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K-contractions, and perturbations of Toeplitz operators

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Abstract

The present thesis is concerned with two different problems from multivariable operator theory on Hilbert spaces; the model theory for commuting contractive operator tuples, and perturbations of (analytic) Toeplitz operators.

The first part develops a generalization of the model theory of Agler, Müller-Vasilescu, Pott, Arveson, Ambrozie-Engliš-Müller, Arazy-Engliš and Olofsson for a class of reproducing kernel Hilbert spaces on the open unit ball in $\mathbb{C}^d$. Here, we examine two classes of commuting tuples which coincide for the case of weighted Bergman spaces with m-hypercontractions and for suitable Nevanlinna-Pick spaces with a class of commuting tuples recently studied by Clouâtre-Hartz. As an application, we obtain a Beurling-type theorem, where we characterize the invariant subspaces of the shift operator which arise as the image of suitable partially isometric multipliers. As a second consequence, we extend the work of Arveson and Bhattacharjee et al. on the uniqueness of minimal coextensions.

In the second part we study Toeplitz operators associated with regular A-isometries, a notion introduced by Eschmeier as a generalization of spherical isometries. In this setting, we use results of Didas-Eschmeier-Everard to characterize finite-rank and Schatten-class perturbations of (analytic) Toeplitz operators.

Zusammenfassung

In der vorliegenden Arbeit beschäftigen wir uns mit zwei Teilgebieten der mehrdimensionalen Operatorentheorie auf Hilberträumen; zum einen mit der Modelltheorie für kontraktive Operatortupel, zum anderen mit Störungen von (analytischen) Toeplitzoperatoren.

Der erste Teil stellt eine Verallgemeinerung der Modellsätze von Agler, Müller-Vasilescu, Pott, Arveson, Ambrozie-Engliš-Müller, Arazy-Engliš und Olofsson für eine Klasse von funktionalen Hilberträumen auf der offenen Einheitskugel in $\mathbb{C}^d$ dar. Hierbei untersuchen wir zwei Klassen von kommutierenden Tupeln, welche im Fall von gewichteten Bergmanräumen mit den m-Hyperkontraktionen und im Fall einer geeigneten Teilklasse von vollständigen Nevanlinna-Pick-Räumen mit den von Clouâtre-Hartz untersuchten Tupeln zusammenfallen. Als Folgerung erhalten wir einen Satz vom Beurlingtyp, der die invarianten Teilräume des Shifts, die Bild einer geeigneten partiellen Isometrie sind, charakterisiert. Ebenfalls können wir Resultate von Arveson und Bhattacharjee et al. über die Eindeutigkeit von minimalen Koerweiterungen auf unsere allgemeinere Situation übertragen.

Im Anschluss wenden wir uns einer von Eschmeier eingeführten Verallgemeinerung der Klasse der sphärischen Isometrien, sogenannten A-Isometrien, zu. Wir benutzen Resultate von Didas-Eschmeier-Everard, um Störungen mit endlichem Rang und Schattenklasse-Störungen von (analytischen) Toeplitzoperatoren zu charakterisieren.

Contents

Introduction

In this thesis, we study classes of commuting tuples of bounded linear operators on Hilbert spaces. The first part is concerned with the model and dilation theory of a generalization of contractions, called K-contractions, connected to reproducing kernel Hilbert spaces H_K on the open unit ball in $\mathbb{C}^d$. In the second part we obtain characterizations of Schatten-class perturbations of (analytic) Toeplitz operators in the setting of so-called regular A-isometries.

Part I: K-contractions

Hilbert space contractions T can be characterized by the positivity condition $\mathrm{id} - TT^* \geq 0$. The left-hand side of this inequality is formally obtained by replacing the variables z and $\overline{w}$ in the reciprocal

$$\frac{1}{K(z,w)} = 1 - z\overline{w}$$

of the reproducing kernel K of the Hardy space $H^2(\mathbb{D})$ on the unit disc $\mathbb{D}$ by the operator T and its adjoint T^*. By the Sz.-Nagy dilation theory (cf. [62]), every contraction is unitarily equivalent to a restriction of an operator of the form $((M_z \otimes \mathrm{id}_{\mathcal{D}}) \oplus U)^*$ to an invariant subspace, where $M_z \in B(H^2(\mathbb{D}))$ is the multiplication operator with symbol z, the identity map, on the Hardy space, $\mathcal{D}$ is a Hilbert space, and U is a unitary operator. Hence, to understand contractions, it is sufficient to understand restrictions of operators of the latter kind.

Besides the Hardy space, there exist many other reproducing kernel Hilbert spaces of analytic functions, for example, the (weighted) Bergman spaces or Nevanlinna-Pick spaces (in one and higher dimensions). Therefore, it is natural to ask for a characterization of restrictions of $((M_z \otimes \mathrm{id}_{\mathcal{D}}) \oplus U)^*$, where M_z is now the multiplication operator on a reproducing kernel Hilbert space of analytic functions. For the standard weighted Bergman spaces $H_m(\mathbb{B}_d)$ on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ given by the reproducing kernels

$$K^{(m)}(z,w) = \frac{1}{(1 - \langle z, w \rangle)^m} \quad (m \in \mathbb{N}),$$

such restrictions have been characterized by Agler [2], Müller and Vasilescu [55], and Arveson [9] as those commuting contractive tuples $T \in B(\mathcal{H})^n$ which satisfy the positivity condition

$$\frac{1}{K^{(m)}}(T, T^*) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^j T^{*j} \geq 0.$$

Pott [59], Ambrozie, Engliš and Müller [5], and Arazy and Engliš [6] studied reproducing kernels, where the underlying subset of $\mathbb{C}^d$ can be quite general. This generality goes along with a restriction on the reciprocal of the kernel (often assumed to be a polynomial or a rational function in z and $\overline{w}$). Furthermore, Clouâtre and Hartz [18] characterized the aforementioned restrictions for a certain class of Nevanlinna-Pick spaces; again in terms of the reciprocal of the reproducing kernel. The main goal of this part is to unify the approaches for the (weighted) Bergman spaces and Nevanlinna-Pick spaces. We focus on reproducing kernel Hilbert spaces $H_K \subset \mathcal{O}(\mathbb{B}_d)$ of analytic functions on the open unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ such that the reproducing kernel $K \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is of the form

$$K(z, w) = k(\langle z, w \rangle) \quad (z, w \in \mathbb{B}_d)$$

for some zero-free analytic function $k \colon \mathbb{D} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} a_n z^n$ with $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$, and $\sup_{n \in \mathbb{N}} a_n / a_{n+1} < \infty$. The last condition ensures that the tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ on H_K is a well-defined bounded linear operator. Many classical spaces on $\mathbb{B}_d$, including the Drury-Arveson space, the Hardy space, the Dirichlet space, and weighted Bergman spaces, are of this kind. In particular, the Drury-Arveson space corresponds to the constant sequence $a_k = 1$. To obtain a class of commuting tuples which can be realized as restrictions of $((M_z \otimes \mathrm{id}_{\mathcal{D}}) \oplus U)^*$, we use two different approaches: a geometrical-algebraic one and an analytic one. A basic problem in both approaches is to make sense of the operator $1/K(T, T^*)$. Since by hypothesis $1/k$ has an expansion of the form $1/k(z) = \sum_{n=0}^{\infty} c_n z^n$ $(z \in \mathbb{D})$ for some suitable sequence $(c_n)_{n \in \mathbb{N}}$ of real numbers, a natural idea is to define

$$\frac{1}{K}(T, T^*) = \sum_{n=0}^{\infty} c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}),$$

where the series is asked to converge in the strong operator topology and

$$\sigma_T \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \sum_{i=1}^{d} T_i X T_i^*.$$

For the commuting tuple $M_z \in B(H_K)^d$, this condition is satisfied, for instance, if almost all of the coefficients c_n ($n \in \mathbb{N}$) have the same sign (see [17]), which will be assumed further on. We call a commuting tuple T a *K-contraction* if $1/K(T,T^*)$ exists and defines a positive operator. Furthermore, as an analogue of the class $C_{\cdot 0}$ in the case of classical contractions, we introduce for a K-contraction T the notion of *K-pureness*, which means by definition that the SOT-limit $\Sigma(T)$ of the sequence $(\Sigma_N(T))_{N\in\mathbb{N}}$ defined by

$$\Sigma_N(T) = \mathrm{id}_{\mathcal{H}} - \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T,T^*) \right)$$

is zero. With these preparations, we can state our model theorem (see also [6]).

1 Theorem (Theorem 2.15)**.** *Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is K-pure,

(ii) there exist a Hilbert space $\mathcal{D}$ and an isometry $\Pi \colon \mathcal{H} \to H_K \otimes \mathcal{D}$ such that

$$\Pi T_i^* = (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}})^* \Pi \quad (i = 1, \ldots, d).$$

If a K-contraction T is *strong*, i.e., $\Sigma(T)$ defines a positive operator and satisfies $\sigma_T(\Sigma(T)) = \Sigma(T)$, one can show that there are a Hilbert space $\mathcal{L}$ and a spherical unitary $W \in B(\mathcal{L})^d$ such that there exists an isometry

$$\Psi_T \colon \mathcal{H} \to (H_K \otimes \mathcal{D}_T) \oplus \mathcal{L}$$

which intertwines the tuples $T^* = (T_1^*, \ldots, T_d^*)$ on $\mathcal{H}$ and $((M_z \otimes \mathrm{id}_{\mathcal{D}_T}) \oplus W)^*$ on $(H_K \otimes \mathcal{D}_T) \oplus \mathcal{L}$ componentwise. Here, $\mathcal{D}_T = (1/K(T,T^*))^{1/2}$ is the defect operator and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$ is the defect space of T.

2 Theorem (Theorem 2.30)**.** *Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is a strong K-contraction,

(ii) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi \colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi \quad (i = 1, \ldots, d).$$

If in addition H_K is regular, i.e., $\lim_{n\to\infty} a_n/a_{n+1} = 1$, *then the above are also equivalent to*

(iii) there exists a unital, completely contractive linear map

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \ ; \ i = 1, \ldots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \textit{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_iT_i^*$$

for all $i = 1, \ldots, d$.

Condition (iii) shows that M_z plays the role of a universal tuple for the class defined in (i).

For m-hypercontractions, or the Nevanlinna-Pick case, the above theorem reduces to the mentioned results of Müller-Vasilescu, and Clouâtre-Hartz, respectively.

The analytic approach was inspired by Agler [1], Vasilescu [63], Arveson [9], Olofsson [56], and Clouâtre and Hartz [18]. The idea is to study the family rT $(0 < r < 1)$ instead of the single tuple T. The advantage of this approach is that one can easily make sense of the operators $1/K(rT, rT^*)$ via Taylor's analytic functional calculus. We call a commuting tuple T with spectrum in the closed unit ball $\overline{\mathbb{B}}_d$ a *radial K-hypercontraction* if all operators $1/K(rT, rT^*)$ $(0 < r < 1)$ are positive. Olofsson stated in [56] a condition which guarantees the existence of the limit $\lim_{r\to 1} 1/K(rT, rT)$ in the strong operator topology for a radial K-hypercontraction. This condition, which will be assumed to hold further on, is fulfilled in the cases mentioned before. If we suppose that H_K is regular, then Theorem 2 holds if we replace strong K-contractions with radial K-hypercontractions (cf. Theorem 3.21).

Let ν be a positive real number. We call a commuting tuple T a *ν-hypercontraction* if T is a $K^{(1)}$-contraction and a $K^{(\nu)}$-contraction, where

$$K^{(\nu)}(z, w) = \frac{1}{(1 - \langle z, w\rangle)^\nu} \quad (z, w \in \mathbb{B}_d).$$

This is a natural generalization of m-hypercontractions, where m is a positive natural number. Our main result about ν-hypercontractions shows that they coincide with strong $K^{(\nu)}$-contractions and with radial $K^{(\nu)}$-hypercontractions. Furthermore, for a ν-hypercontraction $T \in B(\mathcal{H})^d$, the operator $\Sigma(T)$ is the limit in the strong operator topology of the sequence $(\sigma_T^N(\mathrm{id}_{\mathcal{H}}))_{N\in\mathbb{N}}$.

In 1949, Beurling [14] gave the following characterization of invariant subspaces of the shift operator M_z on the Hardy space $H^2(\mathbb{D})$.

3 Theorem (Beurling)**.** *For $\mathcal{S} \subset H^2(\mathbb{D})$, the following are equivalent:*

(i) $\mathcal{S}$ is a closed invariant subspace for $M_z \in B(H^2(\mathbb{D}))$,

(ii) there exists a bounded analytic multiplier θ on $H^2(\mathbb{D})$ such that the induced multiplication operator M_θ is an isometry with image $\mathcal{S}$.

A similar result for the multiplication tuple $M_z = (M_{z_1}, \dots, M_{z_n})$ on the Drury-Arveson space was given by McCullough and Trent [54] (see also [61]). As an application of our model theorem for pure K-contractions and multiplier characterizations due to Barbian [12], we obtain the following result.

4 Theorem (Theorem 4.6)**.** *Let $\mathcal{E}$ be a Hilbert space, $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_d, \mathcal{E})$ a reproducing kernel Hilbert space, and suppose that $M_z \in B(H(\mathcal{E}))^d$ is K-pure. For $\mathcal{S} \subset H(\mathcal{E})$, the following statements are equivalent:*

(i) $\mathcal{S}$ is a closed invariant subspace for M_z and $M_z|_{\mathcal{S}}$ is K-pure,

(ii) there exist a Hilbert space $\mathcal{D}$ and a bounded analytic multiplier θ between $H_K(\mathcal{D})$ and $H(\mathcal{E})$ such that M_θ is a partial isometry with image $\mathcal{S}$.

As a second application, we prove a uniqueness result for minimal isometric coextensions of strong K-contractions $T \in B(\mathcal{H})^d$ for suitable reproducing kernels K. Thus, we extend corresponding uniqueness results proved by Arveson [9] and Bhattacharjee et al [15] to our more general setting.

5 Theorem (Corollary 5.17)**.** *Suppose that H_K is regular and that $P_{\mathbb{C}}$ belongs to the closed linear span of $\left\{M_z^\alpha M_z^{*\beta} \;;\; \alpha, \beta \in \mathbb{N}^d\right\}$. Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. Furthermore, let $\mathcal{D}$ and $\mathcal{K}$ be Hilbert spaces, $U \in B(\mathcal{K})^d$ a spherical unitary, and let $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ be an isometry which intertwines T^* with $((M_z \otimes \mathrm{id}_{\mathcal{D}}) \oplus U)^*$ componentwise. Then there exist isometries $V_s \in B(\mathcal{D}_T, \mathcal{D})$ and $V_u \in B(\mathcal{L}, \mathcal{K})$ such that the diagram*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Pi"] \arrow[d, "\Psi_T"'] & H_K(\mathcal{D}) \oplus \mathcal{K} \\ H_K(\mathcal{D}_T) \oplus \mathcal{L} \arrow[ur, "(\mathrm{id}_{H_K} \otimes V_s) \oplus V_u"'] & \end{tikzcd}$$

commutes.

In particular, the last result holds if $K = K^{(\nu)}$ for a positive real number ν.

Part II: Perturbations of Toeplitz operators

Another way to look at the Hardy space is to identify $H^2(\mathbb{D})$ with the subspace $H^2(m)$ of $L^2(m)$ consisting of all functions which have vanishing negative Fourier coefficients, where m is the canonical probability measure on $\mathbb{T}$. The *Toeplitz operator with symbol* $f \in L^\infty(m)$ is defined as the compression $T_f = P_{H^2(m)} M_f|_{H^2(m)}$ of the multiplication operator $M_f \in B(L^2(m))$ to the closed subspace $H^2(m) \subset L^2(m)$. Via the above identification, the operator M_z on $H^2(\mathbb{D})$ corresponds to the Toeplitz operator T_z on $H^2(m)$. Furthermore, we call a Toeplitz operator *analytic* if the corresponding symbol belongs to $H^\infty(m) = H^2(m) \cap L^\infty(m)$. Brown and Halmos obtained in [16] the following characterization of Toeplitz operators and analytic Toeplitz operators.

6 Theorem (Brown, Halmos)**.** *Let $X \in B(H^2(m))$.*

(i) The operator X is a Toeplitz operator if and only if $T_z^* X T_z = X$.

(ii) The operator X is an analytic Toeplitz operator if and only if $XT_z = T_z X$.

As a corollary, one obtains the following result which can be seen as the starting point for our study of perturbations of Toeplitz operators. Here, we call a function $u \in H^\infty(m)$ *inner* if $|u| = 1$ m-almost everywhere.

7 Corollary. *Let $X \in B(H^2(m))$.*

(i) The operator X is a Toeplitz operator if and only if $T_u^* X T_u - X = 0$ for all inner functions $u \in H^\infty(m)$.

(ii) The operator X is an analytic Toeplitz operator if and only if $[X, T_u] = XT_u - T_u X = 0$ for all $u \in H^\infty(m)$.

If $\mathcal{J} \subset B(H^2(m))$ is an ideal, then each operator $X = T_f + J$ ($f \in L^\infty(m)$, $J \in \mathcal{J}$) satisfies the condition $T_u^* X T_u - X \in \mathcal{J}$ for all inner functions u. Hence, the question naturally arises for which ideals $\mathcal{J}$ conversely the latter condition implies that $X = T_f + J$ with $f \in L^\infty(m)$ and $J \in \mathcal{J}$ (cf. [32, Exercise 7.38]). The ideals of finite-rank operators (cf. [42] by Gu) and compact operators (cf. [64] by Xia) enjoy this property. For the ideal of Schatten-class operators, we obtain the following analogue.

8 Theorem (Corollary 8.13)**.** *For $p \in [1, \infty)$, an operator X on $H^2(m)$ can be written as $X = T_f + S$, where $f \in L^\infty(m)$ and S belongs to the Schatten-p-class, if and only if $T_u^* X T_u - X$ lies in the Schatten-p-class for every inner function u.*

In [25], Davidson showed that, for an operator $X \in B(H^2(m))$, $XT_u - T_uX$ is compact for all $u \in H^\infty(m)$ if and only if X is a compact perturbation of a Toeplitz operator T_f with symbol $f \in H^\infty(m)+C(\mathbb{T})$. By a result of Hartman [47], the set $H^\infty(m) + C(\mathbb{T})$ coincides with $\{f \in L^\infty(m)\ ;\ H_f \text{ is compact}\}$, where $H_f = (\mathrm{id}_{L^2(m)} - P_{H^2(m)})M_f|_{H^2(m)}$ is the *Hankel operator with symbol* $f \in L^\infty(m)$. A generalization of this result to the Hardy space on the unit sphere was obtained by Ding and Sun [31] in 1997. The modified question where the ideal of compact operators is replaced with the ideal of finite-rank operators was studied by Guo and Wang in [45]. Their result characterizes the operators X on the Hardy space on the unit sphere or the distinguished boundary of the unit polydisc which satisfy the property that $XT_u - T_uX$ is of finite rank for all $u \in H^\infty(\sigma)$ as the finite-rank perturbations $X = T_f + F$ for some $f \in L^\infty(\sigma)$ such that H_f has finite rank and F is a finite-rank operator. Here, σ denotes the canonical probability measure on the unit sphere or the distinguished boundary of the unit polydisc, respectively, and $H^\infty(\sigma) = L^\infty(\sigma) \cap H^2(\sigma)$, where $H^2(\sigma)$ is the corresponding Hardy space.

