0} \Gamma(T_\varphi)(z) = \varphi(z_0) \text{ for all } z_0 \in \partial\Omega.$$

By Corollary 2.36, we conclude that the compactness of T_φ is equivalent to

$$\varphi(z_0) = \lim_{z\to z_0} \Gamma(T_\varphi)(z) = 0 \text{ for all } z_0 \in \partial\Omega.$$

□

Note that in the setting of Corollary 2.37, the Toeplitz operators

$$T_\varphi \in L(\mathcal{H}^2_{A(\mathbb{B}_n)}(\nu)), \quad \varphi \in C(\overline{\mathbb{B}}_n),$$

are canonically unitarily equivalent to the Toeplitz operators

$$T_\varphi = P_{L^2_a(\overline{\mathbb{B}}_n,\nu)} M_\varphi|_{L^2_a(\overline{\mathbb{B}}_n,\nu)} \in L(L^2_a(\overline{\mathbb{B}}_n,\nu))$$

via the unitary map

$$\rho : L^2_a(\overline{\mathbb{B}}_n,\nu) \to \mathcal{H}^2_{A(\mathbb{B}_n)}(\nu).$$

3 Exact Toeplitz Sequences

In Theorem 2.6, we showed that, on suitable analytic functional Hilbert spaces, a Toeplitz operator T_f with $f \in C(\overline{\Omega})$ is compact if its symbol vanishes on the boundary of Ω. In this chapter, we will prove that if the multiplication tuple T_z is essentially normal, it suffices to know that the restriction of the symbol to a certain subset of the boundary, namely the essential spectrum of the operator tuple T_z, is the zero-function. This enables us to replace the requirement $\overline{\partial_p A} = \partial\Omega$ in Theorem 2.16 by a weaker condition.

We consider a functional Hilbert space of the form $\mathcal{H}^2_A(\mu)$ on a bounded domain $\Omega \subset \mathbb{C}^n$ as in Section 2.1 and the corresponding Toeplitz algebra $\mathcal{T}_A = C^*(T_z)$. Furthermore, we assume from now on that the multiplication tuple $T_z \in L(\mathcal{H}^2_A(\mu))^n$ is essentially normal.

3.1 Joint spectra

Since we want to determine the compact operators in $\mathcal{T}_A$, it will be useful to introduce the Calkin algebra on $\mathcal{H}^2_A(\mu)$, which is the quotient of the bounded operators on $\mathcal{H}^2_A(\mu)$ modulo the compact operators.

Definition 3.1. *Let $\mathcal{K}$ denote the set of compact operators on $\mathcal{H}^2_A(\mu)$. Then we call*

$$\mathcal{C}(\mathcal{H}^2_A(\mu)) = L(\mathcal{H}^2_A(\mu))/\mathcal{K}$$

the Calkin algebra for $\mathcal{H}^2_A(\mu)$ and we denote by

$$\pi : L(\mathcal{H}^2_A(\mu)) \to \mathcal{C}(\mathcal{H}^2_A(\mu)), T \mapsto \pi(T) = [T]$$

the quotient mapping.

We now look at the range of the C^*-algebra $\mathcal{T}_A$ under the quotient mapping π into the Calkin algebra.

Remark 3.2. *The set*

$$\mathcal{C}_A = \pi(\mathcal{T}_A) \subset \mathcal{C}(\mathcal{H}^2_A(\mu))$$

is a C^*-algebra, as π is a $*$-homomorphism. The essential normality of T_z ensures that $\mathcal{C}_A$ is commutative, since by Lemma 2.13, operators of the form

$$T_f T_g - T_g T_f \text{ for } f, g \in C(\overline{\Omega})$$

are compact. Since

$$\mathcal{C}_A = \pi(\mathcal{T}_A) = \pi(C^*(T_z)) = C^*(\pi(T_z)) = C^*([T_z]) \subset \mathcal{C}(\mathcal{H}^2_A(\mu)),$$

$\mathcal{C}_A$ is generated by $[T_z] = ([T_{z_1}], \dots, [T_{z_n}])$.

It turns out that there exists a strong connection between the elements of $\mathcal{C}_A$ and the continuous functions on the joint spectrum of the tuple $[T_z]$ in $\mathcal{C}_A$. The latter is introduced in the next definition.

Definition 3.3. *Let C be a commutative unital Banach algebra and let $x = (x_1, \dots, x_n) \in C^n$ be a finite tuple. We define the joint spectrum of x by*

$$\sigma_C(x) = \{\lambda \in \mathbb{C}^n; 1 \notin \sum_{i=1}^{n} (\lambda_i - x_i)C\}.$$

In a first step, we observe that there exists a homeomorphism between the Gelfand space of C and the joint spectrum of a generating tuple of C.

Lemma 3.4. *Let C be a commutative unital C^*- algebra and let Δ_C denote the Gelfand space of C, which is defined as the space of the non-trivial multiplicative linear functionals of C equipped with the relative w^*-topology of C'. Suppose that the C^*-algebra C is generated by a finite tuple $x = (x_1, \dots, x_n) \in C^n$. Then the map*

$$\kappa : \Delta_C \longrightarrow \sigma_C(x), \lambda \mapsto \lambda(x) = (\lambda(x_1), \dots, \lambda(x_n))$$

is a well-defined homeomorphism.

Proof. Let $\phi \in \Delta_C$. Suppose $1 \in \sum_{i=1}^{n} (\phi(x_i) - x_i)C$. Then there exist $c_1, \dots, c_n \in C$ satisfying

$$1 = \sum_{i=1}^{n} (\phi(x_i) - x_i)c_i.$$

Hence

$$1 = \phi(1) = \phi(\sum_{i=1}^{n} (\phi(x_i) - x_i)c_i) = \sum_{i=1}^{n} (\phi(x_i) - \phi(x_i))\phi(c_i) = 0,$$

so our assumption was wrong. Therefore we conclude $1 \notin \sum_{i=1}^{n} (\phi(x_i) - x_i)C$, which means $\phi(x) \in \sigma_C(x)$. So the map κ is well-defined. In order to show that it is also surjective, let $\lambda \in \sigma_C(x)$. Then

$$\mathcal{I} = \sum_{i=1}^{n} (\lambda_i - x_i)C \subset C$$

is a proper ideal. By Zorn's lemma, every proper ideal is contained in a maximal ideal. We can therefore find a maximal ideal $\mathcal{I}_M$ in C with