We prove generalizations of Theorem 8 and the results of Davidson, and Guo and Wang in the setting of regular A-isometries, which were first introduced by Eschmeier in [35]. The idea of the general notion of an A-isometry originates from the well-known characterization of spherical isometries by Athavale [10] as those tuples which are subnormal such that the Taylor spectrum of the minimal normal extension lies in the unit sphere. We will now introduce some basic facts about regular A-isometries.

Let $T \in B(\mathcal{H})^d$ be a subnormal commuting tuple on a Hilbert space $\mathcal{H}$, and let $U \in B(\hat{\mathcal{H}})^d$ be a minimal normal extension of T on some Hilbert space $\hat{\mathcal{H}} \supset \mathcal{H}$. Then the scalar spectral measure μ of U is a finite positive Borel measure on $\sigma_n(T) = \sigma(U)$ which is, up to mutual absolute continuity, independent of the choice of U. We call μ a *scalar spectral measure of* T. Let $K \subset \mathbb{C}^d$ be a compact set and let A be a closed subalgebra of $C(K)$ which contains the polynomials. Furthermore, we denote by ∂_A the Shilov boundary of A. We call T an *A-isometry* if the normal spectrum $\sigma_n(T)$ is a subset of the Shilov boundary ∂_A and A is contained in the restriction algebra $\mathcal{R}_T = \{f \in L^\infty(\mu)\ ;\ \Psi_U(f)\mathcal{H} \subset \mathcal{H}\}$. Here, the map $\Psi_U \colon L^\infty(\mu) \to B(\hat{\mathcal{H}})$ denotes the associated L^∞-functional calculus of U. In that case, the measure μ can be viewed as a measure on ∂_A by trivial extension. An A-isometry $T \in B(\mathcal{H})^d$ is called *regular* if the triple $(A|_{\partial_A}, \partial_A, \mu)$ is regular in the sense of Aleksandrov (cf. [4]) which means by definition that, for every continuous function $\varphi \in C(\partial_A)$ with $\varphi > 0$, there exists a sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ in A such that $|\varphi_k| < \varphi$ on ∂_A for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} |\varphi_k| = \varphi$ holds μ-almost everywhere on ∂_A. If $D \subset \mathbb{C}^d$ is a strictly pseudoconvex domain or a bounded symmetric domain, one can show that for the domain algebra

$$A = A(D) = \big\{f \in C(\overline{D}) \ ; \ f|_D \text{ is analytic}\big\}$$

the triple $(A|_{\partial_A}, \partial_A, \mu)$ is regular for all finite positive regular Borel measures μ on ∂_A (see [4, 26, 27]). For each such measure μ, the abstract Hardy space multiplication tuple

$$T_z = (T_{z_1}, \ldots, T_{z_d}) \in B(H^2_A(\mu))^d \quad \text{with} \quad H^2_A(\mu) = \overline{A|_{\partial_A}}^{\tau_{\|\cdot\|_{L^2(\mu)}}} \subset L^2(\mu)$$

is a regular A-isometry. If we choose μ to be the canonical probability measure σ on $\partial_{A(D)}$, then we obtain the usual Hardy spaces.

We are now able to define Toeplitz operators associated with a regular A-isometry. To this end, let $T \in B(\mathcal{H})^d$ be a regular A-isometry, $U \in B(\hat{\mathcal{H}})^d$ a minimal normal extension of T, and μ a scalar spectral measure of T. For $f \in L^\infty(\mu)$, we call the operator $T_f = P_{\mathcal{H}}\Psi_U(f)|_{\mathcal{H}}$ the *concrete Toeplitz operator with symbol* f. If $f \in H^\infty_A(\mu) = \overline{A}^{\tau_{w^*}} \subset L^\infty(\mu)$, then T_f is called *analytic*. Since A is regular, Aleksandrov's results on the existence of abstract inner functions (cf. [4]) guarentee that the set

$$I_\mu = \{\theta \in H^\infty_A(\mu) \ ; \ |\theta| = 1 \ \mu\text{-a.e. on } \partial_A\}$$

of all μ-inner functions generates $L^\infty(\mu)$ as a von Neumann algebra (see [28, Corollary 2.5]). In the spirit of Brown and Halmos, we call an operator $X \in B(\mathcal{H})$ an *abstract Toeplitz operator* if

$$T^*_\theta X T_\theta - X = 0$$

holds for all $\theta \in I_\mu$. The joint work of Eschmeier and Everard [37] shows that under the condition that the von Neumann algebra $W^*(U)$ is maximal abelian the sets of abstract and concrete Toeplitz operators coincide. This is, for instance, true in the case $T = T_z \in B(H^2_A(\mu))^d$ with $U = M_z \in B(L^2(\mu))^d$. Since in the results of Davidson, and Guo and Wang Hankel operators appear, we define the *Hankel operator with symbol* $f \in L^\infty(\mu)$ by

$$H_f = (\mathrm{id}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})\Psi_U(f)|_{\mathcal{H}} \in B(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H}).$$

The generalization of the result by Guo and Wang reads as follows.

9 Theorem (Theorem 7.14; Theorem 8 in [30]). *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with empty point spectrum, minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M^+_1(\partial_A)$. Suppose that $W^*(U)$ is a maximal abelian von Neumann algebra. For $X \in B(\mathcal{H})$, the following statements are equivalent:*

(i) *$XT_f - T_fX$ is of finite rank for all $f \in H_A^\infty(\mu)$,*

(ii) *$X = T_f + F$ for some finite rank operator $F \in B(\mathcal{H})$ and $f \in L^\infty(\mu)$ such that $H_f \in B(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$ has finite rank.*

In our setting, the result of Davidson takes the following form.

10 Theorem (Theorem 7.18; Corollary 4 in [30]). *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$. Suppose that $W^*(U)$ is a maximal abelian von Neumann algebra. For $p \in [1, \infty)$ and $X \in B(\mathcal{H})$, the following statements are equivalent:*

(i) *$XT_f - T_fX$ is in the Schatten-p-class for all $f \in H_A^\infty(\mu)$,*

(ii) *$X = T_f + S$ for some Schatten-p-class operator $S \in B(\mathcal{H})$ and $f \in L^\infty(\mu)$ such that $H_f \in B(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$ lies in the Schatten-p-class.*

To generalize Theorem 8, we focus on the case when $A = A(D)$ for some bounded domain $D \subset \mathbb{C}^d$. Note that map

$$r_m \colon H^\infty(\mathbb{D}) \to L^\infty(m),\ f \mapsto f^*,$$

where f^* denotes the non-tangential boundary value of $f \in H^\infty(\mathbb{D})$, is isometric, τ_{w^*}-continuous and satisfies $r_m(f|_{\mathbb{D}}) = f|_{\mathbb{T}}$ for all $f \in A(\mathbb{D})$. We call a scalar spectral measure μ of a regular $A(D)$-isometry $T \in B(\mathcal{H})^d$ a *faithful Henkin measure* if there exists an isometric τ_{w^*}-continuous algebra homomorphism

$$r_\mu \colon H^\infty(D) \to L^\infty(\mu),\ f \mapsto r_\mu(f) =: f^*$$

with $r_\mu(f|_D) = f|_{\partial_{A(D)}}$ for all $f \in A(D)$.

11 Theorem (Theorem 8.12; Theorem 2 in [30]). *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with respect to $A = A(D)$, where $D \subset \mathbb{C}^d$ is a bounded domain such that the associated scalar spectral measure $\mu \in M_1^+(\partial_{A(D)})$ is a faithful Henkin probability measure. Suppose that $W^*(U)$ is a maximal abelian von Neumann algebra. For $p \in [1, \infty)$ and $X \in B(\mathcal{H})$, the following statements are equivalent:*

(i) *$T_\theta^* X T_\theta - X$ is in the Schatten-p-class for all $\theta \in I_\mu$,*

(ii) *$X = T_f + S$ for some Schatten-p-class operator $S \in B(\mathcal{H})$ and $f \in L^\infty(\mu)$.*

Part I.

K-contractions

1. Preliminaries

In this chapter we recall some fundamental facts about subnormal operators and reproducing kernel Hilbert spaces which will be needed in the sequel. All Hilbert spaces are supposed to be complex. In the whole thesis, d will always denote a positive integer.

1.1. Subnormal operators

Fix two Hilbert spaces $\mathcal{H}$ and $\hat{\mathcal{H}}$. We call a tuple $T \in B(\mathcal{H})^d$ of bounded linear operators *commuting* if

$$T_i T_j = T_j T_i$$

for all $i, j = 1, \ldots, d$. Furthermore, we use the notation $T^* = (T_1^*, \ldots, T_d^*) \in B(\mathcal{H})^d$.

We start with a well-known fact about the C^*-algebra and the von Neumann-algebra generated by a commuting tuple of normal operators.

1.1 Lemma. *Let $N \in B(\mathcal{H})^d$ be a commuting tuple of normal operators. Then*

$$N_i^* N_j = N_j N_i^*$$

for all $i, j = 1, \ldots, d$, and hence, $C^*(N)$ and $W^*(N)$ are abelian.

The statement follows from the Putnam-Fuglede theorem (cf. [19, Theorem IX.6.7]).

1.2 Definition. Let $T \in B(\mathcal{H})^d$ be a tuple of commuting operators, and let $\mathcal{M} \subset \mathcal{H}$ be a closed subspace.

(i) We call $\mathcal{M}$ an *invariant subspace of* T if $T_i\mathcal{M} \subset \mathcal{M}$ for all $i = 1, \ldots, d$. We write $\mathrm{Lat}(T)$ for the set of all closed invariant subspaces of T.

(ii) We call $\mathcal{M}$ a *reducing subspace of* T if $\mathcal{M}$ is an invariant subspace of T and T^*. We write $\mathrm{Red}(T)$ for the set of all closed reducing subspaces of T.

Since commuting tuples of normal operators admits a very rich spectral theory, we are interested in commuting tuples which possess commuting normal extensions or dilations. The following definiton can be found in [51, Definition 2 & 3].

1.3 Definition. Let $T \in B(\mathcal{H})^d$ and $S \in B(\hat{\mathcal{H}})^d$ be commuting tuples.

(i) We call S an *extension* of T if $\mathcal{H} \subset \hat{\mathcal{H}}$ and $S_i h = T_i h$ for all $i = 1, \ldots, d$ and $h \in \mathcal{H}$.

(ii) We call S a *normal extension of* T if S is an extension of T which consists of normal operators. If such an extension exists, we call T *subnormal.*

(iii) If S is a normal extension of T, we call S *minimal* if the only reducing subspace for S that contains $\mathcal{H}$ is the space $\hat{\mathcal{H}}$.

(iv) We call S a *([minimal] normal) coextension* of T if S^* is a ([minimal] normal) extension of T^*. If a normal coextension exists, we call T *cosubnormal.*

Let $T \in B(\mathcal{H})^d$ and $S \in B(\hat{\mathcal{H}})^d$ be commuting tuples. If there exists an isometry $\psi\colon \mathcal{H} \to \hat{\mathcal{H}}$, by an abuse of language, we call S an extension of T if

$$S_i \psi h = \psi T_i h$$

for all $i = 1, \ldots, d$ and $h \in \mathcal{H}$. In other words, we identify $\mathcal{H}$ with $\psi(\mathcal{H})$ and T with $S|_{\mathrm{Im}(\psi)} \in B(\mathrm{Im}(\psi))^d$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. We say that commuting tuples $T \in B(\mathcal{H}_1)^d$ and $S \in B(\mathcal{H}_2)^d$ are *similar (unitarily equivalent)* if there exists an invertible (a unitary) operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$UT_i = S_i U$$

for all $i = 1, \ldots, d$.

The following result guarantees the existence and uniqueness of minimal normal extensions.

1.4 Proposition. *Let $T \in B(\mathcal{H})^d$ be a subnormal commuting tuple with normal extension $N \in B(\hat{\mathcal{H}})^d$. Then N is a minimal normal extension of T if and only if*

$$\hat{\mathcal{H}} = \bigvee \left\{ N^{*\alpha} h \ ; \ \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\}.$$

Furthermore, minimal normal extensions are unique up to unitary equivalence modulo a unitary operator which acts as the identity operator on $\mathcal{H}$.

Proof. By Lemma 1.1, the closed linear span on the right is reducing for N. Obviously, it is the smallest reducing subspace for N that contains $\mathcal{H}$.

The second part follows from [51, Theorem 2]. □

In the following, we will denote by $\sigma(T)$ the *Taylor spectrum* of a commuting tuple $T \in B(\mathcal{H})^d$. For further reading, we recommend [38].

1.5 Lemma. *Similar commuting tuples $T \in B(\mathcal{H}_1)^d$ and $S \in B(\mathcal{H}_2)^d$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, possess the same Taylor spectrum.*

A proof of this lemma can be found in [38, Lemma 2.2.3].

Let T be a commuting tuple and N be a minimal normal extension of T. We call

$$\sigma_n(T) = \sigma(N)$$

the *normal spectrum of* T, which is well-defined by the last two results.

The following lemma is a reformulation of a well-known result by Athavale (cf. [10, Proposition 2] and [9, Corollary 1 on p. 217]).

1.6 Lemma (Athavale)**.** *Let $V \in B(\mathcal{H})^d$ be a commuting tuple. Then V satisfies $\sum_{i=1}^d V_i V_i^* = \mathrm{id}_{\mathcal{H}}$ if and only if V is cosubnormal with $\sigma_n(V^*) \subset \mathbb{S}_d$, where $\mathbb{S}_d$ denotes the unit sphere in $\mathbb{C}^d$.*

1.2. Reproducing kernel Hilbert spaces

Since reproducing kernel Hilbert spaces play an important role throughout this thesis, we provide some basic results. Here, we use [48, 11, 12] as guidelines. For further reading, we recommend [7] and the books [58, 3].

Let X be a non-empty set and let $\mathcal{E}$ be Hilbert space.

1.7 Definition. We call a Hilbert space $\mathcal{H} \subset \mathcal{E}^X$ a *reproducing kernel Hilbert space* if the point evaluations

$$\delta_x \colon \mathcal{H} \to \mathcal{E}, \ f \mapsto f(x)$$

are continuous for all $x \in X$.

The following proposition follows from the Riesz representation theorem and justifies the terminology introduced above.

1.8 Proposition. *Let $\mathcal{H} \subset \mathcal{E}^X$ be a reproducing kernel Hilbert space. Then there exists a unique function $K \colon X \times X \to B(\mathcal{E})$, called the* reproducing kernel for $\mathcal{H}$*, which satisfies*

(i) $K(\cdot, y)\eta \in \mathcal{H}$ *for all* $y \in X$ *and* $\eta \in \mathcal{E}$,

(ii) $\langle f, K(\cdot, y)\eta\rangle_{\mathcal{H}} = \langle f(y), \eta\rangle_{\mathcal{E}}$ *for all* $f \in \mathcal{H}$, $y \in X$, *and* $\eta \in \mathcal{E}$.

1.9 Proposition. *Let* $\mathcal{H} \subset \mathcal{E}^X$ *be a reproducing kernel Hilbert space with reproducing kernel* K. *Then*

(i) $K(x, y) = \delta_x \delta_y^*$ *for all* $x, y \in X$,

(ii) *the set*

$$\{K(\cdot, y)\eta \ ; \ y \in X, \eta \in \mathcal{E}\} \subset \mathcal{H}$$

is total, i.e., the closed linear span of $\{K(\cdot, y)\eta \ ; \ y \in X, \eta \in \mathcal{E}\}$ *is* $\mathcal{H}$.

Proof. (i) This is an easy calculation.

(ii) For $f \in \{K(\cdot, y)\eta \ ; \ y \in X, \eta \in \mathcal{E}\}^{\perp}$, we have

$$0 = \langle f, K(\cdot, y)\eta\rangle = \langle f(y), \eta\rangle$$

for all $y \in X$ and $\eta \in \mathcal{E}$, and hence, $f(y) = 0$ for all $y \in X$. But this means $f = 0$, and thus, $\{K(\cdot, y)\eta \ ; \ y \in X, \eta \in \mathcal{E}\}^{\perp} = \{0\}$. The result follows. □

It is well known that the reproducing kernel of a scalar-valued reproducing kernel Hilbert space can be calculated using an arbitrary orthonormal basis.

1.10 Lemma. *Let* $\mathcal{H} \subset \mathbb{C}^X$ *be a reproducing kernel Hilbert space with reproducing kernel* K *and orthonormal basis* $(e_i)_{i \in I}$. *Then*

$$K(\cdot, y) = \tau_{\|\cdot\|}\text{-}\sum_{i \in I} \overline{e_i(y)} e_i$$

for all $y \in X$, *where* $\tau_{\|\cdot\|}$ *denotes the norm topology.*

Since the reproducing kernel Hilbert spaces in the next chapters will often be defined on open subsets of $\mathbb{C}^n$, the following example, which can be found in [12, Example 1.1.3 (b)], will be useful.

1.11 Example. Let $D \subset \mathbb{C}^d$ be open and let $\mathcal{H} \subset \mathcal{E}^D$ be a reproducing kernel Hilbert space with reproducing kernel K. The following statements are equivalent:

(i) $\mathcal{H} \subset \mathcal{O}(D, \mathcal{E})$,

(ii) the map $D \to B(\mathcal{H}, \mathcal{E}),\ z \mapsto \delta_z$ is analytic,

(iii) the kernel K is sesquianalytic, i.e., analytic in the first component and antianalytic in the second component.

In this case, if $\mathcal{E}$ is separable, then $\mathcal{H}$ is also separable.

1.12 Definition. A function $K\colon X \times X \to B(\mathcal{E})$ is called *positive definite* if, for all $x_1, \dots, x_n \in X$,

$$(K(x_i, x_j))_{i,j=1}^n \in B(\mathcal{E}^n)$$

is a positive operator.

We often use the trivial identification $\mathbb{C} \cong B(\mathbb{C})$ given by the isomorphism

$$B(\mathbb{C}) \to \mathbb{C}, \ A \mapsto A(1).$$

The connection between reproducing kernel Hilbert spaces and positive definite functions is due to Moore (cf. [12, Theorem 1.1.5]).

1.13 Theorem (Moore)**.** *(i) The reproducing kernel of a reproducing kernel Hilbert space is a positive definite function.*

(ii) Every positive definite function is the reproducing kernel of a unique reproducing kernel Hilbert space.