$$\mathcal{I} \subset \mathcal{I}_M.$$

By standard Gelfand theory, there exists a non-trivial multiplicative linear functional $\phi \in \Delta_C$ such that

$$\ker(\phi) = \mathcal{I}_{\mathrm{M}}.$$

Hence

$$\mathcal{I} \subset \ker(\phi) \subset \mathrm{C}.$$

Therefore,

$$\phi(\sum_{i=1}^{n} (\lambda_i - x_i)c_i) = 0 \text{ for all } c_i \in C.$$

For $k \in \{1, \dots, n\}$, choosing $c_i = 0$ for $i \neq k$ and $c_k = 1$ leads to

$$\lambda_k - \phi(x_k) = 0,$$

and we obtain

$$\lambda = \phi(x).$$

Since the functionals $\lambda \in \Delta_C$ are $*$-homomorphisms, and in particular also continuous, they are uniquely determined by their action on the generators $x_1, \dots, x_n$of the C^*-algebra C. Hence κ is injective.
We would now like to show that κ is also continuous. Therefore, we consider a net $(\lambda_i)_{i \in I}$ in Δ_C converging to a non-trivial multiplicative linear form $\lambda \in \Delta_C$ with respect to the Gelfand topology. Since this topology is the restriction of the w^*-topology on C' to Δ_C, it follows

$$\lambda_i(x_j) \overset{i \to \infty}{\longrightarrow} \lambda(x_j) \text{ for all } j \in \{1, \dots, n\}.$$

Hence

$$\kappa(\lambda_i) = \lambda_i(x) \stackrel{k\to\infty}{\longrightarrow} \lambda(x) = \kappa(\lambda),$$

thus κ is continuous.
The Gelfand space Δ_C is a compact Hausdorff space and $\sigma_C(x)$ is a Hausdorff space. This implies that the continuous and bijective map κ is already a homeomorphism.

□

In particular, it follows that $\sigma_{\mathcal{C}}(x) \subset \mathbb{C}^n$ is a non-empty compact subset of $\mathbb{C}^n$.
The next lemma yields an isomorphism between $\mathcal{C}_A$ and the continuous functions on $\sigma_{\mathcal{C}_A}([T_z])$.

Lemma 3.5. *For an element $[x] \in \mathcal{C}_A$, we denote its Gelfand transform by $\widehat{[x]}$. Then the map*

$$i : \mathcal{C}_A \longrightarrow C(\sigma_{\mathcal{C}_A}([T_z])), [x] \mapsto \widehat{[x]} \circ \kappa^{-1}$$

is an isomorphism between C^*-algebras.

Proof. By Remark 3.2, the tuple $[T_z]$ generates $\mathcal{C}_A$ as a commutative unital C^*-algebra. Hence, Lemma 3.4 implies that the map

$$\kappa : \Delta_{\mathcal{C}_A} \to \sigma_{\mathcal{C}_A}([T_z]), \lambda \mapsto \lambda([T_z])$$

is a homeomorphism. It is easy to see that the function

$$\tilde{\kappa} : C(\Delta_{\mathcal{C}_A}) \to C(\sigma_{\mathcal{C}_A}([T_z])), f \mapsto f \circ \kappa^{-1}$$

is then an isomorphism of C^*-algebras. Furthermore, the Gelfand-Naimark theorem (see e.g. Theorem 1.1.1 in [Arv76]) states that the map

$$\gamma : \mathcal{C}_A \to C(\Delta_{\mathcal{C}_A}), [x] \mapsto \widehat{[x]}$$

is a $*$-isomorphism. We conclude that the composition

$$i = \tilde{\kappa} \circ \gamma : \mathcal{C}_A \longrightarrow C(\sigma_{\mathcal{C}_A}([T_z])), [x] \mapsto \widehat{[x]} \circ \kappa^{-1}$$

is then an isomorphism between C^*-algebras. □

We now obtain a C^*-homomorphism from $C(\overline{\Omega})$ into the continuous functions on $\sigma_{\mathcal{C}_A}([T_z])$.

Lemma 3.6. *The composition*

$$\Phi : C(\overline{\Omega}) \longrightarrow \mathcal{C}_A \stackrel{\sim}{\longrightarrow} C(\sigma_{\mathcal{C}_A([T_z])}), f \mapsto i([T_f])$$

with i as in Lemma 3.5 is a unital C^*-homomorphism with

$$\Phi(z_i) = z_i|_{\sigma_{\mathcal{C}_A}([T_z])} \text{ for } i \in \{1, \dots, n\},$$

where the right-hand side stands for the i-th coordinate function of $\mathbb{C}^n$ restricted to the compact set $\sigma_{\mathcal{C}_A([T_z])} \subset \mathbb{C}^n$.

Proof. By the essential normality of the multiplication tuple $T_z \in L(\mathcal{H}^2_A(\mu))^n$, it follows from Lemma 2.13 that Φ is multiplicative, since operators of the form

$$T_{fg} - T_f T_g \in L(\mathcal{H}^2_A(\mu))$$

for $f, g \in C(\overline{\Omega})$ are compact. Furthermore, since

$$T_f^* = T_{\overline{f}} \text{ for all } f \in C(\overline{\Omega}),$$

the map

$$C(\overline{\Omega}) \to \mathcal{C}_A, f \mapsto [T_f]$$

is a unital homomorphism between C^*-algebras. Together with Lemma 3.5, it follows that the map Φ is a unital $*-$homomorphism as well. For $\lambda \in \sigma_{\mathcal{C}_A}([T_z])$, we can find a $\phi \in \Delta_{\mathcal{C}_A}$ such that

$$\lambda = \kappa(\phi) = \phi([T_z]).$$

This leads to

$$\Phi(z_i)(\lambda) = \left(\widehat{[T_{z_i}]} \circ \kappa^{-1}\right)(\lambda) = \widehat{[T_{z_i}]}(\kappa^{-1}(\lambda)) = \widehat{[T_{z_i}]}(\phi) = \phi([T_{z_i}]) = \lambda_i.$$

Hence

$$\Phi(z_i) = z_i|_{\sigma_{\mathcal{C}_A}([T_z])}.$$

$\square$

We can now conclude that the joint spectrum of $[T_z]$ in $\mathcal{C}_A$ is entirely contained in the closure of Ω.

Lemma 3.7. *For the joint spectrum of* $[T_z]$*, we have*

$$\sigma_{\mathcal{C}_A}([T_z]) \subset \overline{\Omega}.$$

Proof. Since the map

$$C(\overline{\Omega}) \to \mathcal{C}_A, f \mapsto [T_f],$$

is a unital algebra homomorphism, it decreases joint spectra. Hence

$$\sigma_{\mathcal{C}_A}([T_z]) \subset \sigma_{C(\overline{\Omega})}(z_1, \ldots, z_n) = \overline{\Omega}.$$

□

We have already seen that the map Φ acts as the restriction map on the coordinate functions. The next lemma shows that this is true on the whole space $C(\overline{\Omega})$.

Corollary 3.8. *The map* Φ *defined in Lemma 3.6 acts as*

$$\Phi(f) = f|_{\sigma_{\mathcal{C}_A}([T_z])}.$$

Proof. The map Φ is a homomorphism between the unital C^*-algebras $C(\overline{\Omega}) = C^*(z_1, \ldots, z_n)$ and $C(\sigma_{\mathcal{C}_A}([T_z]))$ that coincides on $1, z_1, \ldots, z_n$ with the C^*- homomorphism

$$C(\overline{\Omega}) \to C(\sigma_{\mathcal{C}_A}([T_z])), f \mapsto f|_{\sigma_{\mathcal{C}_A}([T_z])}.$$

Since by the Stone-Weierstrass theorem

$$C(\overline{\Omega}) = C^*(z_1, \ldots z_n),$$

it follows that

$$\Phi(f) = f|_{\sigma_{\mathcal{C}_A}([T_z])} \text{ for all } f \in C(\overline{\Omega}).$$

□

3.2 The Toeplitz extension

Note that

$$\mathcal{T}_A + \mathcal{K} = \pi^{-1}(\pi(\mathcal{T}_A)) \subset L(\mathcal{H}_A^2(\mu))$$

is a C^*-subalgebra. We now use the homomorphism Φ from Lemma 3.6 to construct a homomorphism from $\mathcal{T}_A + \mathcal{K}$ into the continuous functions on $\sigma_{\mathcal{C}_A}([T_z])$.

Lemma 3.9. *The composition*

$$\Psi : \mathcal{T}_A + \mathcal{K} \xrightarrow{\pi} \mathcal{C}_A \xrightarrow{\sim} C(\sigma_{\mathcal{C}_A}([T_z]))$$

is the unique C^*-homomorphism with $\Psi|_{\mathcal{K}} = 0$ and

$$\Psi(T_f) = f|_{C(\sigma_{\mathcal{C}_A}([T_z]))} \text{ for all } f \in C(\overline{\Omega}).$$

Furthermore,

$$0 \longrightarrow \mathcal{K} \hookrightarrow \mathcal{T}_A + \mathcal{K} \xrightarrow{\Psi} C(\sigma_{\mathcal{C}_A}([T_z])) \longrightarrow 0$$

is an exact sequence of C^*-algebras. We call this sequence the Toeplitz extension (of $\mathcal{K}$ by $C(\sigma_{\mathcal{C}_A}([T_z]))$). In particular, for $f \in C(\overline{\Omega})$, the equivalence

$$T_f \in \mathcal{K} \Leftrightarrow f|_{\sigma_{\mathcal{C}_A}([T_z])} = 0$$

holds.

Proof. By the definition of Ψ, it is clear that Ψ is a C^*-homomorphism satisfying

$$\Psi|_{\mathcal{K}} = 0,$$

and since $\Psi(T_f) = \Phi(f)$, we obtain

$$\Psi(T_f) = f|_{C(\sigma_{\mathcal{C}_A}([T_z]))} \text{ for all } f \in C(\overline{\Omega}).$$

As every C^*-homomorphism on $\mathcal{T}_A + \mathcal{K}$ is already determined by its values on elements of the form $T_f + K$, where $f \in C(\overline{\Omega})$ and $K \in \mathcal{K}$, it follows that Ψ is unique. For $T \in \ker(\Psi)$, it follows $[T] = 0$ in $\mathcal{C}_A$, hence

$$\ker(\Psi) = \mathcal{K}$$

and, since π is surjective, Ψ is a surjection as well. This proves the exactness of the sequence

$$0 \longrightarrow \mathcal{K} \hookrightarrow \mathcal{T}_A + \mathcal{K} \xrightarrow{\Psi} C(\sigma_{\mathcal{C}_A}([T_z])) \longrightarrow 0$$

and the last equivalence. □

Under certain circumstances, this leads to the joint spectrum being contained in the boundary of Ω. In particular, if $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$, Theorem 2.6 yields that the condition

$$T_f \in \mathcal{K} \text{ for all } f \in C(\overline{\Omega}) \text{ with } f|_{\partial\Omega} = 0$$

in the first part of the next lemma is always satisfied.

Lemma 3.10. *If $T_f \in \mathcal{K}$ holds for all $f \in C(\overline{\Omega})$ with $f|_{\partial\Omega} = 0$, then $\sigma_{\mathcal{C}_A}([T_z])$ is entirely contained in the boundary of Ω. Furthermore, if the equivalence*

$$T_f \in \mathcal{K} \Leftrightarrow f|_{\partial\Omega} = 0$$

holds, we have

$$\sigma_{\mathcal{C}_A}([T_z]) = \partial\Omega.$$

Proof. Suppose that

$$T_f \in \mathcal{K} \text{ for all } f \in C(\overline{\Omega}) \text{ with } f|_{\partial\Omega} = 0.$$

Assume there was $\lambda \in \sigma_{\mathcal{C}_A}([T_z])$ with $\lambda \notin \partial\Omega$. By Urysohn's Lemma, we can find a function $f \in C(\overline{\Omega})$ with $f(\lambda) = 1$ and $f|_{\partial\Omega} = 0$. But then $T_f \in \mathcal{K}$ and hence, by the preceding lemma,

$$f|_{\sigma_{\mathcal{C}_A}([T_z])} = 0,$$

which contradicts $f(\lambda) = 1$. So we conclude that $\sigma_{\mathcal{C}_A([T_z])} \subset \partial\Omega$. In case even the equivalence

$$T_f \in \mathcal{K} \Leftrightarrow f|_{\partial\Omega} = 0$$

holds, a similar argument shows $\partial\Omega \subset \sigma_{\mathcal{C}_A}([T_z])$. $\square$

The next lemma shows that every operator in $\mathcal{T}_A + \mathcal{K}$ can be expressed as a sum of a Toeplitz operator with continuous symbol and a compact operator.