If $K\colon X \times X \to B(\mathcal{E})$ is a positive definite function, we write $H_K(\mathcal{E})$ for the unique reproducing kernel Hilbert space from Moore's theorem. In the case $\mathcal{E} = \mathbb{C}$, we use the abbreviation H_K.

For a scalar-valued positive definite function $K\colon X \times X \to \mathbb{C}$ and a Hilbert space $\mathcal{E}$, the map

$$K_{\mathcal{E}} = K \cdot \mathrm{id}_{\mathcal{E}}$$

is positive definite again. In this case, we use the abbreviation $H_{K_{\mathcal{E}}} = H_{K_{\mathcal{E}}}(\mathcal{E})$ and say that $H_{K_{\mathcal{E}}}$ is an *inflation (of H_K along $\mathcal{E}$)* and $K_{\mathcal{E}}$ is *elementary (with respect to K)*. The following proposition gives us another perspective of such reproducing kernel Hilbert spaces. A proof of this statement can be found in [12, Proposition 1.2.2].

1.14 Proposition. *Let $K\colon X \times X \to \mathbb{C}$ be a scalar-valued positive definite function. Then there exists a unique Hilbert space isomorphism $U\colon H_K \otimes \mathcal{E} \to H_{K_{\mathcal{E}}}$ with*

$$U(f \otimes \eta) = f \cdot \eta$$

for all $f \in H_K$ and $\eta \in \mathcal{E}$.

1.15 Definition. Let $K\colon X \times X \to B(\mathcal{E})$ be a positive definite function.

(i) We call $H_K(\mathcal{E})$ *irreducible* if $K(x,y) \neq 0$ for all $x, y \in X$ and $K(\cdot, x)$ and $K(\cdot, y)$ are linearly independent if $x \neq y$.

(ii) We say that K is *normalized at* $x_0 \in X$ if $K(x, x_0) = \mathrm{id}_{\mathcal{E}}$ for all $x \in X$.

(iii) The space $H_K(\mathcal{E})$ is called *non-degenerate* if, for every $x \in X$, the point evaluation δ_x is onto.

(iv) We say $H_K(\mathcal{E})$ has *no common zeros* if, for every $x \in X$,

$$\mathrm{Im}(\delta_x) = \{f(x)\ ;\ f \in H_K(\mathcal{E})\} \neq \{0\}\,.$$

It is clear that irreducibility implies the absence of common zeros.

1.16 Lemma. *Let $K\colon X \times X \to B(\mathcal{E})$ be a positive definite function. The following assertions are equivalent:*

(i) $H_K(\mathcal{E})$ has no common zeros,

(ii) $K(x,x) \neq 0$ for all $x \in X$.

Proof. First suppose that (ii) holds and let $x \in X$. Then there exists $\eta \in \mathcal{E}$ such that $K(x,x)\eta \neq 0$. Hence,

$$\delta_x(K(\cdot, x)\eta) = K(x,x)\eta \neq 0,$$

i.e., $\mathrm{Im}(\delta_x) \neq \{0\}$.

Now suppose that (i) holds. We observe that

$$|\langle f(x), \eta\rangle|^2 = |\langle f, K(\cdot, x)\eta\rangle|^2 \leq \|f\|^2\,\|K(\cdot, x)\eta\|^2 = \|f\|^2\,\langle K(x,x)\eta, \eta\rangle$$

for all $f \in H_K(\mathcal{E})$ and $\eta \in \mathcal{E}$. Thus, if there exists $x \in X$ such that $K(x,x) = 0$, then $f(x) = 0$ for all $f \in H_K(\mathcal{E})$, i.e., $\mathrm{Im}(\delta_x) = \{0\}$. But this is a contradiction and hence, $K(x,x) \neq 0$ for all $x \in X$. $\square$

1.17 Proposition. *Let $K\colon X \times X \to B(\mathcal{E})$ be a positive definite function. The following assertions are equivalent:*

(i) $H_K(\mathcal{E})$ is non-degenerate,

(ii) for all $x \in X$, the point evaluation δ_x has a right inverse,

(iii) for all $x \in X$, the operator $K(x,x) \in B(\mathcal{E})$ is invertible.

If $H_K(\mathcal{E})$ contains the constant functions, then $H_K(\mathcal{E})$ is non-degenerate. Furthermore, if $H_K(\mathcal{E})$ is an inflation, the above are also equivalent to

(iv) $H_K(\mathcal{E})$ has no common zeros.

A proof of this proposition can be found in [12, Proposition 1.3.2].

We conclude this section with a sufficient criterium for irreducibility.

1.18 Proposition. *Let $K\colon X \times X \to B(\mathcal{E})$ be a non-vanishing positive definite function which is normalized at some point $x_0 \in X$. If $H_K(\mathcal{E})$ is a separating set for X, then $H_K(\mathcal{E})$ is irreducible.*

Proof. Let $x, y \in X$ with $x \neq y$ and let $\alpha, \beta \in \mathbb{C}$ such that

$$\alpha K(\cdot, x) + \beta K(\cdot, y) = 0.$$

Since K is normalized at x_0, we obtain $\beta = -\alpha$ and hence,

$$\alpha(K(\cdot, x) - K(\cdot, y)) = 0.$$

Let $f \in H_K(\mathcal{E})$ be a separating function for x and y. Then

$$0 = \langle f, \alpha(K(\cdot, x) - K(\cdot, y))\eta\rangle = \overline{\alpha}\, \langle f(x) - f(y), \eta\rangle$$

for all $\eta \in \mathcal{E}$. Thus

$$\alpha = 0 = \beta,$$

i.e., $K(\cdot, x)$ and $K(\cdot, y)$ are linearly independent. □

1.3. Multipliers

Let X be a non-empty set, let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ be Hilbert spaces and $\mathcal{H} \subset \mathcal{E}^X, \mathcal{H}_1 \subset \mathcal{E}_1^X, \mathcal{H}_2 \subset \mathcal{E}_2^X$ reproducing kernel Hilbert spaces. The corresponding reproducing kernels are denoted by K, K_1, and K_2.

1.19 Definition. (i) We call a function $\varphi\colon X \to B(\mathcal{E}_1, \mathcal{E}_2)$ a *multiplier* between $\mathcal{H}_1$ and $\mathcal{H}_2$ if, for $f \in \mathcal{H}_1$,

$$\varphi \cdot f\colon X \to \mathcal{E}_2,\ x \mapsto \varphi(x)f(x)$$

belongs to $\mathcal{H}_2$. The set of all multipliers between $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted by $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$. We use the abbreviation $\mathcal{M}(\mathcal{H}) = \mathcal{M}(\mathcal{H}, \mathcal{H})$.

(ii) For $\varphi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, the operator

$$M_\varphi \colon \mathcal{H}_1 \to \mathcal{H}_2,\ f \mapsto \varphi \cdot f$$

is called the *multiplication operator with symbol* φ. The set of all multiplication operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted by $M(\mathcal{H}_1, \mathcal{H}_2)$. We use the abbreviation $M(\mathcal{H}) = M(\mathcal{H}, \mathcal{H})$.

By the closed graph theorem, it is easy to see that $M(\mathcal{H}_1, \mathcal{H}_2)$ is a linear subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$.

If $\mathcal{H}_1$ is non-degenerate, then the map

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \to M(\mathcal{H}_1, \mathcal{H}_2),\ \varphi \mapsto M_\varphi$$

is injective and hence, the map

$$\|\cdot\|_{\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)} : \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \to [0, \infty),\ \varphi \mapsto \|M_\varphi\|$$

is a well-defined norm on $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, the *multiplier norm.*

1.20 Lemma. *For every $\varphi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, we have*

$$M_\varphi^* K_2(\cdot, y)\eta = K_1(\cdot, y)\varphi(y)^*\eta$$

for all $y \in X$ and $\eta \in \mathcal{E}_2$.

Proof. We have

$$\begin{aligned} \left\langle f, M_\varphi^* K_2(\cdot, y)\eta \right\rangle &= \langle \varphi f, K_2(\cdot, y)\eta \rangle \\ &= \langle \varphi(y) f(y), \eta \rangle \\ &= \langle f(y), \varphi(y)^*\eta \rangle \\ &= \langle f, K_1(\cdot, y)\varphi(y)^*\eta \rangle \end{aligned}$$

for all $f \in \mathcal{H}_1$, $y \in X$, and $\eta \in \mathcal{E}_2$. □

The following result, which gives a sufficient condition for multipliers to be bounded, is a special case of [12, Corollary 1.7.7].

1.21 Proposition. *Let $\mathcal{H} \subset \mathbb{C}^X$ be a scalar-valued reproducing kernel Hilbert space and let $\mathcal{H}_1 \subset \mathcal{E}_1^X$, $\mathcal{H}_2 \subset \mathcal{E}_2^X$ be inflations of $\mathcal{H}$. If $\mathcal{H}$ is non-degenerate, then*

$$\|\varphi\|_\infty \leq \|\varphi\|_{\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)}$$

for all $\varphi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$.

1.22 Proposition (Barbian)**.** *Let $\mathcal{H}_1 \subset \mathcal{E}_1^X$, $\mathcal{H}_2 \subset \mathcal{E}_2^X$ be reproducing kernel Hilbert spaces. For $x \in X$ and $i = 1, 2$, we denote by $\delta_{i,x} \colon \mathcal{H}_i \to \mathcal{E}_i$ the point evaluation at x on $\mathcal{H}_i$. Furthermore, we suppose that $\mathcal{H}_1$ is non-degenerate. Then, for $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, the following statements are equivalent:*

(i) $T \ker(\delta_{1,x}) \subset \ker(\delta_{2,x})$ *for all* $x \in X$,

(ii) $T^* \operatorname{Im}(\delta_{2,x}^*) \subset \operatorname{Im}(\delta_{1,x}^*)$ *for all* $x \in X$,

(iii) $T \in M(\mathcal{H}_1, \mathcal{H}_2)$.

A proof can be found in [13, Theorem 2.1] or [12, Proposition 1.7.9]. The following class of multipliers will be important in Chapter 4.

1.23 Definition. We call a multiplier *inner* if the corresponding multiplication operator is a partial isometry.

1.4. Unitarily invariant spaces and regularity

We will now examine special classes of reproducing kernels on the open unit ball $\mathbb{B}_d$ more closely.

1.24 Definition. Let $\mathcal{H}$ be a scalar-valued reproducing kernel Hilbert space on $\mathbb{B}_d$ with reproducing kernel K. We call $\mathcal{H}$ a *unitarily invariant space on* $\mathbb{B}_d$ if K satisfies

$$K(Uz, Uw) = K(z, w)$$

for all $z, w \in \mathbb{B}_d$ and every unitary linear map $U \colon \mathbb{C}^d \to \mathbb{C}^d$.

The reproducing kernel of a unitarily invariant space $\mathcal{H} \subset \mathcal{O}(\mathbb{B}_d)$ has a quite particular form, as the the following propositon shows (cf. [49, Lemma 2.2]).

1.25 Proposition. *Let $K \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ be a function. The following statements are equivalent:*

(i) K is analytic in the first component, normalized at 0, and is the reproducing kernel of a unitarily invariant space on $\mathbb{B}_d$,

(ii) there exists an analytic function

$$k \colon \mathbb{D} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} a_n z^n,$$

where $a_0 = 1$ *and* $a_n \geq 0$ *for all* $n \in \mathbb{N}$*, such that*

$$K(z,w) = k(\langle z,w\rangle) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n$$

for all $z, w \in \mathbb{B}_d$*.*

1.26 Convention. From now on, let H_K be a unitarily invariant space on $\mathbb{B}_d$ whose reproducing kernel is of the form $K(z,w) = k(\langle z,w\rangle)$ $(z, w \in \mathbb{B}_d)$ for some analytic function $k\colon \mathbb{D} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} a_n z^n$ with $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$.

By [44, Proposition 4.1] and Example 1.11, we obtain the following result.

1.27 Proposition. *The family* $(\sqrt{\gamma_\alpha} z^\alpha)_{\alpha \in \mathbb{N}^d}$*, where* $\gamma_\alpha = a_{|\alpha|} \frac{|\alpha|!}{\alpha!}$ *for* $\alpha \in \mathbb{N}^d$*, is an orthonormal basis for* H_K*. In particular, we have*

$$\overline{\mathbb{C}[z]} = H_K \subset \mathcal{O}(\mathbb{B}_d).$$

Let $i \in \{1, \ldots, d\}$. We define

$$(M_{z_i} f)(z) = z_i f(z)$$

for all $f \in H_K$ and $z \in \mathbb{B}_d$. By [44, Corollary 4.4], the map

$$M_{z_i}\colon H_K \to H_K$$

is a well-defined bounded operator on H_K if and only if $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$.

1.28 Convention. From now on, we make the additional assumption that

$$\sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} < \infty.$$

We call the commuting tuple $M_z = (M_{z_1}, \ldots, M_{z_d}) \in B(H_K)^d$ the *K-shift* on H_K.

By

$$\mathbb{H}_n = \left\{ \sum_{|\alpha|=n} f_\alpha z^\alpha \ ;\ f_\alpha \in \mathbb{C} \text{ for } |\alpha| = n \right\} \subset \mathbb{C}[z]$$

we denote the set of all homogeneous polynomials of degree $n \in \mathbb{N}$.

For convenience, we set $a_n = 0$ for all negative integers n and $\gamma_\alpha = 0$ for all $\alpha \in \mathbb{Z}^d$ with $\alpha_i < 0$ for some $i \in \{1, \ldots, d\}$.

1.29 Lemma. *We have*

(i) $M_z^{*\beta} z^\alpha = \frac{\gamma_{\alpha-\beta}}{\gamma_\alpha} z^{\alpha-\beta}$ *for all* $\alpha, \beta \in \mathbb{N}^d$,

(ii) $\sum_{i=1}^d M_{z_i} M_{z_i}^* = \tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^\infty \frac{a_{n-1}}{a_n} P_{\mathbb{H}_n}$, *where* τ_{SOT} *denotes the strong operator topology on* $B(H_K)$.

Proof. (i) Let $\alpha, \beta \in \mathbb{N}^d$. For $\delta \in \mathbb{N}^d$, we have

$$\begin{aligned}
\left\langle M_z^{*\beta} z^\alpha, z^\delta \right\rangle &= \left\langle z^\alpha, M_z^\beta z^\delta \right\rangle \\
&= \left\langle z^\alpha, z^{\delta+\beta} \right\rangle \\
&= \begin{cases} \frac{1}{\gamma_\alpha}, & \text{if } \alpha = \beta + \delta, \\ 0, & \text{if } \alpha \neq \beta + \delta \end{cases} \\
&= \begin{cases} \left\langle \frac{\gamma_{\alpha-\beta}}{\gamma_\alpha} z^{\alpha-\beta}, z^\delta \right\rangle, & \text{if } \alpha = \beta + \delta, \\ 0, & \text{if } \alpha \neq \beta + \delta. \end{cases}
\end{aligned}$$

(ii) Since

$$\frac{\gamma_{\alpha-e_i}}{\gamma_\alpha} = \frac{a_{|\alpha|-1} \frac{(|\alpha|-1)!}{(\alpha-e_i)!}}{a_{|\alpha|} \frac{|\alpha|!}{\alpha!}} = \frac{a_{|\alpha|-1}}{a_{|\alpha|}} \frac{\alpha_i}{|\alpha|}$$

for all $\alpha \geq e_i$ and $i = 1, \ldots, d$, we conclude with (i) that

$$\sum_{i=1}^d M_{z_i} M_{z_i}^* z^\alpha = \sum_{i=1}^d \frac{\gamma_{\alpha-e_i}}{\gamma_\alpha} z^\alpha = \frac{a_{n-1}}{a_n} z^\alpha$$

for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = n$. □

We conclude this chapter by looking at the case when M_z is essentially normal. For this purpose, we first recall the definition and some fundamental results.

1.30 Definition. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. We say that T is *essentially normal* if T_i is essentially normal for all $i = 1, \ldots, d$.

1.31 Remark. A commuting tuple $T \in B(\mathcal{H})^d$ is essentially normal if and only if

$$T_i^* T_j - T_i T_j^*$$

is compact for each $i, j = 1, \ldots, d$. This is a consequence of the C^*-algebra version of Lemma 1.1.

1.32 Lemma. *The K-shift M_z on H_K is essentially normal if and only if*

$$\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}-\frac{a_{n-1}}{a_n}\right)=0.$$

This result was obtained in [44, Corollary 4.4]. By [44, Theorem 4.5], we can calculate the spectrum and essential spectrum of essentially normal K-shifts.

1.33 Lemma. *Suppose that the K-shift M_z on H_K is essentially normal. For*

$$s=\left(\liminf_{n\to\infty}\frac{a_n}{a_{n+1}}\right)^{1/2}\quad \textit{and}\quad t=\left(\limsup_{n\to\infty}\frac{a_n}{a_{n+1}}\right)^{1/2},$$

we have

$$\sigma(M_z)=\left\{z\in\mathbb{C}^d\ ;\ |z|\le t\right\}\quad \textit{and}\quad \sigma_{\mathrm{e}}(M_z)=\left\{z\in\mathbb{C}^d\ ;\ s\le|z|\le t\right\},$$

where $\sigma_{\mathrm{e}}(M_z)$ denotes the essential Taylor spectrum of M_z.

1.34 Remark. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $x_n>0$ for all $n\in\mathbb{N}$. Then

$$\liminf_{n\to\infty}\frac{x_{n+1}}{x_n}\le\liminf_{n\to\infty}\sqrt[n]{x_n}\le\limsup_{n\to\infty}\sqrt[n]{x_n}\le\limsup_{n\to\infty}\frac{x_{n+1}}{x_n}.$$

In the setting of the last lemma, if the radius of convergence of k is 1, we have

$$s^2=\liminf_{n\to\infty}\frac{a_n}{a_{n+1}}\le 1\le\limsup_{n\to\infty}\frac{a_n}{a_{n+1}}=t^2,$$

i.e.,

$$\mathbb{S}_d\subset\sigma_{\mathrm{e}}(M_z)\subset\sigma(M_z).$$

If $M_z\in B(H_K)^d$ is essentially normal, we have that

$$K(H_K)\subset C^*(M_z)$$

by [44, Proposition 2.1] and [8, Corollary 2 of Theorem 1.4.2].

1.35 Definition. We call H_K *regular* if

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1.$$

Note that our notion of regularity is stronger than the one in [44].

1.36 Remark. If H_K is regular, then, by Lemmas 1.32 and 1.33, $M_z\in B(H_K)^d$ is essentially normal with $\sigma(M_z)=\overline{\mathbb{B}}_d$ and $\sigma_{\mathrm{e}}(M_z)=\mathbb{S}_d$ and, by Remark 1.34, the radius of convergence of k is necessarily 1.

The following result is a particular case of [44, Theorem 4.6].

1.37 Theorem (Guo, Hu, Xu)**.** *If H_K is regular, then there is an exact sequence of C^*-algebras*

$$0 \to K(H_K) \hookrightarrow C^*(M_z) \xrightarrow{\pi} C(\mathbb{S}_d) \to 0,$$

where π is a unital $*$-homomorphism uniquely determined by $\pi(M_{z_i}) = z_i|_{\mathbb{S}_d}$ for $i = 1, \dots, d$. In particular, the operator $\mathrm{id}_{H_K} - \sum_{i=1}^d M_{z_i} M_{z_i}^*$ is compact.

1.5. The setting and examples

The irreducibility of our unitarily invariant space H_K can easily be characterized in terms of the function k.

1.38 Lemma. *For H_K as before, the following are equivalent:*

(i) the function $k\colon \mathbb{D} \to \mathbb{C}$ has no zeros in $\mathbb{D}$,

(ii) H_K is irreducible.

Proof. Since k has no zeros if and only if the reproducing kernel K is non-vanishing, the result follows by the fact that

$$\mathbb{C}[z] \subset \mathcal{H}$$

holds and Proposition 1.18. □

1.39 Convention. For the rest of Part I of this thesis, we always suppose that, in addition to Conventions 1.26 and 1.28, the function $k\colon \mathbb{D} \to \mathbb{C}$ has no zeros.

1.40 Remark. Since $1 \in H_K$, the space H_K is non-degenerate.

By Lemma 1.10 and Proposition 1.27, we have that

$$K(z,w) = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha z^\alpha \overline{w}^\alpha$$

for all $z, w \in \mathbb{B}_d$. One can show that

$$H_{K_\mathcal{E}} = \left\{ f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}_d, \mathcal{E}) \ ; \ \|f\|^2 = \sum_{\alpha \in \mathbb{N}^d} \frac{\|f_\alpha\|^2}{\gamma_\alpha} < \infty \right\}.$$

Since k has no zeros in the unit disc, the function

$$\frac{1}{k}\colon \mathbb{D} \to \mathbb{C},\ z \mapsto \frac{1}{k(z)}$$

is again analytic and hence admits a Taylor expansion

$$\frac{1}{k}(z) = \sum_{n=0}^{\infty} c_n z^n \quad (z \in \mathbb{D})$$

with a suitable sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Note that $c_0 = 1$.

In the following, we will use the convention

$$\frac{1}{\infty} = 0.$$

1.41 Remark. (i) By standard results on Abel-summability, we always have

$$\lim_{r\to 1} \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} a_n \in (1, \infty].$$

(ii) If $\sum_{n=0}^{\infty} c_n$ is convergent, then

$$\sum_{n=0}^{\infty} c_n = \frac{1}{\sum_{n=0}^{\infty} a_n} \in [0, 1).$$

Indeed, by Abel's limit theorem, we have

$$\sum_{n=0}^{\infty} c_n = \lim_{r\to 1} \sum_{n=0}^{\infty} c_n r^n = \lim_{r\to 1} \frac{1}{\sum_{n=0}^{\infty} a_n r^n} = \frac{1}{\sum_{n=0}^{\infty} a_n}.$$

1.5.1. Complete Nevanlinna-Pick spaces

The following definition is a combination of the Definitions 5.12 & 5.13 and Exercise 5.14 in [3].

1.42 Definition. Let $K\colon X \times X \to \mathbb{C}$ be a scalar-valued positive definite function. We call H_K a *complete Nevanlinna-Pick space* if, whenever $x_1, \ldots, x_n \in X$ and $W_1, \ldots, W_n \in B(\ell^2(\mathbb{N}))$ such that

$$\left((\mathrm{id}_{\ell^2(\mathbb{N})} - W_i W_j^*)K(x_i, x_j)\right)_{i,j=1}^n \in B(\ell^2(\mathbb{N})^n)$$

is positive, then there exists a multiplier φ in the closed norm unit ball of $\mathcal{M}(H_K(\ell^2(\mathbb{N})))$ such that

$$\varphi(x_i) = W_i$$

for all $i = 1, \ldots, n$.

The following generalization of [3, Theorem 7.33] by Hartz (cf. [49, Lemma 2.3]) gives us a criterion for our space to be a complete Nevanlinna-Pick space by the means of the Taylor coefficients of the reciprocal of the reproducing kernel.

1.43 Proposition. *Let H_K be a reproducing kernel Hilbert space that satisfies Convention 1.39. Then the following assertions are equivalent:*

(i) H_K is a complete Nevanlinna-Pick space,

(ii) we have $c_n \leq 0$ for all $n \geq 1$.

1.44 Example. Let $\sigma \leq 0$. The reproducing kernel Hilbert space $\mathcal{H}_\sigma$ with reproducing kernel

$$K_\sigma \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z,w) \mapsto \sum_{n=0}^{\infty} (n+1)^\sigma \langle z, w\rangle^n$$

is an irreducible complete Nevanlinna-Pick space by [3, Corollary 7.41]. The space $\mathcal{H}_0$ corresponds to the Hardy space ($d = 1$) or the Drury-Arveson space ($d \geq 2$), and the space $\mathcal{H}_{-1}$ coincides with the Dirichlet space. Furthermore, $\mathcal{H}_\sigma$ is regular.

If our space H_K is a complete Nevanlinna-Pick space, then the following result by Greene, Richter and Sundberg [41, Proposition 4.5] provides a sufficient condition for H_K to be regular.

1.45 Lemma. *Let H_K be a reproducing kernel Hilbert space that satisfies Convention 1.39 and is a complete Nevanlinna-Pick space. Suppose that $\sum_{n=0}^\infty c_n = 0$ and either $\sum_{n=0}^\infty nc_n > -\infty$ or $(a_n)_{n\in\mathbb{N}}$ is eventually nonincreasing. Then H_K is regular.*

Further results on Nevannlinna-Pick spaces can be found in [3].

1.5.2. Weighted Bergman spaces

Other important examples are generalized weighted Bergman spaces. These spaces are the irreducible unitarily invariant spaces on $\mathbb{B}_d$ with reproducing kernels

$$K = K^{(\nu)} \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z,w) \mapsto \frac{1}{(1 - \langle z, w\rangle)^\nu},$$

where ν is a positive real number. In this case, we have

$$a_n = a_n^{(\nu)} = (-1)^n \binom{-\nu}{n} = \prod_{j=1}^{n} \frac{\nu + (j-1)}{j}$$

and

$$c_n = c_n^{(\nu)} = (-1)^n \binom{\nu}{n} = \prod_{j=1}^{n} \frac{-\nu + (j-1)}{j}$$

for all $n \in \mathbb{N}$. Since

$$\frac{a_n^{(\nu)}}{a_{n+1}^{(\nu)}} = \frac{n+1}{n+\nu} \to 1$$

as $n \to \infty$, we see that $H_{K^{(\nu)}}$ is regular. Furthermore, if we define

$$n_0 = \min\{n \in \mathbb{N}\ ;\ (n-1) - \nu \geq 0\},$$

then, for all $n \geq n_0$, we have

$$c_n^{(\nu)} \leq 0$$

if n_0 is even and

$$c_n^{(\nu)} \geq 0$$

if n_0 is odd.

If $\nu > d$, then

$$H_{K^{(\nu)}} = \mathcal{O}(\mathbb{B}_d) \cap L^2(\mathbb{B}_d, \mathrm{d}v_\nu),$$

where

$$\mathrm{d}v_\nu(z) = \frac{\Gamma(\nu)}{\Gamma(\nu - d)\pi^d}(1 - |z|^2)^{\nu - d - 1}\mathrm{d}\lambda(z) \qquad (z \in \mathbb{B}_d)$$

is a probability measure on $\mathbb{B}_d$ which is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{B}_d$. These spaces are studied in [65].

2. A geometric model

Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ a bounded linear operator. Then T is a contraction if and only if $\mathrm{id}_{\mathcal{H}} - TT^* \geq 0$. Let

$$K_{H^2(\mathbb{D})} \colon \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ (z, w) \mapsto \frac{1}{1 - z\overline{w}}$$

be the reproducing kernel of the Hardy space on the unit disc $H^2(\mathbb{D})$. If we replace z with T and $\overline{w}$ with T^*, we see that the inequality above can be written as

$$\frac{1}{K_{H^2(\mathbb{D})}}(T, T^*) \geq 0.$$

Hence, contractions are related to the Hardy space on the unit disc. Furthermore, the theory by Sz.-Nagy and Foias [62] shows that every contraction is unitarily equivalent to a restriction of an operator of the form $((M_z \otimes \mathrm{id}_{\mathcal{D}}) \oplus U)^*$ to an invariant subspace, where M_z is the shift operator on $H^2(\mathbb{D})$, $\mathcal{D}$ is a Hilbert space, and U is a unitary operator. The problem of characterizing operators which satisfy an inequality obtained in a similar way from other reproducing kernel Hilbert spaces has received considerable attention over the last couple of decades. One of the main problems is to give sense to the expression $1/K(T, T^*)$ for an arbitrary reproducing kernel K. In this chapter, the upcoming definition of the aforementioned term is inspired by the work of Arazy and Engliš [6], and Clouâtre and Hartz [18].

Let $\mathcal{H}$ be a Hilbert space, $T \in B(\mathcal{H})^d$ a commuting tuple, and let $H_K \subset \mathcal{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space whose kernel $K \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is of the form $K(z, w) = k(\langle z, w \rangle)$ $(z, w \in \mathbb{B}_d)$ with a zero-free analytic function $k \colon \mathbb{D} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} a_n z^n$ such that $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$, and $\sup_{n \in \mathbb{N}} a_n / a_{n+1} < \infty$.

For $N \in \mathbb{N}$, let

$$\left(\frac{1}{K}\right)_N (z, w) = \sum_{n=0}^{N} c_n \langle z, w \rangle^n = \sum_{|\alpha| \leq N} c_{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha \qquad (z, w \in \mathbb{B}_d)$$

be the Nth partial sum of $1/K$. We define

$$\left(\frac{1}{K}\right)_N (T, T^*) = \sum_{n=0}^{N} c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) = \sum_{|\alpha| \leq N} c_{|\alpha|} \frac{|\alpha|!}{\alpha!} T^\alpha T^{*\alpha}$$

for all $N \in \mathbb{N}$, where

$$\sigma_T\colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \sum_{i=1}^{d} T_i X T_i^{\,*}.$$

With these preparations, we set

$$\frac{1}{K}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \left(\frac{1}{K}\right)_N (T, T^*)$$

if the latter exists. Note that in case where $1/K$ is a polynomial in z and $\overline{w}$, the limit always exists.

2.1 Definition. We call a commuting tuple $T \in B(\mathcal{H})^d$ a *K-contraction* if $1/K(T, T^*)$ exists and defines a positive operator.

Note that, for $n \in \mathbb{N}^*$ and the kernel $K^{(n)}$ defined as in Section 1.5.2, the identity

$$\frac{1}{K^{(n)}}(T, T^*) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} T^\alpha T^{*\alpha}$$

holds.

For $m \in \mathbb{N}^*$, we call a commuting tuple $T \in B(\mathcal{H})^d$ an *m-hypercontraction* if T is a $K^{(n)}$-contraction for $n = 1, \ldots, m$ (cf. [55]).

2.2 Remark. Let $T \in B(\mathcal{H})^d$ be a commuting tuple.

(i) If $d = 1$, the $K^{(1)}$-contractions are precisely the contractions.

(ii) Let $m \in \mathbb{N}^*$. By [55, Lemma 2], T is an m-hypercontraction if and only if T is a $K^{(1)}$-contraction as well as a $K^{(m)}$-contraction.

In the one-dimensional case, a contraction $T \in B(\mathcal{H})$ is said to be of class $C_{\cdot 0}$ if

$$\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} (T^*)^N = 0.$$

We extend this notion to the case of K-contractions following ideas of [5, 6]. First we recall a well-known convergence result for sequences of selfadjoint operators.

2.3 Lemma. *Let $(A_N)_{N\in\mathbb{N}}$ be a decreasing sequence of selfadjoint bounded linear operators on $\mathcal{H}$. The following are equivalent:*

(i) $(A_N)_{N\in\mathbb{N}}$ *is norm-bounded,*

(ii) $\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} A_N$ *exists.*

Furthermore, the following statements are equivalent:

(iii) $A_N \geq 0$ *for all* $N \in \mathbb{N}$,

(iv) $\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} A_N \geq 0$.

Proof. (i) $\implies$ (ii): By assumption, there exists a non-negative real number C such that

$$\|A_N\| \leq C$$

for all $N \in \mathbb{N}$. Define $B_N = A_0 - A_N$ for all $N \in \mathbb{N}$. The sequence $(B_N)_{N\in\mathbb{N}}$ fulfills

$$0 \leq B_N \leq B_{N+1} \leq 2C \cdot \mathrm{id}_{\mathcal{H}}$$

for all $N \in \mathbb{N}$. Hence, there exists a selfadjoint operator $B \in B(\mathcal{H})$ such that

$$\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} B_N = B.$$

Therefore, we obtain

$$\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} A_N = A_0 - B.$$

(ii) $\implies$ (i): This follows immediately from the Banach-Steinhaus theorem.
(iii) $\iff$ (iv): This is clear. □

The following definition originates from [5] and [6].
Let $T \in B(\mathcal{H})^d$ be a K-contraction and define

$$\Sigma_{K,N}(T) = \mathrm{id}_{\mathcal{H}} - \sum_{n=0}^{N} a_n \sigma_T^n\left(\frac{1}{K}(T,T^*)\right) = \mathrm{id}_{\mathcal{H}} - \sum_{|\alpha|\leq N} \gamma_\alpha T^\alpha\left(\frac{1}{K}(T,T^*)\right)T^{*\alpha}$$

for all $N \in \mathbb{N}$. We suppress the superscript K if the reproducing kernel is clear from the context. Since we have

$$\Sigma_N(T) - \Sigma_{N+1}(T) = a_{N+1}\sigma_T^{N+1}\left(\frac{1}{K}(T,T^*)\right) \geq 0$$

for all $N \in \mathbb{N}$, $(\Sigma_N(T))_{N\in\mathbb{N}}$ is a decreasing sequence of selfadjoint bounded linear operators. Furthermore, we write

$$\Sigma_K(T) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \Sigma_{K,N}(T)$$

if the latter exists. Again, if the reproducing kernel is clear from the context, we suppress the superscript K.

2.4 Corollary. *Let $T \in B(\mathcal{H})^d$ be a K-contraction. The following are equivalent:*

(i) $(\Sigma_N(T))_{N\in\mathbb{N}}$ *is norm-bounded,*

(ii) $\Sigma(T)$ *exists.*

Furthermore, the following statements are equivalent:

(iii) $\Sigma_N(T) \geq 0$ *for all* $N \in \mathbb{N}$,

(iv) $\Sigma(T) \geq 0$.

2.5 Remark. If $T \in B(\mathcal{H})^d$ is a $K^{(1)}$-contraction, then

$$\begin{aligned}\Sigma_{K^{(1)},N}(T) &= \mathrm{id}_{\mathcal{H}} - \sum_{n=0}^{N} \sigma_T^n\left(\mathrm{id}_{\mathcal{H}} - \sigma_T(\mathrm{id}_{\mathcal{H}})\right)\\ &= \mathrm{id}_{\mathcal{H}} - \left(\mathrm{id}_{\mathcal{H}} - \sigma_T^{N+1}(\mathrm{id}_{\mathcal{H}})\right) = \sigma_T^{N+1}(\mathrm{id}_{\mathcal{H}})\end{aligned}$$

for all $N \in \mathbb{N}$, and hence,

$$\Sigma_{K^{(1)}}(T) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \sigma_T^{N+1}(\mathrm{id}_{\mathcal{H}}) \geq 0.$$

The following proposition is the cornerstone for our model theory which is a generalization of the one-dimensional case mentioned at the beginning of this chapter.

2.6 Proposition. *Let $T \in B(\mathcal{H})^d$ be a K-contraction such that $\Sigma(T)$ exists. The map*

$$\psi_T \colon \mathcal{H} \to H_K \otimes \mathcal{D}_T,\ h \mapsto \sum_{\alpha\in\mathbb{N}^d} \gamma_\alpha(z^\alpha \otimes D_T T^{*\alpha} h),$$

where $D_T = \left(1/K(T,T^*)\right)^{1/2}$ and $\mathcal{D}_T = \overline{D_T\mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\psi_T h\|^2 = \|h\|^2 - \langle \Sigma(T)h, h\rangle$$

for all $h \in \mathcal{H}$ and

$$\psi_T T_i^* = (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}_T})^* \psi_T$$

for all $i = 1, \ldots, d$.

Proof. Let $h \in \mathcal{H}$. With

$$
\begin{aligned}
\sum_{|\alpha| \le N} \|\gamma_\alpha (z^\alpha \otimes D_T T^{*\alpha} h)\|^2 &= \sum_{|\alpha| \le N} \gamma_\alpha \left\langle T^\alpha \frac{1}{K}(T, T^*) T^{*\alpha} h, h \right\rangle \\
&= \sum_{n=0}^{N} a_n \left\langle {\sigma_T}^n \left(\frac{1}{K}(T, T^*) \right) h, h \right\rangle \\
&= \|h\|^2 - \left\langle \left(\mathrm{id}_{\mathcal{H}} - \sum_{n=0}^{N} a_n {\sigma_T}^n \left(\frac{1}{K}(T, T^*) \right) \right) h, h \right\rangle \\
&= \|h\|^2 - \langle \Sigma_N(T) h, h \rangle \\
&\to \|h\|^2 - \langle \Sigma(T) h, h \rangle
\end{aligned}
$$

as $N \to \infty$ and the paragraph after Remark 1.40 it follows that the map ψ_T is a well-defined bounded linear operator with

$$
\|\psi_T h\|^2 = \|h\|^2 - \langle \Sigma(T) h, h \rangle \le (1 + \|\Sigma(T)\|) \|h\|^2
$$

for all $h \in \mathcal{H}$. In view of

$$
\begin{aligned}
\psi_T {T_i}^* h &= \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha (z^\alpha \otimes D_T T^{*\alpha + e_i} h) \\
&= \sum_{\alpha \ge e_i} \gamma_{\alpha - e_i} (z^{\alpha - e_i} \otimes D_T T^{*\alpha} h) \\
&= \sum_{\alpha \ge e_i} \gamma_\alpha \left(\frac{\gamma_{\alpha - e_i}}{\gamma_\alpha} z^{\alpha - e_i} \otimes D_T T^{*\alpha} h \right) \\
&= (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}_T})^* \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha (z^\alpha \otimes D_T T^{*\alpha} h) \\
&= (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}_T})^* \psi_T h
\end{aligned}
$$

for all $i = 1, \dots, d$ and $h \in \mathcal{H}$, the remaining assertion follows. □

2.7 Definition. Let $T \in B(\mathcal{H})^d$ be a K-contraction. If $\Sigma(T) = 0$, we call T *K-pure.*

2.8 Remark. In the setting of Proposition 2.6, if T is K-pure, then ψ_T is an isometry. Conversely, if ψ_T is a well-defined isometry, then the proof of Proposition 2.6 shows that

$$
\Sigma(T) = \tau_{\mathrm{SOT}}\text{-}\lim_{N \to \infty} \Sigma_N(T) = 0.
$$

We are now interested in sufficient conditions for M_z to be a K-contraction. The following proposition, which originates from [5, Proposition 13], reduces our problem to the bare existence of $1/K(M_z, M_z^*)$.