Lemma 3.11. *We have*

$$\mathcal{T}_A + \mathcal{K} = \{T_f + K; f \in C(\overline{\Omega}) \text{ and } K \in \mathcal{K}\}.$$

Proof. Consider an operator $T \in \mathcal{T}_A + \mathcal{K}$. Then

$$\Psi(T) \in C(\sigma_{\mathcal{C}_A}([T_z]))$$

and, by the Tietze extension theorem, we can choose a function $f \in C(\overline{\Omega})$ with

$$\Psi(T) = f|_{\sigma_{\mathcal{C}_A}([T_z])}.$$

If we apply the C^*-homomorphism Ψ to the Toeplitz operator with symbol f, we obtain

$$\Psi(T_f) = f|_{\sigma_{\mathcal{C}_A}([T_z])} = \Psi(T).$$

The exactness of the sequence

$$0 \longrightarrow \mathcal{K} \hookrightarrow \mathcal{T}_A + \mathcal{K} \xrightarrow{\Psi} C(\sigma_{\mathcal{C}_A([T_z])}) \longrightarrow 0$$

then yields $T - T_f \in \mathcal{K}$. $\square$

Remark 3.12. *If $T_z \in L(\mathcal{H}^2_A(\mu))^n$ is essentially normal and*

$$(*) \lim_{z\to\partial\Omega} \Gamma(K)(z) = 0$$

for every compact operator $K \in \mathcal{K}(\mathcal{H}^2_A(\mu))$, then Lemma 3.11 shows that the validity of the following two statements

- *For $f \in C(\overline{\Omega}) : T_f$ compact $\Leftrightarrow \lim_{z\to\partial\Omega} \Gamma(T_f)(z) = 0$.*
- *For $T \in \mathcal{T}_A : T$ compact $\Leftrightarrow \lim_{z\to\partial\Omega} \Gamma(T)(z) = 0$.*

is equivalent. Recall that the condition $(*)$ is satisfied whenever

$$\mathrm{w}-\lim_{z\to\partial\Omega} k(\cdot, z) = 0.$$

3.3 Essential spectrum and Shilov boundary

We know from Proposition 1.14 in [Con85] that, for a bounded operator T on $\mathcal{H}^2_A(\mu)$, the identity

$$\sigma_{\mathcal{C}_A}([T]) = \sigma_{\mathcal{C}(\mathcal{H}^2_A(\mu))}([T])$$

holds. In addition, Theorem 1.4.16 in [Mur90] yields that the spectrum of $[T]$ in the Calkin algebra $\mathcal{C}(\mathcal{H}^2_A(\mu))$ coincides with the essential spectrum of the operator T, which is defined as follows.

Definition 3.13. *Let H be a Hilbert space and $T \in L(H)$ a bounded operator. We define the essential spectrum of T as*

$$\{\lambda \in \mathbb{C}; \lambda I - T \text{ is not a Fredholm operator }\}$$

As mentioned above, for an operator $T \in \mathcal{T}_A$, we have

$$\sigma_e(T) = \sigma_{\mathcal{C}_A}([T]).$$

In [EP96], Eschmeier and Putinar use the Koszul complex to define the essential Taylor spectrum $\sigma_e(T)$ of a commuting tuple $T = (T_1, \ldots, T_n)$ of bounded operators on a Hilbert space H (or, more generally, on Banach spaces). By Corollary 2.6.11 in [EP96], the essential spectrum of a commuting tuple $T \in L(H)^n$ coincides with the Taylor spectrum $\sigma(L_T, \mathcal{C}(H))$ of the induced tuple of left multiplication operators

$$L_{T_i} : \mathcal{C}(H) \to \mathcal{C}(H), [x] \mapsto [T_i x] \qquad (1 \le i \le n)$$

on the Calkin algebra.

By a result of Curto (Theorem 1 in [Cur82]), it follows that, for $T_z \in L(\mathcal{H}^2_A(\mu))^n$, the identity

$$\sigma(L_{T_z}, \mathcal{C}(\mathcal{H}^2_A(\mu))) = \sigma(L_{T_z}, \mathcal{C}_A)$$

holds. If T_z is essentially normal, then $\mathcal{C}_A$ is commutative and one can show that

$$\sigma(L_{T_z}, \mathcal{C}_A) = \sigma_{\mathcal{C}_A}([T_z]).$$

Hence, in the setting of Section 3.1 and Section 3.2 the identity

$$\sigma_e(T_z) = \sigma_{\mathcal{C}_A}([T_z])$$

holds. Under certain conditions, the essential spectrum of T_z contains the Shilov boundary of A, which is defined as follows.

Definition 3.14. *Let X be a compact Hausdorff space and let A be an algebra of continuous $\mathbb{C}$-valued functions on X which separates the points of X. We define the Shilov boundary of A as*

$$\partial_{Shilov}(A) = \bigcap (S \subset X \textit{ closed}\ ;\ \sup\{|\mathrm{a(x)}|; \mathrm{x} \in \mathrm{X}\} = \sup\{|\mathrm{a(x)}|; \mathrm{x} \in \mathrm{S}\} \textit{ for all } \mathrm{a} \in \mathrm{A}).$$

One can show (Theorem 9.1 in [AW98]), that

$$\|a\|_{\infty, \partial_{Shilov}} = \|a\|_{\infty, X} \text{ for all } a \in A.$$

In case that all functions in $\mathcal{H}^2_A(\mu)$ are holomorphic and Ω is connected, the essential spectrum contains the Shilov boundary of A.

Theorem 3.15. *Suppose that the multiplication tuple $T_z \in L(\mathcal{H}^2_A(\mu))^n$ is essentially normal and that $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$. Suppose in addition that Ω is connected. Then*

$$\partial_{Shilov}(A) \subset \sigma_e(T_z).$$

Proof. Since by Lemma 3.6 and Corollary 3.8 ,

$$i([T_f]) = \Phi(f) = f|_{\sigma_e(T_z)} \text{ for all } f \in C(\overline{\Omega}),$$

we obtain the equality

$$f(\sigma_e(T_z)) = \sigma_{C(\sigma_e(T_z))}(f|_{\sigma_e(T_z)}) = \sigma_{\mathcal{C}_A}([T_f]) = \sigma_{\mathcal{C}(\mathcal{H}^2_A(\mu))}([T_f]) = \sigma_e(T_f)$$

for every function $f \in C(\overline{\Omega})$. For a function $f \in A$, the inclusion

$$f(\overline{\Omega}) \subset \sigma(T_f)$$

holds.