2.9 Proposition. *Suppose that $1/K(M_z, M_z^*)$ exists and let $\mathcal{E}$ be a Hilbert space. Then we have*

$$\frac{1}{K}(M_z \otimes \mathrm{id}_{\mathcal{E}}, (M_z \otimes \mathrm{id}_{\mathcal{E}})^*) = P_{\mathbb{C}} \otimes \mathrm{id}_{\mathcal{E}} \geq 0.$$

Proof. By Lemma 1.20, we have

$$\begin{aligned}&\left\langle \left(\frac{1}{K}\right)_N (M_z \otimes \mathrm{id}_{\mathcal{E}}, (M_z \otimes \mathrm{id}_{\mathcal{E}})^*)(K(\cdot, w) \otimes \eta), K(\cdot, z) \otimes \zeta \right\rangle \\ &= \left(\sum_{n=0}^{N} c_n \langle z, w \rangle^n \right) K(z, w) \langle \eta, \zeta \rangle\end{aligned}$$

for all $N \in \mathbb{N}$, $z, w \in \mathbb{B}_d$ and $\eta, \zeta \in \mathcal{E}$, and hence,

$$\left\langle \frac{1}{K}(M_z \otimes \mathrm{id}_{\mathcal{E}}, (M_z \otimes \mathrm{id}_{\mathcal{E}})^*)(K(\cdot, w) \otimes \eta), K(\cdot, z) \otimes \zeta \right\rangle = \langle \eta, \zeta \rangle$$

for all $z, w \in \mathbb{B}_d$ and $\eta, \zeta \in \mathcal{E}$. Furthermore, we see that

$$\langle (P_{\mathbb{C}} \otimes \mathrm{id}_{\mathcal{E}})(K(\cdot, w) \otimes \eta), K(\cdot, z) \otimes \zeta \rangle = \langle 1, K(\cdot, z) \rangle \langle \eta, \zeta \rangle = \langle \eta, \zeta \rangle$$

for all $z, w \in \mathbb{B}_d$ and $\eta, \zeta \in \mathcal{E}$. Since $\{K(\cdot, w) \otimes \eta \ ; \ w \in \mathbb{B}_d, \eta \in \mathcal{E}\} \subset H_K \otimes \mathcal{E}$ is total by Proposition 1.9, the result follows. $\square$

Let $(d_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$. We write $[d_n]_n \in B(H_K)$ for the diagonal operator with respect to the orthogonal decomposition $H_K = \bigoplus_{n \in \mathbb{N}} \mathbb{H}_n$:

$$[d_n]_n : H_K \to H_K, \ \sum_{n=0}^{\infty} f_n \mapsto \sum_{n=0}^{\infty} d_n f_n.$$

Since

$$M_{z_i} [d_n]_n f = \sum_{n=0}^{\infty} d_n (z_i f_n) = [d_{n-1}]_n M_{z_i} f$$

for $i = 1, \dots, d$ and $f = \sum_{n=0}^{\infty} f_n \in H_K$, it follows from Lemma 1.29 that

$$\sigma_{M_z}^j(\mathrm{id}_{H_K}) = \sigma_{M_z}^{j-1}\left(\left[\frac{a_{n-1}}{a_n}\right]_n\right) = \sigma_{M_z}^{j-2}\left(\left[\frac{a_{n-2}}{a_{n-1}}\frac{a_{n-1}}{a_n}\right]_n\right) = \cdots = \left[\frac{a_{n-j}}{a_n}\right]_n$$

for $j \in \mathbb{N}$. Hence

$$\sum_{j=0}^{N} c_j \sigma_{M_z}^j(\mathrm{id}_{H_K}) = \left[\sum_{j=0}^{N} c_j \frac{a_{n-j}}{a_n}\right]_n$$

for $N \in \mathbb{N}$.

Using these observations, we give an alternative proof of a result due to Chen (cf. [17, Proposition 2.1 & Lemma 2.2]).

2.10 Proposition (Chen)**.** *Suppose that there exists a natural number $p \in \mathbb{N}$ such that*

$$c_n \geq 0 \text{ for all } n \geq p \quad \text{or} \quad c_n \leq 0 \text{ for all } n \geq p$$

holds. Then $1/K(M_z, M_z^*)$ exists and $\sum_{n=0}^{\infty} c_n$ converges absolutely.

Proof. Let us suppose that there is an index $p \geq 1$ such that c_j $(j \geq p)$ have the same sign. Then, by standard results on Abel-summability, the series $\sum_{j=0}^{\infty} c_j$ converges absolutely, and, by Remark 1.41 (ii), $\sum_{j=0}^{\infty} c_j = 1/\sum_{j=0}^{\infty} a_j \in [0, 1)$. For $N \geq p$, we obtain

$$\left\| \sum_{j=p}^{N} c_j \sigma_{M_z}^j(\mathrm{id}_{H_K}) \right\| = \left\| \left[\sum_{j=p}^{N} c_j \frac{a_{n-j}}{a_n}\right]_n \right\| = \sup_{n \geq p} \left| \sum_{j=p}^{N} c_j \frac{a_{n-j}}{a_n} \right| = \sup_{n \geq p} \sum_{j=p}^{N} |c_j| \, \frac{a_{n-j}}{a_n}.$$

Using the fact that

$$\sum_{j=0}^{n} c_j a_{n-j} = 0$$

for $n \geq 1$, and the estimates

$$\frac{a_{n-j}}{a_n} = \frac{a_{n-j}}{a_{n-j+1}} \frac{a_{n-j+1}}{a_{n-j+2}} \cdots \frac{a_{n-1}}{a_n} \leq s^j$$

for $0 \leq j \leq n$, where $s = \sup_{n\in\mathbb{N}} a_n/a_{n+1}$, we find that

$$\begin{aligned}\sum_{j=p}^{N} |c_j| \frac{a_{n-j}}{a_n} &\leq \sum_{j=p}^{n} |c_j| \frac{a_{n-j}}{a_n} \\ &= \left| \sum_{j=p}^{n} c_j \frac{a_{n-j}}{a_n} \right| \\ &= \left| - \sum_{j=0}^{p-1} c_j \frac{a_{n-j}}{a_n} \right| \\ &\leq \sum_{j=0}^{p-1} |c_j| \frac{a_{n-j}}{a_n} \\ &\leq \left(\sum_{j=0}^{p-1} |c_j| \right) \max(s,1)^{p-1}\end{aligned}$$

for $n, N \geq p$. But then

$$\sup_{N\in\mathbb{N}} \left\| \sum_{j=p}^{N} c_j \sigma_{M_z}^{j}(\mathrm{id}_{H_K}) \right\| \leq \left(\sum_{j=0}^{p-1} |c_j| \right) \max(s,1)^{p-1} < \infty,$$

and hence, by Lemma 2.3, $1/K(M_z, M_z^*)$ exists. $\square$

2.11 Property. There exists a natural number $p \in \mathbb{N}$ such that

$$c_n \geq 0 \text{ for all } n \geq p \quad \text{or} \quad c_n \leq 0 \text{ for all } n \geq p$$

holds.

Note that the examples in Section 1.5 satisfy Property 2.11 (cf. the corresponding subsections).

2.12 Proposition. *Suppose that $1/K(M_z, M_z^*)$ exists and let $\mathcal{E}$ be a Hilbert space. Then $M_z \otimes \mathrm{id}_{\mathcal{E}} \in B(H_K \otimes \mathcal{E})^d$ is K-pure.*

Proof. Let $N \in \mathbb{N}$, $w \in \mathbb{B}_d$ and $\eta \in \mathcal{E}$. By Lemma 1.20 and Proposition 2.9, we have

$$
\begin{aligned}
&\Sigma_N(M_z \otimes \mathrm{id}_{\mathcal{E}})(K(\cdot,w)\otimes\eta)\\
&= \left(\left(\mathrm{id}_{H_K} - \sum_{|\alpha|\le N} \gamma_\alpha M_z^\alpha \frac{1}{K}(M_z, M_z^*) M_z^{*\alpha}\right) \otimes \mathrm{id}_{\mathcal{E}}\right)(K(\cdot,w)\otimes\eta)\\
&= \left(\left(\mathrm{id}_{H_K} - \sum_{|\alpha|\le N} \gamma_\alpha M_z^\alpha P_{\mathbb{C}} M_z^{*\alpha}\right) K(\cdot,w)\right)\otimes\eta\\
&= \left(K(\cdot,w) - \sum_{|\alpha|\le N} \gamma_\alpha M_z^\alpha P_{\mathbb{C}} \overline{w}^\alpha K(\cdot,w)\right)\otimes\eta\\
&= \left(K(\cdot,w) - \sum_{|\alpha|\le N} \gamma_\alpha M_z^\alpha \overline{w}^\alpha\right)\otimes\eta\\
&= \left(\left(\mathrm{id}_{H_K} - \sum_{n=0}^{N} P_{\mathbb{H}_n}\right)\otimes \mathrm{id}_{\mathcal{E}}\right)(K(\cdot,w)\otimes\eta).
\end{aligned}
$$

Since $\{K(\cdot,w)\otimes\eta\ ;\ w\in\mathbb{B}_d, \eta\in\mathcal{E}\}\subset H_K\otimes\mathcal{E}$ is a total subset by Proposition 1.9, and $\Sigma_N(M_z\otimes\mathrm{id}_{\mathcal{E}})$ is a bounded operator, we conclude that

$$\Sigma_N(M_z\otimes\mathrm{id}_{\mathcal{E}}) = \left(\mathrm{id}_{H_K} - \sum_{n=0}^{N} P_{\mathbb{H}_n}\right)\otimes\mathrm{id}_{\mathcal{E}}\,.$$

From this equality we see that $(\Sigma_N(M_z\otimes\mathrm{id}_{\mathcal{E}}))_{N\in\mathbb{N}}$ is a decreasing sequence of positive operators which is τ_{SOT}-convergent to 0. □

2.13 Lemma. *Let $T\in B(\mathcal{H})^d$ and $S\in B(\tilde{\mathcal{H}})^d$ be commuting tuples on Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, and suppose that there exists an isometry $\Pi\colon\mathcal{H}\to\tilde{\mathcal{H}}$ such that $\Pi T_i^* = S_i^*\Pi$ for all $i=1,\dots,d$. If S is a (K-pure) K-contraction, then T is a (K-pure) K-contraction.*

Proof. Let $S\in B(\tilde{\mathcal{H}})^d$ be a (K-pure) K-contraction. We have

$$T^\alpha T^{*\alpha} = T^\alpha\Pi^*\Pi T^{*\alpha} = \Pi^* S^\alpha S^{*\alpha}\Pi$$

for all $\alpha\in\mathbb{N}^d$ and hence,

$$\left(\frac{1}{K}\right)_N (T,T^*) = \Pi^*\left(\frac{1}{K}\right)_N (S,S^*)\Pi$$

for all $N \in \mathbb{N}$. By taking limits, we find that

$$\frac{1}{K}(T, T^*) = \Pi^* \frac{1}{K}(S, S^*)\Pi$$

and hence that

$$\Sigma_N(T) = \Pi^* \Sigma_N(S)\Pi$$

for all $N \in \mathbb{N}$. It follows that T is a K-contraction and that T is K-pure if S is K-pure. □

2.14 Remark. If we suppose that Property 2.11 holds, then H_K admits a strong $1/K$-calculus in the sense of [6, Definition 1.1].

The next result can also be deduced from [6, Theorem 1.3 or Corollary 3.2].

2.15 Theorem. *Suppose that $1/K(M_z, M_z^*)$ exists, and let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is K-pure,

(ii) there exist a Hilbert space $\mathcal{D}$ and an isometry $\Pi\colon \mathcal{H} \to H_K \otimes \mathcal{D}$ such that

$$\Pi T_i^* = (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}})^* \Pi$$

for all $i = 1, \dots, d$.

Proof. (i) $\implies$ (ii): This follows from Remark 2.8.
(ii) $\implies$ (i): This follows from Proposition 2.12 and Lemma 2.13. □

For $K = K^{(\nu)}$ ($\nu > 0$), the last result follows also from [33, Corollary on p. 59].

Besides K-pure commuting tuples, the following class of K-contractions will turn out to be useful.

2.16 Definition. We call a K-contraction $T \in B(\mathcal{H})^d$ *strong* if $\Sigma(T)$ exists, is a positive operator, and satisfies the identity

$$\Sigma(T) = \sigma_T(\Sigma(T)) = \sum_{i=1}^{d} T_i \Sigma(T) T_i^*.$$

2.17 Remark. In the above definition, the positivity condition corresponds to [6, (5.7)], and the identity to the last calculation on p. 857 in [6].

Every K-pure commuting tuple is a strong K-contraction. Hence, by Proposition 2.12, the K-shift $M_z \in B(H_K)^d$ is a strong K-contraction if we suppose that $1/K(M_z, M_z^*)$ exists.

2.18 Definition. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. We call T

(i) a *row contraction* if

$$\sigma_T(\mathrm{id}_{\mathcal{H}}) = \sum_{i=1}^{d} T_i T_i^* \leq \mathrm{id}_{\mathcal{H}},$$

i.e., T is a $K^{(1)}$-contraction,

(ii) a *spherical coisometry* if $\sigma_T(\mathrm{id}_{\mathcal{H}}) = \mathrm{id}_{\mathcal{H}}$,

(iii) a *spherical unitary* if T is a spherical coisometry and a tuple of normal operators.

By definition, it is clear that if $V \in B(\mathcal{H})^d$ is a strong K-contraction satisfying $1/K(V, V^*) = 0$, then V is a spherical coisometry. For a converse, the property $\sum_{n=0}^{\infty} c_n = 0$ will be sufficient, as the following proposition shows.

2.19 Proposition. *Suppose that $\sum_{n=0}^{\infty} c_n$ converges, and let $V \in B(\mathcal{H})^d$ be a spherical coisometry. Then:*

(i) V is a strong K-contraction.

(ii) If $\sum_{n=0}^{\infty} c_n = 0$, then V satisfies $1/K(V, V^*) = 0$.

(iii) If $\sum_{n=0}^{\infty} c_n > 0$, then V is K-pure.

Proof. Suppose that $\sum_{n=0}^{\infty} c_n$ converges. By Remark 1.41, it follows that $\sum_{n=0}^{\infty} c_n \in [0, 1)$. Since

$$\sum_{n=0}^{N} c_n \sigma_V^n(\mathrm{id}_{\mathcal{H}}) h = \left(\sum_{n=0}^{N} c_n \right) h$$

for all $h \in \mathcal{H}$ and all $N \in \mathbb{N}$, we see that

$$\frac{1}{K}(V, V^*) = \left(\sum_{n=0}^{\infty} c_n \right) \mathrm{id}_{\mathcal{H}} .$$

By Remark 1.41, we have that

$$\frac{1}{K}(V, V^*) = \left(\sum_{n=0}^{\infty} c_n \right) \mathrm{id}_{\mathcal{H}} = \left(\frac{1}{\sum_{n=0}^{\infty} a_n} \right) \mathrm{id}_{\mathcal{H}} \geq 0.$$

Furthermore, we observe that

$$\Sigma_N(V) = \mathrm{id}_{\mathcal{H}} - \sum_{j=0}^{N} a_j \sigma_V^j \left(\frac{1}{K}(V, V^*) \right) = \left(1 - \frac{\sum_{j=0}^{N} a_j}{\sum_{n=0}^{\infty} a_n} \right) \mathrm{id}_{\mathcal{H}}$$

for all $N \in \mathbb{N}$.

If $\sum_{j=0}^{\infty} a_j = \infty$, then $\Sigma_N(V) = \mathrm{id}_{\mathcal{H}}$ for all $N \in \mathbb{N}$ and

$$\sigma_V(\Sigma(V)) = \sigma_V(\mathrm{id}_{\mathcal{H}}) = \mathrm{id}_{\mathcal{H}} = \Sigma(V).$$

If $\sum_{j=0}^{\infty} a_j < \infty$, then $\Sigma(V) = 0$ and V is K-pure. □

2.20 Lemma. *Let $S_1 \in B(\mathcal{H}_1)^d$ and $S_2 \in B(\mathcal{H}_2)^d$ be (K-pure/strong) K-contractions on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $S_1 \oplus S_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)^d$ is a (K-pure/strong) K-contraction.*

Proof. The result follows from the observations that

$$\frac{1}{K}(S_1 \oplus S_2, (S_1 \oplus S_2)^*) = \frac{1}{K}(S_1, S_1^*) \oplus \frac{1}{K}(S_2, S_2^*)$$

and that

$$\sigma_{S_1 \oplus S_2}\left(\Sigma_N(S_1 \oplus S_2)\right) = \sigma_{S_1}\left(\Sigma_N(S_1)\right) \oplus \sigma_{S_2}\left(\Sigma_N(S_2)\right)$$

for all $N \in \mathbb{N}$ hold. □

2.21 Lemma. *Let $T \in B(\mathcal{H})^d$ and $S \in B(\tilde{\mathcal{H}})^d$ be commuting tuples on Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, and suppose that there exists an isometry $\Pi \colon \mathcal{H} \to \tilde{\mathcal{H}}$ such that $\Pi T_i^* = S_i^* \Pi$ for all $i = 1, \ldots, d$. If S is a (K-pure/strong) K-contraction, then T is a (K-pure/strong) K-contraction.*

Proof. The result follows from Lemma 2.13 (see also its proof) and the identities

$$\sigma_T(\Sigma_N(T)) = \Pi^* \sigma_S(\Sigma_N(S)) \Pi$$

for all $N \in \mathbb{N}$. □

2.22 Proposition. *Suppose $1/K(M_z, M_z^*)$ exists and that $\sum_{n=0}^{\infty} c_n$ converges. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. If there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical coisometry $U \in B(\mathcal{K})^d$, and an isometry $\Pi \colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$, then T is a strong K-contraction.

Proof. By Propositions 2.12 and 2.19, we have that $M_z \in B(H_K)^d$ and $U \in B(\mathcal{K})^d$ are strong K-contractions. The result follows now from Lemmas 2.20 and 2.21. $\square$

Our goal is to show that the statements in the last proposition are actually equivalent. Furthermore, we can achieve U to be a spherical unitary.

The next result is an adaption of [55, Lemma 10] and [6, Theorem 5.1].