Indeed, for $\lambda \in \Omega$ and $g \in \mathcal{H}^2_A(\mu)$,

$$(f(\lambda) - T_f)g = (f(\lambda) - f)g \in \{h \in \mathcal{H}^2_A(\mu)\,; h(\lambda) = 0\} \neq \mathcal{H}^2_A(\mu)\,,$$

since $\mathcal{H}^2_A(\mu)$ contains the constant functions. So $(f(\lambda) - T_f)$ is not surjective and thus, $f(\lambda)$ is an element of the spectrum. Hence,

$$f(\overline{\Omega}) = \overline{f(\Omega)} \subset \sigma(T_f).$$

For $f \in A$, we consider a point $\mu \in \sigma(T_f)$ with

$$|\mu| = r(T_f)$$

where $r(T_f)$ denotes the spectral radius of T_f. By Proposition 6.7 in [Con85],

$$\partial\sigma(T_f) \subset \sigma_\pi(T_f),$$

where $\sigma_\pi(T_f)$ is the approximate point spectrum of T_f. Hence μ is an element of $\sigma_\pi(T_f)$. From the preceding parts of this proof, we know that

$$\sigma_e(T_f) = f(\sigma_e(T_z)) \subset f(\overline{\Omega}) \subset \sigma(T_f).$$

Let us now assume that μ wasn't in the essential spectrum of T_f. Then $\mu - T_f$ is a Fredholm operator, which implies that the symbol f doesn't equal the constant function μ, since the image of $\mu - T_f$ has finite codimension in $\mathcal{H}^2_A(\mu)$. In addition, the range of $\mu - T_f$ is closed in $\mathcal{H}^2_A(\mu)$, so by Propostion 6.4 in [Con85], if $\mu - T_f$ were injective it would also be bounded below. As this is impossible, since $\mu \in \sigma_\pi(T_f)$, we conclude that $\mu \in \sigma_p(T_f)$. Hence, there exists a function $g \in \mathcal{H}^2_A(\mu) \setminus \{0\}$ with

$$(\mu - f|_\Omega)g = (\mu - T_f)g = 0$$

on Ω. The function g vanishes on $\Omega \setminus Z(\mu - f, \Omega)$, where

$$Z(\mu - f, \Omega) = \{z \in \Omega; \mu - f(z) = 0\}.$$

Since Ω is connected and $f \neq \mu$ on Ω, $Z(\mu - f, \Omega)$ is a thin set. Thus the continuity of g as an element of $\mathcal{H}^2_A(\mu) \subset \mathcal{O}(\Omega)$ implies

$$g(z) = 0 \text{ for all } z \in \Omega,$$

which contradicts $g \in \mathcal{H}^2_A(\mu) \setminus \{0\}$. The above argument still holds for an arbitrary element of $\sigma_\pi(T_f)$, thus it follows that for each $f \in A$

$$\sigma_\pi(T_f) \subset \sigma_e(T_f).$$

Altogether, we obtain

$$f(\overline{\Omega}) \subset \sigma(T_f) \subset \overline{D}_{r(T_f)}(0) = \overline{D}_{r_e(T_f)}(0) = \overline{D}_{\|f\|_{\sigma_e(T_z)}}(0).$$

This means

$$\|f\|_{\overline{\Omega}} = \|f\|_{\sigma_e(T_z)}.$$

By the definition of the Shilov boundary, this implies

$$\partial_{Shilov}(A) \subset \sigma_e(T_z).$$

□

We are now able to formulate a more general version of Theorem 2.16.

Theorem 3.16. *Suppose that* $T_z \in L(\mathcal{H}_A^2(\mu))^n$ *is essentially normal and that*

(i) $w - \lim\limits_{z \to \partial\Omega} k(\cdot, z) = 0$,

(ii) $A|_\Omega \subset \mathcal{O}(\Omega)$,

(iii) $\partial_{Shilov}(A) \supset \sigma_e(T_z)$.

Then, for $T \in \mathcal{T}_A$*, we have*

$$T \text{ compact } \Leftrightarrow \lim_{z \to \partial\Omega} \Gamma(X)(z) = 0.$$

Proof. Suppose that $T \in \mathcal{T}_A$ is a compact operator. Since by hypothesis

$$w - \lim_{z \to \partial\Omega} k(\cdot, z) = 0,$$

we obtain that

$$\lim_{z \to \partial\Omega} \Gamma(T)(z) = 0.$$

Conversely, suppose that $T \in \mathcal{T}_A$ is an operator with

$$\lim_{z \to \partial\Omega} \Gamma(T)(z) = 0.$$

By Remark 3.12, we may assume that $T = T_f$ for some $f \in C(\overline{\Omega})$. For a peak point $z_0 \in \partial_p A$, Satz 6.18 in [Kre11] yields

$$\lim_{z \to z_0} \Gamma(T_f)(z) = f(z_0).$$

Consider the set of peak points of A as a uniform algebra which is defined as the set of all $\lambda \in \overline{\Omega}$ for which there exists a function $f \in A$ with

$$f(\lambda) = 1 > |f(z)| \text{ for all } z \in \overline{\Omega} \setminus \{\lambda\}.$$

Since all functions in $A|_\Omega$ are holomorphic, every point in the above set is a boundary point of Ω. To see this, assume there were $\lambda \in \Omega$ and $f \in A \subset \mathcal{O}(\Omega)$ such that

$$f(\lambda) = 1 > |f(z)| \text{ for all } z \in \overline{\Omega} \setminus \{\lambda\}.$$

Then the image of the connected component $C(\lambda)$ of λ in Ω would be open with

$$1 \in f(C(\lambda)) \subset \overline{D}_1(0),$$

which is not possible. Hence, the set of peak points of A as a uniform algebra is contained in the boundary of Ω and thus coincides with our usual definition of the peak set of A. We may then apply Corollary 4.3.7 (ii) in [Dal00] which yields that the peak points for A form a dense subset of the Shilov boundary. The continuity of f leads to

$$f = 0 \text{ on } \partial_{Shilov}(A) \supset \sigma_e(T_z).$$

The compactness of T_f then follows from Lemma 3.9. □

Remark 3.17. *Note that the hypotheses of Theorem 2.16 imply that $A|_\Omega \subset \mathcal{O}(\Omega)$ and that $\sigma_e(T_z) \subset \partial\Omega = \partial_{Shilov}(A)$.*