2.23 Lemma. *Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. Then there exist a Hilbert space $\mathcal{L}$ with $\Sigma(T)^{1/2}\mathcal{H} \subset \mathcal{L}$, and a spherical unitary $W \in B(\mathcal{L})^d$ such that*

$$\Sigma(T)^{1/2}T_i^*h = W_i^*\Sigma(T)^{1/2}h$$

for all $h \in \mathcal{H}$ and $i = 1, \ldots, d$. Furthermore, $\mathcal{L}$ and W can be chosen such that

$$\mathcal{L} = \bigvee \left\{W^\alpha \Sigma(T)^{1/2}h \ ; \ \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H}\right\}$$

holds.

Proof. We can decompose

$$\mathcal{H} = \ker\left(\Sigma(T)^{1/2}\right) \oplus \overline{\operatorname{Im}\left(\Sigma(T)^{1/2}\right)}$$

and we set $\mathcal{L}_0 = \overline{\operatorname{Im}\left(\Sigma(T)^{1/2}\right)}$.

For $h \in \mathcal{H}$,

$$\sum_{i=1}^d \left\|\Sigma(T)^{1/2}T_i^*h\right\| = \langle\sigma_T(\Sigma(T))h, h\rangle = \langle\Sigma(T)h, h\rangle = \left\|\Sigma(T)^{1/2}h\right\|^2.$$

Hence, there are bounded linear operators $V_i \colon \mathcal{L}_0 \to \mathcal{L}_0$ with

$$V_i^*\Sigma(T)^{1/2}h = \Sigma(T)^{1/2}T_i^*h \quad (h \in \mathcal{H}, i = 1, \ldots, d).$$

The tuple $V = (V_1, \ldots, V_d) \in B(\mathcal{L}_0)^d$ is commuting and satisfies

$$\sum_{i=1}^d \left\|V_i^*\Sigma(T)^{1/2}h\right\|_{\mathcal{L}_0}^2 = \left\|\Sigma(T)^{1/2}h\right\|_{\mathcal{L}_0}^2$$

for all $h \in \mathcal{H}$. Since $\Sigma(T)^{1/2}\mathcal{H} \subset \mathcal{L}_0$ is dense, we conclude that

$$\mathcal{L}_0 \to \mathcal{L}_0^d, \ h \mapsto (V_i^*h)_{i=1}^d$$

is an isometry. But then $V \in B(\mathcal{L}_0)^d$ is a spherical coisometry. By Lemma 1.6, there exist a larger Hilbert space $\mathcal{L} \supset \mathcal{L}_0$ and a tuple $W \in B(\mathcal{L})^d$ which is a spherical unitary such that W^* is the minimal normal extension of V^*, i.e.,

$$\mathcal{L} = \bigvee \left\{ W^\alpha \Sigma(T)^{1/2} h \ ; \ \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\} \supset \mathcal{L}_0.$$

The calculation

$$W_i^* \Sigma(T)^{1/2} h = V_i^* \Sigma(T)^{1/2} h = \Sigma(T)^{1/2} T_i^* h$$

for all $i = 1, \ldots, d$ and $h \in \mathcal{H}$ ends the proof. □

In the setting of Lemma 2.23, if T is K-pure, one can choose $\mathcal{L} = \{0\}$.

2.24 Remark. Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. Using the notations from Proposition 2.6 and Lemma 2.23, we define

$$\Psi_T = \left(\psi_T \oplus \Sigma(T)^{1/2} \right) \circ j \colon \mathcal{H} \to (H_K \otimes \mathcal{D}_T) \oplus \mathcal{L},$$

where

$$j \colon \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}, \ h \mapsto h \oplus h.$$

Then Ψ_T is an isometry with

$$\Psi_T T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}_T}) \oplus W_i)^* \Psi_T$$

for $i = 1, \ldots, d$. Furthermore, one can achieve that

$$\mathcal{L} = \bigvee \left\{ W^\alpha \Sigma(T)^{1/2} h \ ; \ \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\}.$$

If we combine Remark 2.24 and Proposition 2.22, we obtain our model theorem for strong K-contractions.

2.25 Theorem. *Suppose that $1/K(M_z, M_z^*)$ exists and that $\sum_{n=0}^\infty c_n$ converges. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is a strong K-contraction,

(ii) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi \colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$.

In particular, by Proposition 2.10, the above result holds if we suppose that Property 2.11 holds.

2.26 Remark. Suppose that the setting of Theorem 2.25 holds. If $\sum_{n=0}^{\infty} c_n > 0$, then Propositions 2.12 and 2.19 show that the classes of K-pure commuting tuples and strong K-contractions coincide.

By specializing Theorem 6.3 from [6] to the case of the unit ball, one obtains:

2.27 Theorem. *Let $\nu > 0$ and let $T \in B(\mathcal{H})^d$ be a commuting tuple such that $1/K^{(\nu)}(T, T^*)$ exists, $\Sigma_{K^{(\nu)}}(T) \geq 0$, and there exists $c \geq 0$ such that*

$$\sigma_T(\Sigma_{K^{(\nu)}}(T)) \leq c \cdot \Sigma_{K^{(\nu)}}(T).$$

Then there exist a Hilbert space $\mathcal{K}$, a commuting tuple $U \in B(\mathcal{K})^d$ with $1/K^{(\nu)}(U, U^*) = 0$, and an isometry $\Pi\colon \mathcal{H} \to (H_{K^{(\nu)}} \otimes \mathcal{H}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{H}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$.

2.28 Remark. Let $K = K^{(\nu)}$ with $\nu > 0$. One can show that

$$\sum_{n=0}^{\infty} c_n = \frac{1}{\sum_{n=0}^{\infty} a_n} = 0$$

in this case. Hence, by Proposition 2.19, every spherical coisometry $V \in B(\mathcal{H})^d$ satisfies the condition

$$\frac{1}{K^{(\nu)}}(V, V^*) = 0.$$

Thus, the case $\sigma_T(\Sigma_{K^{(\nu)}}) = \Sigma_{K^{(\nu)}}$ in Theorem 2.27 is contained in Remark 2.24 (and Theorem 2.25).

To strengthen Theorem 2.25, we elaborate the cases when M_z is essentially normal or H_K is even regular.

2.29 Lemma. *Suppose that $M_z \in B(H_K)^d$ is essentially normal and let $T \in B(\mathcal{H})^d$ be a commuting tuple. If there exists a unital, completely contractive linear map*

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \ ; \ i = 1, \ldots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \text{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_iT_i^*$$

for all $i = 1, \ldots, d$, then there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a tuple of normal operators $U \in B(\mathcal{K})^d$, and an isometry $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$.

If H_K is regular, U can be chosen to be a spherical unitary.

Proof. By Arveson's extension theorem (cf. [57, Theorem 7.5]), we find a unital, completely positive map $\Phi\colon B(H_K) \to B(\mathcal{H})$ such that Φ extends φ. By Stinespring's dilation theorem (cf. [57, Theorem 4.1]), there exist a Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$, and a C^*-homomorphism $\pi\colon B(H_K) \to B(\tilde{\mathcal{H}})$ such that

$$\Phi(X) = P_{\mathcal{H}} \pi(X)|_{\mathcal{H}}$$

for all $X \in B(H_K)$. By [18, Lemma 3.3], there exist a decomposition $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}_2$ of $\tilde{\mathcal{H}}$ into reducing subspaces for π, an index set I, and a unitary operator $V \in B(\tilde{\mathcal{H}}_1, H_K \otimes \ell^2(I))$ such that

$$V\pi(X)|_{\tilde{\mathcal{H}}_1} = (X \otimes \mathrm{id}_{\ell^2(I)})V$$

for all $X \in C^*(M_z)$ and

$$\pi(K(H_K))|_{\tilde{\mathcal{H}}_2} = \{0\}\,.$$

We set $\mathcal{D} = \ell^2(I)$, $\mathcal{K} = \tilde{\mathcal{H}}_2$, and $\Pi = (V \oplus \mathrm{id}_{\mathcal{K}})|_{\mathcal{H}}$. Define the commuting tuple

$$U = (\pi(M_{z_1})|_{\mathcal{K}}, \ldots, \pi(M_{z_d})|_{\mathcal{K}}) \in B(\mathcal{K})^d$$

and observe that U is a tuple of normal operators since M_z is essentially normal. Finally, we obtain

$$\begin{aligned}
\Pi T_i^* h &= (V \oplus \mathrm{id}_{\mathcal{K}}) P_{\mathcal{H}} \pi(M_{z_i})^* h \\
&= (V \oplus \mathrm{id}_{\mathcal{K}}) \pi(M_{z_i})^* h \\
&= ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \,(V \oplus \mathrm{id}_{\mathcal{K}}) h \\
&= ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \,\Pi h
\end{aligned}$$

for all $i = 1, \ldots, d$ and $h \in \mathcal{H}$, since $\mathcal{H}$ is invariant for $\pi(M_{z_i})^*$ for all $i = 1, \ldots, d$ (cf. [18, Lemma 3.2]).

Suppose now that H_K is regular. With Theorem 1.37 we see that

$$\mathrm{id}_{\mathcal{K}} - \sum_{i=1}^{d} U_i U_i^* = \pi\left(\mathrm{id}_{H_K} - \sum_{i=1}^{d} M_{z_i} M_{z_i}^*\right)|_{\mathcal{K}} = 0,$$

i.e., U is a spherical coisometry, and hence, U is a spherical unitary. $\square$

With these preparations, we are now able to state our main result about strong K-contractions.

2.30 Theorem. *Suppose that k has radius of convergence 1, that the operator $1/K(M_z, M_z^*)$ exists, that $M_z \in B(H_K)^d$ is essentially normal, and that $\sum_{n=0}^{\infty} c_n$ converges. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. Consider the following statements:*

(i) T is a strong K-contraction,

(ii) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \dots, d$,

(iii) there exists a unital, completely contractive linear map

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \ ; \ i = 1, \dots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \textit{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_iT_i^*$$

for all $i = 1, \dots, d$.

The implications (i) $\iff$ (ii) $\implies$ (iii) hold.

If in addition H_K is regular, then all statements are equivalent.

Proof. (i) $\iff$ (ii): This is Theorem 2.25.

(ii) $\implies$ (iii): By Lemma 1.1, the C^*-algebra $C^*(M_z)/K(H_K) = C^*(M_z + K(H_K))$ is abelian. Since $\sigma_{\mathrm{e}}(M_z)$ coincides with the joint spectrum of the tuple $M_z + K(H_K)$ in the abelian Banach algebra $C^*(M_z + K(H_K))$, Gelfand theory yields an isomorphism of C^*-algebras $\varphi_1\colon C^*(M_z + K(H_K)) \to C(\sigma_{\mathrm{e}}(M_z))$ with $\varphi_1(M_{z_i} + K(H_K)) = z_i|_{\sigma_{\mathrm{e}}(M_z)}$ for $i = 1, \dots, d$. Since $\sigma(U) \subset \mathbb{S}_d$ and $C^*(U)$ is abelian, there exists a C^*-algebra homomorphism $\varphi_2\colon C(\mathbb{S}_d) \to C^*(U)$ with $\varphi_2(z_i|_{\mathbb{S}_d}) = U_i$ for $i = 1, \dots, d$. Denoting by $\iota\colon C(\sigma_{\mathrm{e}}(M_z)) \to C(\mathbb{S}_d),\ f \mapsto f|_{\mathbb{S}_d}$ the restriction map and $q\colon C^*(M_z) \to C^*(M_z)/K(H_K)$ the quotient map, the function

$$\pi_u = \varphi_2 \circ \iota \circ \varphi_1 \circ q\colon C^*(M_z) \to C^*(U)$$

is a C^*-algebra homomorphism with

$$\pi_u(M_{z_i}) = U_i$$

for all $i = 1, \dots, d$ and $\pi_u(K(H_K)) = \{0\}$. Hence, setting

$$\pi_s \colon C^*(M_z) \to B(H_K \otimes \mathcal{D}),\ X \mapsto X \otimes \mathrm{id}_{\mathcal{D}},$$

the map

$$\pi \colon C^*(M_z) \to B((H_K \otimes \mathcal{D}) \oplus \mathcal{K}),\ X \mapsto \pi_s(X) \oplus \pi_u(X)$$

is a unital C^*-algebra homomorphism. Finally, the map

$$\varphi \colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \ ;\ i = 1, \dots, d\right\} \to B(\mathcal{H}),\ X \mapsto \Pi^*\pi(X)\Pi$$

is completely positive, unital, and satisfies

$$\varphi(M_{z_i}) = \Pi^*\pi(M_{z_i})\Pi = \Pi^*((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)\Pi = T_i\Pi^*\Pi = T_i$$

and

$$\varphi(M_{z_i}M_{z_i}^*) = \Pi^*\pi(M_{z_i}M_{z_i}^*)\Pi = \Pi^*\pi(M_{z_i})\pi(M_{z_i})^*\Pi = T_i\Pi^*\Pi T_i^* = T_iT_i^*$$

for all $i = 1, \dots, d$.
(iii) $\implies$ (ii): This follows from Lemma 2.29. $\square$

3. An analytic model

In this chapter, we use another approach to define the operator $1/K(T,T^*)$. The idea originates from [1] and was also used in, e.g., [63] and [56]. In the first section we elaborate this approach in general and obtain similar results to the preceding chapter. The second section is concerned with the case when our commuting tuple is a row contraction. This will enable us take a closer look at m-hypercontractions. As before, let $\mathcal{H}$ be a Hilbert space, $T \in B(\mathcal{H})^d$ a commuting tuple, and let $H_K \subset \mathcal{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space whose kernel $K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is of the form $K(z,w) = k(\langle z,w\rangle)$ $(z,w \in \mathbb{B}_d)$ with a zero-free analytic function $k\colon \mathbb{D} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} a_n z^n$ such that $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$.

3.1. Radial K-hypercontractions I

For $S \in B(\mathcal{H})$, we define

$$L_S\colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto SX \quad \text{and} \quad R_S\colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto XS$$

and, for a commuting tuple $T \in B(\mathcal{H})^d$, we set

$$L_T = (L_{T_1}, \dots, L_{T_d}) \in B(B(\mathcal{H}))^d \quad \text{and} \quad R_T = (R_{T_1}, \dots, R_{T_d}) \in B(B(\mathcal{H}))^d$$

as well as

$$M_T = (L_T, R_{T^*}) \in B(B(\mathcal{H}))^{2d}.$$

For a commuting tuple $T \in B(\mathcal{H})^d$ with Taylor spectrum $\sigma(T)$, we have $\sigma(M_T) = \sigma(T) \times \sigma(T^*)$ (Theorem 3.1 in [34]).

Let $T \in B(\mathcal{H})^d$ be a commuting tuple with $\sigma(T) \subset \overline{\mathbb{B}}_d$. Then $\sigma(M_T) \subset \overline{\mathbb{B}}_d \times \overline{\mathbb{B}}_d$. For $g \in \mathcal{O}(\overline{\mathbb{D}})$, the function

$$G\colon U \to \mathbb{C},\ (z,w) \mapsto g\left(\sum_{i=1}^{d} z_i w_i\right)$$

is analytic on a suitable open neighborhood U of $\overline{\mathbb{B}}_d \times \overline{\mathbb{B}}_d$. Using Taylor's analytic functional calculus, we define

$$G[T] = G(M_T) \in B(B(H)) \quad \text{and} \quad G(T,T^*) = G[T](\mathrm{id}_{\mathcal{H}}) \in B(H).$$

If $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is the Taylor expansion of g at $z = 0$, then

$$G(z, w) = \sum_{n=0}^{\infty} g_n \left(\sum_{i=1}^{d} z_i w_i \right)^n = \sum_{\alpha \in \mathbb{N}^d} g_{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha w^\alpha,$$

where the series converges locally uniformly on an open neighborhood of $\overline{\mathbb{B}}_d \times \overline{\mathbb{B}}_d$. By using the continuity of Taylor's analytic functional calculus, one obtains that

$$G[T] = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} g_n \sigma_T^n \in B(B(\mathcal{H})) \text{ and } G(T, T^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} g_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) \in B(\mathcal{H}).$$

For $0 < r \leq 1$ and $h \in \mathcal{O}(\mathbb{D})$, we use the notation

$$h_r \colon D_{\frac{1}{r}}(0) \to \mathbb{C}, \; z \mapsto h(rz).$$

By applying the above remarks to the function $g = 1/k_r \in \mathcal{O}(\overline{\mathbb{D}})$, one obtains

$$\frac{1}{K_r}(T, T^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} r^n c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}})$$

for $0 < r < 1$.

3.1 Definition. We call a commuting tuple $T \in B(\mathcal{H})^d$ with $\sigma(T) \subset \overline{\mathbb{B}}_d$ a *radial K-hypercontraction* if

$$\frac{1}{K_r}(T, T^*) \geq 0$$

for all $0 < r < 1$.

3.2 Example. Let $V \in B(\mathcal{H})^d$ be a spherical coisometry. Then

$$\left\langle \frac{1}{K_r}(V, V^*)h, h \right\rangle = \sum_{n=0}^{\infty} c_n r^n \langle h, h \rangle = \frac{1}{k(r)} \langle h, h \rangle \geq 0$$

for all $0 < r < 1$ and $h \in \mathcal{H}$, i.e., V is a radial K-hypercontraction.

Let $r, s \in (0, 1]$. Since k is non-vanishing on $\mathbb{D}$, the function k_r/k_s is well-defined and its Taylor series

$$\frac{k_r(z)}{k_s(z)} = \sum_{n=0}^{\infty} a_n(r, s) z^n$$

converges for all $|z| < \min\left(\frac{1}{r}, \frac{1}{s}\right)$. In particular, we see that

$$a_n = a_n(1,0) \quad \text{and} \quad c_n = a_n(0,1)$$

for all $n \in \mathbb{N}$.

As before, the K-shift M_z will play an important role in a model theory for radial K-hypercontractions. To obtain a condition for M_z to be a radial K-hypercontraction, we will need the following lemma.

3.3 Lemma. *For* $0 < r \leq 1$ *and* $\alpha \in \mathbb{N}^d$, *we have*

$$\frac{|\alpha|!}{\alpha!} a_{|\alpha|}(1,r) = \sum_{\beta \leq \alpha} c_{|\beta|} r^{|\beta|} \frac{|\beta|!}{\beta!} \gamma_{\alpha-\beta}.$$

Proof. For $z \in \mathbb{C}^d$ with $|z|$ small enough, the power series expansions

$$\begin{aligned}
k\left(\sum_{i=1}^d z_i\right) &= \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha, \\
\frac{1}{k_r}\left(\sum_{i=1}^d z_i\right) &= \sum_{\alpha \in \mathbb{N}^d} c_\alpha r^{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha, \\
\frac{k}{k_r}\left(\sum_{i=1}^d z_i\right) &= \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|}(1,r) \frac{|\alpha|!}{\alpha!} z^\alpha
\end{aligned}$$

hold. The Cauchy product formula yields that

$$\frac{k}{k_r}\left(\sum_{i=1}^d z_i\right) = \sum_{\alpha \in \mathbb{N}^d} \left(\sum_{\beta \leq \alpha} c_{|\beta|} r^{|\beta|} \frac{|\beta|!}{\beta!} a_{|\alpha-\beta|} \frac{|\alpha-\beta|!}{(\alpha-\beta)!}\right) z^\alpha$$

for z as above. Thus, the assertion follows from comparising the coefficients of the above power series expansions. $\square$

If $M_z \in B(H_K)^d$ is essentially normal, and

$$\limsup_{n \to \infty} \frac{a_n}{a_{n+1}} \leq 1,$$

then $\sigma(M_z) \subset \overline{\mathbb{B}}_d$ holds (cf. Lemma 1.33). In particular, this is the case if H_K is regular (cf. Remark 1.36).