Example 3.18. *Consider $\Omega = \mathbb{B}_n \setminus \{0\}$ and let $\mu \in M(\overline{\Omega})$ be the trivial extension of the Lebesgue measure λ on Ω to $\overline{\Omega} = \mathbb{B}_n$.*
Then $A(\Omega)$ coincides with $A = A(\mathbb{B}_n)$ and the corresponding functional Hilbert space constructed as in Section 2.1 is the usual (unweighted) Bergman space $L^2_a(\mathbb{B}_n, \lambda)$ on the unit ball $\mathbb{B}_n$.
Since the Shilov boundary for $A(\mathbb{B}_n)$ is the unit sphere in $\mathbb{C}^n$, it is not dense in the topological boundary of Ω. One can check that the multiplication tuple T_z is essentially normal and its essential spectrum coincides with the unit sphere.
Hence $\mathcal{H}^2_A(\mu)$ presents an elementary example of a situation where Theorem 3.16 is applicable, but Theorem 2.16 is not.

4 Appendix

4.1 The $\overline{\partial}$-Neumann problem and pseudoregular sets

In Section 2.4, we consider functional Hilbert spaces on pseudoregular domains in $\mathbb{C}^n$. The definition of these sets is strongly connected with the $\overline{\partial}$-Neumann problem. This problem, first studied by D.C. Spencer, deals with the invertability of the complex Laplacian on the square integrable $(0,1)$-forms on Ω. It was first solved by Hörmander and Kohn. A survey on the $\overline{\partial}$-problem can be found for example in [Str10], but we want to collect some of the most important results on the existence and compactness of the $\overline{\partial}$-Neumann operator in this appendix. We begin by introducing the square integrable forms on a bounded set $\Omega \subset \mathbb{C}^n$.

Let $E = \mathbb{C}^n, p \in \mathbb{N}$ and let

$$\Lambda_0 = \mathbb{C} \text{ and } \Lambda^p = \{w; w : E^p \to \mathbb{C} \ \mathbb{R}\text{-multilinear and alternating}\} \text{ for } p \geq 1$$

denote the $\mathbb{C}$-vector space of alternating $\mathbb{R}$-multilinear maps on E^p. Recall that we can define a $\mathbb{C}$-multilinear alternating map by

$$(\Lambda^1)^p \to \Lambda^p, (\varphi_1, \ldots, \varphi_p) \mapsto \varphi_1 \wedge \ldots \wedge \varphi_p,$$

where

$$(\varphi_1 \wedge \ldots \wedge \varphi_p)(v_1, \ldots, v_p) = \det((\varphi_i(v_j))_{(1 < \leq i,j \leq p)})$$

For $p, q \leq 1$, this induces unique $\mathbb{C}$- bilinear maps

$$\wedge : \Lambda^p \times \Lambda^q \to \Lambda^{p+q}, (v, w) \mapsto v \wedge w,$$

with

$$(v_1 \wedge \ldots \wedge v_p) \wedge (w_1 \wedge \ldots \wedge w_q) = (v_1 \wedge \ldots \wedge v_p \wedge w_1 \wedge \ldots \wedge w_q).$$

For $a \in \mathbb{C}$, we define

$$a \wedge w = w \wedge a = aw.$$

One can show that the map $\wedge$ is associative with $v \wedge w = (-1)^{pq}(w \wedge v)$.

We can now introduce differential forms. Let $\Omega \subset \mathbb{C}^n$ be an open set. We call a map

$$w : \Omega \to \Lambda^r$$

an r-form over Ω in n coordinates.
If we consider Λ^1 as an $\mathbb{R}$-vector space, then the maps

$$dz_j : \mathbb{C}^n \to \mathbb{C}, z \mapsto z_j \text{ and } d\overline{z}_j : \mathbb{C}^n \to \mathbb{C}, z \mapsto \overline{z_j} \quad (1 \leq j \leq n)$$

form an $\mathbb{R}$-basis of Λ^1. For $r \in \{1, \ldots, n\}$, a basis of Λ^r is given by the forms

$$dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\overline{z}_{j_1}, \ldots d\overline{z}_{j_q}$$

$$(p, q \in \{1, \ldots n\} \text{ with } p + q = r, 1 \leq i_1 < \ldots < i_p \leq n, 1 \leq j_1 < \ldots < j_q \leq n).$$

This leads to the definition of a (p, q)-form over Ω.
For $p, q \in \mathbb{N}$ with $p + q = r$, we call an r-form w with basis representation

$$w = \sum_I \sum_J f_{I,J} d_{z_I} \wedge d\overline{z_J},$$

where the sums are formed over all strictly increasing index tuples

$$I = (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p \;\; J = (j_1, \ldots, j_q) \in \{1, \ldots, n\}^q$$

and

$$dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}, \;\; d\overline{z}_J = d\overline{z}j_1 \wedge \ldots \wedge d\overline{z}j_q,$$

a (p, q)-form over Ω.
Let from now on $\Omega \subset \mathbb{C}^n$ be a bounded open set and let λ denote the restriction of the usual Lebesgue measure to Ω.
We consider the (0,q)-forms

$$w = \sum_I w_I d\overline{z}_I,$$

with coefficients $w_I \in L^2(\Omega, \lambda)$. Then we define a vector space

$$L^2_{(0,q)}(\Omega, \lambda) = \{w; w(0, q)\text{-form with } \sum_I \|w_I\|^2_{L^2(\Omega,\lambda)} < \infty\},$$

which we can equip with the inner product

$$\langle v, w \rangle_{L^2_{(0,q)}} = \sum_I \langle v_I, w_I \rangle_{L^2(\Omega)} \text{ for all } v, w \in L^2_{(0,q)}(\Omega).$$