3.4 Lemma. *Let* $M_z \in B(H_K)^d$ *be the* K*-shift,* $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in H_K$, *and* $0 < r < 1$. *Suppose that* $\sigma(M_z) \subset \overline{\mathbb{B}}_d$. *The following statements hold:*

(i) $1/K_r(M_z, M_z^*)f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha \frac{a_{|\alpha|}(1,r)}{a_{|\alpha|}} z^\alpha$,

(ii) $\langle 1/K_r(M_z, M_z^*)f, f\rangle = \sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2}{\gamma_\alpha}\frac{a_{|\alpha|}(1,r)}{a_{|\alpha|}}$,

(iii) $\|1/K_r(M_z, M_z^*)\| = \sup_{n\in\mathbb{N}} \frac{|a_n(1,r)|}{a_n}$.

Proof. Let $0 < r < 1$ and $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in H_K$.

(i) Observe that

$$\begin{aligned}
\frac{1}{K_r}(M_z, M_z^*)f &= \sum_{\alpha\in\mathbb{N}^d} f_\alpha \sum_{n=0}^{\infty} c_n r^n \sum_{|\beta|=n} \frac{|\beta|!}{\beta!} M_z^\beta M_z^{*\beta} z^\alpha \\
&= \sum_{\alpha\in\mathbb{N}^d} f_\alpha \sum_{n=0}^{\infty} c_n r^n \sum_{|\beta|=n,\beta\le\alpha} \frac{|\beta|!}{\beta!}\frac{\gamma_{\alpha-\beta}}{\gamma_\alpha} z^\alpha \\
&= \sum_{\alpha\in\mathbb{N}^d} f_\alpha \sum_{\beta\le\alpha} c_{|\beta|} r^{|\beta|} \frac{|\beta|!}{\beta!}\frac{\gamma_{\alpha-\beta}}{\gamma_\alpha} z^\alpha \\
&= \sum_{\alpha\in\mathbb{N}^d} \left(\frac{f_\alpha}{\gamma_\alpha} \sum_{\beta\le\alpha} c_{|\beta|} r^{|\beta|} \frac{|\beta|!}{\beta!} \gamma_{\alpha-\beta} \right) z^\alpha \\
&= \sum_{\alpha\in\mathbb{N}^d} \left(f_\alpha \frac{1}{\gamma_\alpha} \frac{|\alpha|!}{\alpha!} a_{|\alpha|}(1,r) \right) z^\alpha \\
&= \sum_{\alpha\in\mathbb{N}^d} f_\alpha \frac{a_{|\alpha|}(1,r)}{a_{|\alpha|}} z^\alpha,
\end{aligned}$$

where we have used Lemmas 1.29 and 3.3.

(ii) With (i) we obtain

$$\left\langle \frac{1}{K_r}(M_z, M_z^*)f, f \right\rangle = \sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2}{\gamma_\alpha} \frac{a_{|\alpha|}(1,r)}{a_{|\alpha|}}.$$

(iii) Again with (i) we conclude that $1/K_r(M_z, M_z^*)$ is a diagonal operator with respect to the decomposition

$$H_K = \bigoplus_{n=0}^{\infty} \mathbb{H}_n.$$

Since the norm of a diagonal operator is the supremum of the diagonal, the result follows. $\square$

3.5 Corollary. *Suppose that $\sigma(M_z) \subset \overline{\mathbb{B}}_d$. The K-shift M_z is a radial K-hypercontraction if and only if*

$$a_n(1, r) \geq 0$$

for all $n \in \mathbb{N}$ and $0 < r < 1$.

Proof. The assertion follows directly from Lemma 3.4 (ii). □

3.6 Property. Let

$$a_n(1, r) \geq 0$$

for all $n \in \mathbb{N}$ and $0 < r < 1$.

3.7 Remark. Property 3.6 holds if and only if we have

$$a_n(s, r) \geq 0$$

for all $n \in \mathbb{N}$ and $0 < r < s \leq 1$.

Proof. If $0 < r < s \leq 1$, then

$$\frac{k_s(z)}{k_r(z)} = \frac{k(sz)}{k(\frac{r}{s}(sz))} = \sum_{n=0}^{\infty} a_n\left(1, \frac{r}{s}\right) s^n z^n$$

for all $|z| < \min\left(\frac{1}{r}, \frac{1}{s}\right)$ and hence,

$$a_n(s, r) = a_n\left(1, \frac{r}{s}\right) s^n$$

for all $n \in \mathbb{N}$. □

Olofsson stated in [56, Proposition 5.1] the following sufficient condition for k to satisfy Property 3.6.

3.8 Proposition. *If the function* $\log(k)$ *has non-negative Taylor coefficients, then Property 3.6 holds.*

In particular, if K is the reproducing kernel of a space mentioned in Sections 1.5.1 and 1.5.2, then Property 3.6 holds.

3.9 Remark. The following example, which was also mentioned in [56, Section 5], shows that Property 3.6 does not imply Property 2.11.

Consider

$$k \colon \mathbb{D} \to \mathbb{C}, \; z \mapsto \frac{1+z}{1-z}.$$

Since

$$\log(k(z)) = \log\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n} z^n$$

for all $z \in \mathbb{D}$, Proposition 3.8 implies that Property 3.6 hold. But

$$\frac{1}{k(z)} = 1 + 2\sum_{n=1}^{\infty} (-1)^n z^n$$

for all $z \in \mathbb{D}$, i.e., Property 2.11 does not hold.

3.10 Lemma. *Let $T \in B(\mathcal{H})^d$ be a commuting tuple such that $\sigma(T) \subset \overline{\mathbb{B}}_d$ and $r, s \in (0,1)$. Then*

$$\begin{aligned}\frac{1}{K_r}(T,T^*) &= \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} a_n(s,r)\sigma_T^n\left(\frac{1}{K_s}(T,T^*)\right)\\ &= \frac{1}{K_s}(T,T^*) + \tau_{\|\cdot\|}\text{-}\sum_{n=1}^{\infty} a_n(s,r)\sigma_T^n\left(\frac{1}{K_s}(T,T^*)\right).\end{aligned}$$

Proof. Let $r, s \in (0,1)$. The identity

$$\frac{1}{k_r(z)} = \frac{k_s(z)}{k_r(z)}\frac{1}{k_s(z)} \quad (z \in \mathbb{D})$$

together with the multiplicativity of Taylor's analytic functional calculus yields that

$$\begin{aligned}\frac{1}{K_r}(T,T^*) &= \left(\frac{1}{K_r}[T]\right)(\mathrm{id}_{\mathcal{H}}) = \left(\frac{K_s}{K_r}[T]\frac{1}{K_s}[T]\right)(\mathrm{id}_{\mathcal{H}}) = \frac{K_s}{K_r}[T]\frac{1}{K_s}(T,T^*)\\ &= \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} a_n(s,r)\sigma_T^n\left(\frac{1}{K_s}(T,T^*)\right)\end{aligned}$$

holds. □

3.11 Lemma. *Suppose that Property 3.6 holds. Let $T \in B(\mathcal{H})^d$ be a commuting tuple such that $\sigma(T) \subset \overline{\mathbb{B}}_d$ and such that there exists $s \in (0,1)$ with $1/K_s(T,T^*) \geq 0$. Then*

$$\frac{1}{K_r}(T,T^*) \geq \frac{1}{K_s}(T,T^*) \geq 0$$

for all $0 < r < s$.

Proof. In view of Remark 3.7 this follows from Lemma 3.10. $\square$

3.12 Corollary. *Suppose that Property 3.6 holds. Then a commuting tuple $T \in B(\mathcal{H})^d$ is a radial K-hypercontraction if and only if there exists a sequence $(r_n)_{n\in\mathbb{N}}$ in $(0,1)$ with limit 1 such that $1/K_{r_n}(T,T^*) \geq 0$ for all $n \in \mathbb{N}$.*

3.13 Proposition. *Suppose that Property 3.6 holds. Let $T \in B(\mathcal{H})^d$ be a commuting tuple with $\sigma(T) \subset \overline{\mathbb{B}}_d$. If T is a radial K-hypercontraction then*

$$\frac{1}{K}_{\mathrm{rad}}(T,T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{r\to 1} \frac{1}{K_r}(T,T^*)$$

exists and defines a positive operator.

Proof. This follows from Corollary 3.12 and Lemma 2.3. $\square$

3.14 Remark. Since

$$\frac{k}{k_r} \to 1$$

converges locally uniformly as $r \to 1$, we see that

$$\lim_{r\to 1} a_n(1,r) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases}$$

holds.

3.15 Proposition. *Suppose that $\sigma(M_z) \subset \overline{\mathbb{B}}_d$ and that Property 3.6 holds. Then*

$$\frac{1}{K}_{\mathrm{rad}}(M_z, M_z^*) = P_{\mathbb{C}}.$$

Proof. By Corollary 3.5 and Proposition 3.13, the limit $1/K_{\mathrm{rad}}(M_z, M_z^*)$ exists. Furthermore, by Lemma 3.4, we have

$$\frac{1}{K}_{\mathrm{rad}}(M_z, M_z^*)p = p(0) = P_{\mathbb{C}}p$$

for every polynomial p, since, by Remark 3.14, $\lim_{r\to 1} a_n(1,r) = 0$ for all $n \geq 1$ and $a_0(1,r) = 1$ for $0 < r < 1$. Hence,

$$\frac{1}{K}_{\mathrm{rad}}(M_z, M_z^*) = P_{\mathbb{C}},$$

since the polynomials are dense in H_K. $\square$

3.16 Corollary. *Suppose that $1/K(M_z, M_z^*)$ exists, $\sigma(M_z) \subset \overline{\mathbb{B}}_d$, and that Property 3.6 hold. Then*

$$\frac{1}{K}(M_z, M_z^*) = \frac{1}{K}_{\text{rad}}(M_z, M_z^*) = P_{\mathbb{C}} \geq 0.$$

In Theorem 2.30 we have seen that the existence of a certain completely contractive map is connected with the property of strong K-contractiveness. In our new approach, with have an analogue.

3.17 Proposition. *Let $T \in B(\mathcal{H})^d$ be a radial K-hypercontraction. Then there exists a unital, completely contractive linear map*

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \ ; \ i = 1, \ldots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \textit{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_iT_i^*$$

for all $i = 1, \ldots, d$.

Proof. Let $0 < r < 1$. Then

$$\frac{1}{K}(rT, rT^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} c_n \sigma_{rT}^n(\mathrm{id}_{\mathcal{H}}) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} c_n r^{2n} \sigma_T^n(\mathrm{id}_{\mathcal{H}}) = \frac{1}{K_{r^2}}(T, T^*) \geq 0.$$

Since

$$\Sigma(rT) = \mathrm{id}_{\mathcal{H}} - \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} a_n \sigma_{rT}^n\left(\frac{1}{K}(rT, rT^*)\right) = \mathrm{id}_{\mathcal{H}} - K[rT]\frac{1}{K}[rT](\mathrm{id}_{\mathcal{H}}) = 0,$$

the tuple $rT \in B(\mathcal{H})^d$ is a K-pure commuting tuple. By Proposition 2.6, the map

$$\psi_{rT}\colon \mathcal{H} \to H_K \otimes \mathcal{D}_{rT}, \ h \mapsto \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha r^{|\alpha|}(z^\alpha \otimes D_{rT}T^{*\alpha}h)$$

is an isometry which intertwines the tuple $rT^* \in B(\mathcal{H})^d$ and $(M_z \otimes \mathrm{id}_{\mathcal{D}_{rT}})^* \in B(H_K \otimes \mathcal{D}_{rT})^d$ componentwise. It follows that the map

$$\varphi_r\colon B(H_K) \to B(\mathcal{H}), \ X \mapsto \psi_{rT}^*(X \otimes \mathrm{id}_{\mathcal{D}_{rT}})\psi_{rT}$$

is unital and completely positive with

$$\varphi_r(M_z^\alpha M_z^{*\beta}) = r^{|\alpha|+|\beta|}T^\alpha T^{*\beta}$$

for all $\alpha, \beta \in \mathbb{N}^d$. Since the subset

$$\{\psi \in B(B(H_K), B(\mathcal{H})) \ ; \ \psi \text{ is unital, completely positive, and } \|\psi\|_{\mathrm{cb}} \leq 1\}$$

of $B(B(H_K), B(\mathcal{H}))$ is compact in the bounded weak topology τ_{BW} (cf. [57, Theorem 7.4]), there is a net $(r_i)_{i\in I}$ in $(0,1)$ with $\lim_{i\in I} r_i = 1$ such that the limit

$$\varphi = \tau_{BW}\text{-}\lim_{i\in I} \varphi_{r_i} \in B(B(H_K), B(\mathcal{H}))$$

exists. Since norm-bounded τ_{BW}-convergent nets in $B(B(H_K), B(\mathcal{H}))$ are pointwise convergent in the weak operator topology τ_{WOT} (cf. [57, Proposition 7.3]), it follows that $\varphi\colon B(H_K) \to B(\mathcal{H})$ is unital and completely positive with

$$\varphi(M_z^\alpha M_z^{*\beta}) = \tau_{BW}\text{-}\lim_{i\in I} \varphi_{r_i}(M_z^\alpha M_z^{*\beta}) = T^\alpha T^{*\beta}$$

for all $\alpha, \beta \in \mathbb{N}^d$. □

The proofs of the next two lemmas are similar to the proofs of Lemmas 2.20 and 2.21, where we use the facts that $\sigma(S_1 \oplus S_2) = \sigma(S_1) \cup \sigma(S_2)$ for commuting tuples S_1 and S_2, and that $\sigma(T^*|_M)$ lies in the polynomial convex hull of $\sigma(T^*)$ for a commuting tuple T and a coinvariant subspace M.

3.18 Lemma. *Let $S_1 \in B(\mathcal{H}_1)^d$ and $S_2 \in B(\mathcal{H}_2)^d$ be two radial K-hypercontractions on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $S_1 \oplus S_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)^d$ is a radial K-hypercontraction.*

3.19 Lemma. *Let $T \in B(\mathcal{H})^d$ and $S \in B(\tilde{\mathcal{H}})^d$ be commuting tuples on Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, and suppose that there exists an isometry $\Pi\colon \mathcal{H} \to \tilde{\mathcal{H}}$ such that $\Pi T_i^* = S_i^* \Pi$ for all $i = 1, \ldots, d$. If S is a radial K-hypercontraction, then T is a radial K-hypercontraction.*

3.20 Remark. If we suppose that H_K is regular and that Property 3.6 holds, then H_K admits a strong $1/K$-calculus in the sense of [6, Definition 1.1].

The following theorem is our model theorem for radial K-hypercontractions.

3.21 Theorem. *Suppose that Property 3.6 holds and that H_K is regular. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is a radial K-hypercontraction,

(ii) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$,

(iii) there exists a unital, completely contractive linear map

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \;;\; i = 1, \ldots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \text{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_iT_i^*$$

for all $i = 1, \ldots, d$.

If in addition we suppose that $1/K(M_z, M_z^*)$ *exists and that* $\sum_{n=0}^{\infty} c_n$ *converges, then the above are also equivalent to*

(iv) T *is a strong* K*-contraction.*

Proof. (i) $\implies$ (iii): Proposition 3.17.
(iii) $\implies$ (ii): Lemma 2.29.
(ii) $\implies$ (i): This follows from the vector-valued version of Corollary 3.5, Example 3.2, and Lemmas 3.18 and 3.19.
The rest follows from Theorem 2.30. □

As a consequence of the last result, we obtain a version of [18, Theorem 5.6].

3.22 Theorem (Clouâtre, Hartz). *Let* H_K *be a regular complete Nevanlinna-Pick space and let* $T \in B(\mathcal{H})^d$ *be a commuting tuple. The following statements are equivalent:*

(i) T *is a* K*-contraction,*

(ii) T *is a strong* K*-contraction,*

(iii) T *is a radial* K*-hypercontraction,*

(iv) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$*, a spherical unitary* $U \in B(\mathcal{K})^d$*, and an isometry* $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ *such that*

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$,

(v) there exists a unital, completely contractive linear map

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \;;\; i = 1, \ldots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \text{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_iT_i^*$$

for all $i = 1, \ldots, d$.

Proof. Recall Proposition 3.8.

(ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v): Theorem 3.21.

(ii) $\implies$ (i): This is clear.

(i) $\implies$ (iii): By [18, Lemma 5.3], we have that $\sigma(T) \subset \overline{\mathbb{B}}_d$. Furthermore, we obtain that

$$\frac{1}{K_r}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} c_n r^n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) \geq \tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) = \frac{1}{K}(T, T^*) \geq 0$$

for all $0 < r < 1$, i.e., T is a radial K-hypercontraction. □

Another important case is when H_K is a weighted Bergman space (in the sense of Section 1.5.2).

3.23 Theorem. *Let $\nu > 0$ and let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is a radial $K^{(\nu)}$-hypercontraction,

(ii) T is a strong $K^{(\nu)}$-contraction,

(iii) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi \colon \mathcal{H} \to (H_{K^{(\nu)}} \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \dots, d$,

(iv) there exists a unital, completely contractive linear map

$$\varphi \colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i} M_{z_i}^* \ ; \ i = 1, \dots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \text{and} \quad \varphi(M_{z_i} M_{z_i}^*) = T_i T_i^*$$

for all $i = 1, \dots, d$.

If $0 < \nu \leq 1$, then the above are also equivalent to

(v) T is a K-contraction.

Proof. The result follows from Theorems 3.21 and 3.22. □

3.2. Radial K-hypercontractions II

For this section, we use [56] as a guideline. Our goal is to strengthen Theorem 3.23 in the case $\nu \geq 1$.

We start by characterizing when the K-shift M_z is a row contraction.

3.24 Proposition. *The sequence $(a_n)_{n\in\mathbb{N}}$ of Taylor coefficients of k is increasing if and only if $M_z \in B(H_K)^d$ is a row contraction.*

Proof. By Lemma 1.29 (ii), we have the identity

$$\mathrm{id}_{H_K} - \sigma_{M_z}(\mathrm{id}_{H_K}) = P_{\mathbb{C}} + \tau_{\mathrm{SOT}}\text{-}\sum_{n=1}^{\infty}\left(1 - \frac{a_{n-1}}{a_n}\right) P_{\mathbb{H}_n},$$

which implies the assertion. □

If $T \in B(\mathcal{H})^d$ is a row contraction, then $\sigma(T^*) \subset \overline{\mathbb{B}}_d$ (cf. [55, Remark 7 on p. 988]) and hence, $\sigma(T) \subset \overline{\mathbb{B}}_d$.