For $j \in \{1 \dots, q\}$, we look at the functions $f \in L^2(\Omega, \lambda)$ for which there exists a function $g \in L^2(\Omega, \lambda)$ with

$$\int_\Omega f \frac{\partial \varphi}{\partial \overline{z}_j} d\lambda = - \int_\Omega g \varphi d\lambda \text{ for all } \varphi \in C_c^\infty(\Omega),$$

where

$$C_c^\infty(\Omega) = \{\varphi \in C^\infty(\Omega); \operatorname{supp}(\varphi) \subset \Omega \text{ compact } \}.$$

Since $C_c^\infty(\Omega) \subset L^2(\Omega, \lambda)$ is dense, if such a g exists, it is unique and we then define

$$\frac{\partial f}{\partial \overline{z}_j} = g.$$

For $f \in C_c^\infty(\Omega)$, this coincides with the usual partial derivative $\frac{\partial f}{\partial \overline{z}_j}$ of f. On the set

$$\begin{aligned} dom(\overline{\partial}_q) = \{w = \sum_I w_I d\overline{z}_I \in L^2_{(0,q)}(\Omega); \frac{\partial w_I}{\partial \overline{z}_j} \text{ exists} \\ \text{for all strictly increasing tuples } I \in \{1, \dots, n\}^q\} \end{aligned}$$

we define the $\overline{\partial}$-operator by

$$\overline{\partial}_q w = \sum_{j=1}^n \sum_I \frac{\partial}{\partial \overline{z}_j} w_I d\overline{z_j} \wedge d\overline{z}_I \in L_{(0,q+1)}(\Omega, \lambda).$$

In this way we obtain a linear operator

$$\overline{\partial}_q : dom(\overline{\partial}_q) \to L^2_{(0,q+1)}(\Omega), w \mapsto \overline{\partial} w.$$

Since $C_c^\infty(\Omega)_{(0,q)} \subset dom(\overline{\partial}_q)$, $dom(\overline{\partial}_q)$ is a dense subset of $L^2_{(0,q)}(\Omega, \lambda)$.
Hence, $\overline{\partial}_q$ possesses a unique Hilbert space adjoint

$$\overline{\partial}_q^* : dom(\overline{\partial}_q^*) \to L^2_{(0,q)}(\Omega, \lambda),$$

where

$$\begin{aligned} dom(\overline{\partial}_q^*) = \{u \in L^2_{(0,q+1)}(\Omega, \lambda);\ \text{there exists } v \in L^2_{(0,q)}(\Omega, \lambda) \\ \text{with } \langle u, \overline{\partial}_q w \rangle_{L^2_{(0,q+1)}(\Omega,\lambda)} = \langle v, w \rangle_{L^2_{(0,q+1)}(\Omega,\lambda)} \text{ for all } w \in dom\overline{\partial}_q\}. \end{aligned}$$

This enables us to introduce the complex Laplacian on Ω. Let

$$\begin{aligned} dom(\Box_q) = \{w \in L^2_{(0,q)}(\Omega); w \in dom(\overline{\partial}_{q-1}^*) \cap dom(\overline{\partial}_q) \\ \text{with } \overline{\partial}_{q-1}^* w \in dom(\overline{\partial}_{q-1}) \text{ and } \overline{\partial}_q w \in dom(\overline{\partial}_q^*)\}. \end{aligned}$$

Then , for $q \in \{1, \dots, n\}$, we define the complex Laplacian $\Box_q$ as

$$\Box_q : dom(\Box_q) \to L^2_{(0,q)}(\Omega, \lambda), w \mapsto \overline{\partial}_{q-1}\overline{\partial}^*_{q-1}w + \overline{\partial}^*_q\overline{\partial}_q w.$$

The $\overline{\partial}$-Neumann problem deals with the question of whether the complex Laplacian $\Box_q$ has a bounded inverse on the square integrable $(0, q)$-forms on Ω. The solution goes back to Kohn and Hörmander and can be found for example in the following theorem from [Str10].

Theorem 4.1. *Let $n \geq 2$ and $\Omega \subset \mathbb{C}^n$ a bounded pseudoconvex domain. Then, for $1 \leq q \leq n$, the Laplacian $\Box_q$ is has a bounded inverse, the $\overline{\partial}$-Neumann operator N_q.*

Proof. Theorem 2.9 in [Str10] $\square$

Before we can use the $\overline{\partial}$-Neumann operator to define pseudoregular sets in $\mathbb{C}^n$, we first want to remind the reader of some basic definitions in complex analysis of several variables. The first thing we want to mention are strictly plurisubharmonic functions.

Definition 4.2. *Let $U \subset \mathbb{C}^n$ be an open set.*

- *We call a function $r \in C^2(U, \mathbb{R})$ strictly plurisubharmonic in $p \in U$ if the Levi matrix $L_p(r) = ((\overline{\partial}_j\partial_k r)(p))_{(1\leq j,k\leq n)}$ is positve definite. Furthermore, we call r strictly plurisubharmonic if it is strictly plurisubharmonic in every $p \in U$.*
- *We call a function $\rho : U \to \mathbb{R}$ an exhaustion function for U if , for all $c \in \mathbb{R}$, the set $U_c = \{z \in U; \rho(z) < c\} \subset U$ is relatively compact in U.*

The above definition enables us to introduce pseudoconvex sets.

Definition 4.3. *Let $D \subset \mathbb{C}^n$ be an open subset.*

- *We call D pseudoconvex if there exists a strictly plurisubharmonic exhaustion function for D.*
- *If D is bounded, we call D strictly pseudoconvex in $p \in \partial D$ if there exists an open neighbourhood U of p and a strictly plurisubharmonic function $r \in C^2(U, \mathbb{R})$ with $D \cap U = \{z \in U; r(z) < 0\}$.*
- *If D is bounded, we call D strictly pseudoconvex if there exist an open set $U \supset \partial D$ and a function $r \in C^2(U, \mathbb{R})$ such that r is strictly plurisubharmonic with $D \cap U = \{z \in U; r(z) < 0\}$.*

Strictly pseudoconvex points are peak points for the domain algebra, as the next lemma states.

Lemma 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded pseudoconvex domain which is strictly pseudoconvex in $p \in \partial\Omega$. Then p is a peak point for the domain algebra $A(\Omega)$.*

Proof. The proof follows from Theorem 2.3 in [Noe08]. $\square$

A pseudoregular open set is then defined as follows.

Definition 4.5. *We call a bounded open set $\Omega \subset \mathbb{C}^n$ pseudoregular if Ω is pseudoconvex with smooth boundary and if, in addition, the $\overline{\partial}$-Neumann operator N_1 of Ω is compact.*

The reason why we are interested in pseudoregular sets is that, for the Bergman space $L^2_a(\Omega, \lambda)$ (Definition 2.19), the compactness of the $\overline{\partial}$-Neumann operator N_1 implies a useful property of the Bergman projection on Ω. Note that by Lemma 2.1 in [Str10], $ker(\overline{\partial}_0) \subset L^2(\Omega, \lambda)$ is a closed subset. One can show that

$$ker(\overline{\partial}_0) = L^2_a(\Omega, \lambda).$$

Hence, the projection

$$P : L^2(\Omega, \lambda) \to \ker(\overline{\partial}_0)$$

is the Bergman projection on Ω. Then the following theorem, originally stated in [Str10], yields that the Bergman projection commutes essentially with every multiplication operator with continuous symbol on Ω.

Theorem 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain on which the $\overline{\partial}$-Neumann operator is compact. Then for every $f \in C(\overline{\Omega})$ the operator $PM_f - M_fP$ is compact.*

Proof. Proposition 4.1 in [Str10]. □

By the proof of Lemma 2.9, this yields that for a pseudoregular set $\Omega \subset \mathbb{C}^n$, all Hankel operators on the Bergman space $L^2_a(\Omega, \lambda)$ with continuous symbols are compact.
Lemma 2.13 then implies that the multiplication tuple T_z on $L^2_a(\Omega, \lambda)$ is essentially normal. The next lemma states another important property of pseudoregular domains.

Lemma 4.7. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoregular set. Then the set of strictly pseudoconvex points is dense in $\partial\Omega$.*

Proof. Corollary 1 in [ŞS06]. □

In Remark 1.3 (a) in [KS93], Salinas and Krantz give a list of conditions under which the $\overline{\partial}$-Neumann operator N_1 of Ω is compact. Examples include domains of finite type, smooth convex domains, Reinhardt domains without analytic disk in their boundary, and domains whose boundary satisfies the property (P) defined by Catlin.

Bibliography

[ABR01] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory (2nd edition)*, Springer, New York, 2001.

[Arv76] W.B. Arveson, *An invitation to C^*-algebras*, Springer, New York, 1976.

[AW98] H. Alexander and J. Wermer, *Several complex variables and Banach algebras (3rd edition)*, Spinger, New York, 1998.

[AZ98] S. Axler and D. Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J. **47** (1998), 387–400.

[BS99] H. Boas and E. Straube, *Global regularity of the $\overline{\partial}$-Neumann problem: A survey of the L^2-Sobolev theory*, Several Complex variables, vol. 37, Cambridge University Press, Cambridge, 1999, pp. 79–111.

[ČC99] Z. Čučković and K.C. Chan, *C^*-algebras generated by a subnormal operator*, Trans. Amer. Math. Soc. **351** (1999), 1445–1460.

[Con85] J.B. Conway, *A course in functional analysis*, Springer, New York, 1985.

[ČŞ13] Z. Čučković and S. Şahutoğlu, *Axler-Zheng type theorem on a class of domains in $\mathbb{C}^n$*, Integral Equations Operator Theory **77** (2013), 397–405.

[Cur82] R.E. Curto, *Spectral permanence for joint spectra*, Trans. Amer. Math. Soc. **270** (1982), 659–665.

[DAE+03] H.G. Dales, P. Aiena, J. Eschmeier, K. Laursen, and G. Willis, *Introduction to Banach algebras, operators, and harmonic analysis*, Cambridge University Press, Cambridge, 2003.

[Dal00] H.G. Dales, *Banach algebras and automatic continuity*, Clarendon Press, Oxford, 2000.

[DE12] R.G. Douglas and J. Eschmeier, *Spectral inclusion theorems. Mathematical methods in systems, optimization, and control*, Oper. Theory Adv. Appl. **222** (2012), 113–128.

[DS58] N. Dunford and J. T. Schwartz, *Linear operators. I. General theory*, Interscience Publishers, Inc., New York, 1958.

[Eng99] M. Engliš, *Compact Toeplitz operators via the Berezin transform on bounded symmetric domains*, Integral Equations Operator Theory **33** (1999), 426–455.

[EP96] J. Eschmeier and M. Putinar, *Spectral decompostions and analytic sheaves*, Clarendon Press, Oxford, 1996.

[Gar81] J.B. Garnett, *Bounded analytic functions*, Springer, New York, 1981.

[Hof62] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Inc., Englewood Cliffs, 1962.

[JP00] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*, Walter de Gruyter & Co, Berlin, 2000.

[Kre11] E. Kreutzer, *Das Randverhalten der Berezin-Transformierten auf funktionalen Hilberträumen*, 2011, Bachelor's thesis.

[KS93] S.G. Krantz and N. Salinas, *Proper holomorphic mappings and the Cowen-Douglas class*, Proc. Amer. Math. Soc. **117** (1993), 99–105.

[Le09] T. Le, *Compact Toeplitz operators with continuous symbols*, Glasg. Math. J. **51** (2009), 257–261.

[MSW13] M. Mitkovski, D. Suarez, and B. Wick, *The essential norm of operators on $A^p(\mathbb{B}_n)$*, Integral Equations Operator Theory **75** (2013), 197–233.

[Mur90] G.J. Murphy, *C^*-algebras and operator theory*, Academic Press, London, 1990.

[MW14] M. Mitkovski and B. Wick, *The essential norm of operators on $A^p(\mathbb{D}_n)$*, Conference Proceedings of OT **24** (2014), 165–211.

[Noe08] A. Noell, *Peakpoints for pseudoconvex domains: A survey*, J. Geom. Anal. **18** (2008), 1058–1087.

[Ran86] R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Springer, New York, 1986.

[Rud80] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer, New York, 1980.

[ŞS06] S. Şahutoğlu and E. Straube, *Analytic discs, plurisubharmonic hulls, and non-compactness of the $\overline{\partial}$-Neumann operator*, Math. Ann. **334** (2006), 809–820.

[Str10] E. Straube, *Lecture notes on the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem*, European Mathematical Society (EMS), Zürich, 2010.

[Sua07] D. Suarez, *The essential norm of operators in the Toeplitz algebra on $A^p(\mathbb{B}_n)$*, Indiana Univ. Math. J. **56** (2007), 2185–2232.