Therefore, if $(a_n)_{n\in\mathbb{N}}$ is increasing, then $\sigma(M_z) \subset \overline{\mathbb{B}}_d$.

3.25 Lemma. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Property 3.6 holds. For every row contraction $T \in B(\mathcal{H})^d$ which is a radial K-hypercontraction, we have*

$$\frac{1}{K}_{\mathrm{rad}}(T, T^*) + \tau_{\mathrm{SOT}}\text{-}\sum_{n=1}^{\infty}(a_n - a_{n-1})\sigma_T^n\left(\frac{1}{K}_{\mathrm{rad}}(T, T^*)\right) \leq \frac{1}{K^{(1)}}(T, T^*).$$

Proof. Let $0 < r < 1$. Since

$$\frac{1}{k_r^{(1)}}(z) = 1 - rz = (1 - rz)k_r(z)\frac{1}{k_r}(z) \quad (z \in \mathbb{D}),$$

we obtain with Taylor's analytic functional calculus that

$$\begin{aligned}
\frac{1}{K_r^{(1)}}[T] &= (\mathrm{id}_{B(\mathcal{H})} - r\sigma_T)K_r[T]\frac{1}{K_r}[T] \\
&= (\mathrm{id}_{B(\mathcal{H})} - r\sigma_T)\sum_{n=0}^{\infty} a_n r^n \sigma_T^n \frac{1}{K_r}[T] \\
&= \sum_{n=0}^{\infty} a_n r^n \sigma_T^n \frac{1}{K_r}[T] - \sum_{n=0}^{\infty} a_n r^{n+1}\sigma_T^{n+1}\frac{1}{K_r}[T] \\
&= \frac{1}{K_r}[T] + \sum_{n=1}^{\infty}(a_n - a_{n-1})r^n\sigma_T^n\frac{1}{K_r}[T].
\end{aligned}$$

Hence,

$$\begin{aligned}&\left\langle \frac{1}{K_r^{(1)}}(T,T^*)h,h\right\rangle\\&=\left\langle\left(\frac{1}{K_r}(T,T^*)+\sum_{n=1}^{\infty}(a_n-a_{n-1})r^n\sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)\right)h,h\right\rangle\end{aligned}$$

for all $h\in\mathcal{H}$. The lemma of Fatou implies that

$$\begin{aligned}\left\langle \frac{1}{K^{(1)}}(T,T^*)h,h\right\rangle \geq &\left\langle \frac{1}{K_{\mathrm{rad}}}(T,T^*)h,h\right\rangle\\&+\sum_{n=1}^{\infty}(a_n-a_{n-1})\left\langle \sigma_T^n\left(\frac{1}{K_{\mathrm{rad}}}(T,T^*)\right)h,h\right\rangle\end{aligned}$$

for all $h\in\mathcal{H}$. □

For a commuting tuple $T\in B(\mathcal{H})^d$, we define

$$T_\infty=\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty}\sigma_T^N(\mathrm{id}_{\mathcal{H}})\in B(\mathcal{H}),$$

if the latter exist. The existence is guaranteed if T is a row contraction.

3.26 Lemma. *Let $T\in B(\mathcal{H})^d$ be a row contraction. Then*

$$\tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty}\sigma_T^n\left(\frac{1}{K^{(1)}}(T,T^*)\right)+T_\infty=\mathrm{id}_{\mathcal{H}}\,.$$

Proof. This follows from a combination of Remark 2.5 and Proposition 2.6. □

3.27 Definition. Let $T\in B(\mathcal{H})^d$ be a radial K-hypercontraction. We define

$$\Sigma_{K,N}^{\mathrm{rad}}(T)=\mathrm{id}_{\mathcal{H}}-\sum_{n=0}^{N}a_n\sigma_T^n\left(\frac{1}{K_{\mathrm{rad}}}(T,T^*)\right)$$

for all $N\in\mathbb{N}$, and

$$\Sigma_K^{\mathrm{rad}}(T)=\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty}\Sigma_{K,N}^{\mathrm{rad}}(T)$$

if the latter exists. If K is clear from the context, we suppress the index K.

We call T *radial K-pure* if $\Sigma_K^{\mathrm{rad}}(T)=0$.

To make the proof of Lemma 3.29 below clearer, we state the following remark.

3.28 Remark. Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and let $(b_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers such that the series $\sum_{j=0}^{\infty} b_j$ and $\sum_{n=1}^{\infty}(a_n - a_{n-1})\sum_{j=0}^{\infty} b_{n+j}$ converge. Then, since $a_0 = 1$, we have

$$\begin{aligned}
&\sum_{n=1}^{N}(a_n - a_{n-1})\sum_{j=0}^{\infty} b_{n+j} \\
&= \sum_{n=1}^{N} a_n \sum_{j=0}^{\infty} b_{n+j} - \sum_{n=0}^{N-1} a_n \sum_{j=0}^{\infty} b_{n+1+j} \\
&= \sum_{n=1}^{N-1} a_n \left(\sum_{j=0}^{\infty} b_{n+j} - \sum_{j=0}^{\infty} b_{n+1+j}\right) + a_N \sum_{j=0}^{\infty} b_{N+j} - a_0 \sum_{j=0}^{\infty} b_{j+1} \\
&= \sum_{n=1}^{N-1} a_n b_n + a_N \sum_{j=N}^{\infty} b_j - \sum_{j=1}^{\infty} b_j \\
&= \sum_{n=0}^{N-1} a_n b_n + a_N \sum_{j=N}^{\infty} b_j - \sum_{j=0}^{\infty} b_j
\end{aligned}$$

for all $N \in \mathbb{N}$. From this we see that $\sum_{n=0}^{\infty} a_n b_n$ is convergent and

$$0 \le a_N \sum_{j=N}^{\infty} b_j \le \sum_{j=N}^{\infty} a_j b_j \to 0$$

as $N \to \infty$. Hence,

$$\sum_{n=1}^{\infty}(a_n - a_{n-1})\sum_{j=0}^{\infty} b_{n+j} + \sum_{j=0}^{\infty} b_j = \sum_{n=0}^{\infty} a_n b_n.$$

3.29 Lemma. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Property 3.6 holds. For every row contraction $T \in B(\mathcal{H})^d$ which is a radial K-hypercontraction, we have*

$$\tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} a_n \sigma_T^n \left(\frac{1}{K}_{\mathrm{rad}}(T, T^*)\right) + T_\infty \le \mathrm{id}_{\mathcal{H}}\,.$$

In other words, the assertion

$$\Sigma^{\mathrm{rad}}(T) \ge T_\infty \ge 0$$

holds.

Proof. Let $h \in \mathcal{H}$ and set $b_j = \left\langle \sigma_T^j\,(1/K_{\mathrm{rad}}(T,T^*))\,h, h\right\rangle$ for $j \in \mathbb{N}$. With Lemma 3.25, we conclude that

$$b_j + \sum_{n=1}^{\infty}(a_n - a_{n-1})b_{n+j} \le \left\langle \sigma_T^j\left(\frac{1}{K^{(1)}}(T,T^*)\right) h, h\right\rangle$$

for all $j \in \mathbb{N}$, and hence, using Lemma 3.26, we find that

$$\begin{aligned}\|h\|^2 - \langle T_\infty h, h\rangle &= \sum_{j=0}^{\infty}\left\langle \sigma_T^j\left(\frac{1}{K^{(1)}}(T,T^*)\right) h, h\right\rangle \\ &\ge \sum_{j=0}^{\infty} b_j + \sum_{j=0}^{\infty}\sum_{n=1}^{\infty}(a_n - a_{n-1})b_{n+j} \\ &= \sum_{j=0}^{\infty} b_j + \sum_{n=1}^{\infty}(a_n - a_{n-1})\sum_{j=0}^{\infty} b_{n+j}.\end{aligned}$$

Then, Remark 3.28 yields that

$$\|h\|^2 - \langle T_\infty h, h\rangle \ge \sum_{n=0}^{\infty} a_n b_n = \left\langle \sum_{n=0}^{\infty} a_n \sigma_T^n\left(\frac{1}{K_{\mathrm{rad}}}(T,T^*)\right) h, h\right\rangle.$$

Since $h \in \mathcal{H}$ was arbitrary and the partial sums of the series on the right form an increasing sequence of selfadjoint operators, it follows that

$$\tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} a_n \sigma_T^n\left(\frac{1}{K_{\mathrm{rad}}}(T,T^*)\right) + T_\infty \le \mathrm{id}_{\mathcal{H}}\,. \qquad \square$$

The next proposition is an analogue of Proposition 2.6.

3.30 Proposition. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Property 3.6 holds. Furthermore, let $T \in B(\mathcal{H})^d$ be a row contraction which is a radial K-hypercontraction. The map*

$$\psi_T^{\mathrm{rad}} \colon \mathcal{H} \to H_K \otimes \mathcal{D}_T^{\mathrm{rad}},\; h \mapsto \sum_{\alpha\in\mathbb{N}^d} \gamma_\alpha(z^\alpha \otimes D_T^{\mathrm{rad}} T^{*\alpha} h),$$

where $D_T^{\mathrm{rad}} = (1/K_{\mathrm{rad}}(T,T^*))^{1/2}$ and $\mathcal{D}_T^{\mathrm{rad}} = \overline{D_T^{\mathrm{rad}}\mathcal{H}}$, is a well-defined contraction with

$$\left\|\psi_T^{\mathrm{rad}} h\right\|^2 = \|h\|^2 - \left\langle \Sigma^{\mathrm{rad}}(T)h, h\right\rangle \le \|h\|^2 - \langle T_\infty h, h\rangle$$

for all $h \in \mathcal{H}$, and

$$\psi_T^{\mathrm{rad}} T_i^* = (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}_T^{\mathrm{rad}}})^* \psi_T^{\mathrm{rad}}$$

for all $i = 1, \dots, d$.

Proof. By definition and Lemma 3.29, we have

$$\begin{aligned}\left\|\psi_T^{\mathrm{rad}} h\right\|^2 &= \sum_{\alpha\in\mathbb{N}^d} \gamma_\alpha \left\langle T^\alpha \frac{1}{K}_{\mathrm{rad}}(T,T^*)T^{*\alpha}h, h\right\rangle \\ &= \sum_{n=0}^{\infty} a_n \left\langle \sigma_T^n\left(\frac{1}{K}_{\mathrm{rad}}(T,T^*)\right)h, h\right\rangle \\ &= \|h\|^2 - \left\langle \Sigma^{\mathrm{rad}}(T,T^*)h, h\right\rangle \\ &\leq \|h\|^2 - \langle T_\infty h, h\rangle\,. \end{aligned}$$

for $h \in \mathcal{H}$. The claimed intertwining relation for ψ_T^{rad} follows exactly as in the proof of Proposition 2.6. □

Our next aim is to deduce a condition which implies the equality

$$\Sigma^{\mathrm{rad}}(T) = T_\infty.$$

For this purpose, we have to elaborate some technical results.

3.31 Property. The family of Taylor coefficients of k_r/k $(0 < r < 1)$ is uniformly bounded, i.e.,

$$\sup_{\substack{n\in\mathbb{N}\\ 0<r<1}} |a_n(r,1)| < \infty.$$

3.32 Remark. The following are equivalent:

(i) Property 3.31 holds,

(ii) there exists a real number $C > 0$ such that

$$|a_n(r,1)| \leq C$$

for all $n \in \mathbb{N}$ and $0 < r < 1$,

(iii) there exists a real number $C > 0$ such that $|a_n(r,s)| \leq C$ for all $n \in \mathbb{N}$ and $0 < r \leq s \leq 1$.

To verify this equivalence, it suffices to observe that

$$\sum_{n=0}^{\infty} a_n(r,s)z^n = \frac{k_{\frac{r}{s}}(sz)}{k(sz)} = \sum_{n=0}^{\infty} a_n\left(\frac{r}{s},1\right)s^n z^n$$

for $|z|$ small enough and $0 < r \leq s \leq 1$.

3.33 Lemma. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Properties 3.6 and 3.31 hold. Let $T \in B(\mathcal{H})^d$ be a row contraction and a radial K-hypercontraction, and let $0 < r < 1$. Then we have*

$$\begin{aligned}&\frac{1}{K^{(1)}}(T,T^*) + \tau_{\mathrm{SOT}}\text{-}\sum_{n=1}^{\infty} a_n(r,1)\sigma_T^n\left(\frac{1}{K^{(1)}}(T,T^*)\right)\\ =&\frac{1}{K_{\mathrm{rad}}}(T,T^*) + \tau_{\mathrm{SOT}}\text{-}\sum_{n=1}^{\infty}(r^n a_n - r^{n-1}a_{n-1})\sigma_T^n\left(\frac{1}{K_{\mathrm{rad}}}(T,T)\right).\end{aligned}$$

Proof. Let $0 < r < s < 1$. For $z \in \mathbb{D}$, we have

$$\begin{aligned}\frac{k_r(z)}{k_s(z)}\frac{1}{k^{(1)}(z)} &= (1-z)k_r(z)\frac{1}{k_s(z)}\\ &= \left(\sum_{n=0}^{\infty} a_n r^n z^n - \sum_{n=0}^{\infty} a_n r^n z^{n+1}\right)\frac{1}{k_s(z)}\\ &= \left(1 + \sum_{n=1}^{\infty}(a_n r^n - a_{n-1}r^{n-1})z^n\right)\frac{1}{k_s(z)},\end{aligned}$$

and hence,

$$\begin{aligned}&\frac{1}{K_s}[T] + \sum_{n=1}^{\infty}(a_n r^n - a_{n-1}r^{n-1})\sigma_T^n\frac{1}{K_s}[T]\\ =&\sum_{n=0}^{\infty} a_n(r,s)\sigma_T^n\frac{1}{K^{(1)}}[T]\\ =&\frac{1}{K^{(1)}}[T] + \sum_{n=1}^{\infty} a_n(r,s)\sigma_T^n\frac{1}{K^{(1)}}[T].\end{aligned}$$

Let $h \in \mathcal{H}$. By Lemma 3.26 and Remark 3.32, the dominated convergence theorem yields that

$$\sum_{n=1}^{\infty} a_n(r,s)\left\langle \sigma_T^n\left(\frac{1}{K^{(1)}}(T,T^*)\right)h,h\right\rangle \to \sum_{n=1}^{\infty} a_n(r,1)\left\langle \sigma_T^n\left(\frac{1}{K^{(1)}}(T,T^*)\right)h,h\right\rangle$$

as $s \to 1$. Since

$$\begin{aligned}\left|a_n r^n\left\langle \sigma_T^l\left(\frac{1}{K_s}(T,T^*)\right)h,h\right\rangle\right| &\le a_n r^n\left\|\sigma_T^l\right\|\left\|\frac{1}{K_s}(T,T^*)\right\|\|h\|^2\\ &\le a_n r^n\left\|\frac{1}{K_{s_0}}(T,T^*)\right\|\|h\|^2\end{aligned}$$

for all $k, l \in \mathbb{N}$ and $s_0 \leq s$, we see that

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}(a_n r^n - a_{n-1}r^{n-1}) \left\langle \sigma_T^n \left(\frac{1}{K_s}(T,T^*)\right) h, h\right\rangle\right| \\
\leq &\sum_{n=1}^{\infty} \left(a_n r^n + a_{n-1}r^{n-1}\right) \left|\left\langle \sigma_T^n \left(\frac{1}{K_s}(T,T^*)\right) h, h\right\rangle\right| \\
\leq &2\left\|\frac{1}{K_s}(T,T^*)\right\| \|h\|^2 \sum_{n=0}^{\infty} a_n r^n \\
\leq &2\left\|\frac{1}{K_{s_0}}(T,T^*)\right\| \|h\|^2 k(r)
\end{aligned}$$

for all $s_0 \leq s$. Hence, by the dominated convergence theorem, it follows that

$$\begin{aligned}
&\lim_{s\to 1}\sum_{n=1}^{\infty}(a_n r^n - a_{n-1}r^{n-1}) \left\langle \sigma_T^n \left(\frac{1}{K_s}(T,T^*)\right) h, h\right\rangle \\
= &\sum_{n=1}^{\infty}(a_n r^n - a_{n-1}r^{n-1}) \left\langle \sigma_T^n \left(\frac{1}{K_{\text{rad}}}(T,T^*)\right) h, h\right\rangle .
\end{aligned}$$

Invoking Proposition 3.13 ends the proof. □

3.34 Lemma. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Properties 3.6 and 3.31 hold. Furthermore, let $T \in B(\mathcal{H})^d$ be a row contraction and a radial K-hypercontraction. Then*

$$\frac{1}{K^{(1)}}(T,T^*) = \frac{1}{K_{\text{rad}}}(T,T^*) + \tau_{\text{SOT}}\text{-}\sum_{n=1}^{\infty}(a_n - a_{n-1})\sigma_T^n\left(\frac{1}{K_{\text{rad}}}(T,T^*)\right).$$

Proof. Let $h \in \mathcal{H}$. By Lemma 3.29 and the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{r\to 1}\sum_{n=1}^{\infty}(r^n a_n - r^{n-1}a_{n-1}) \left\langle \sigma_T^n \left(\frac{1}{K_{\text{rad}}}(T,T^*)\right) h, h\right\rangle \\
= &\sum_{n=1}^{\infty}(a_n - a_{n-1}) \left\langle \sigma_T^n \left(\frac{1}{K_{\text{rad}}}(T,T^*)\right) h, h\right\rangle .
\end{aligned}$$

By Lemma 3.26 and the dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{r\to 1}\sum_{n=1}^{\infty} a_n(r,1)\left\langle \sigma_T^n\left(\frac{1}{K^{(1)}}(T,T^*)\right)h,h\right\rangle\\ =&\sum_{n=1}^{\infty} a_n(1,1)\left\langle \sigma_T^n\left(\frac{1}{K^{(1)}}(T,T^*)\right)h,h\right\rangle\\ =&0.\end{aligned}$$

The result follows from Lemma 3.33. □

3.35 Proposition. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Properties 3.6 and 3.31 hold. If $T\in B(\mathcal{H})^d$ is a row contraction and a radial K-hypercontraction, then $\Sigma^{\mathrm{rad}}(T)=T_\infty$ and*

$$\|h\|^2=\left\|\psi_T^{\mathrm{rad}}h\right\|^2+\langle T_\infty h,h\rangle$$

for all $h\in\mathcal{H}$.

Proof. If we replace Lemma 3.25 by Lemma 3.34 in the proof of Lemma 3.29, then the same proof yields that

$$\tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} a_n\sigma_T^n\left(\frac{1}{K_{\mathrm{rad}}}(T,T^*)\right)+T_\infty=\mathrm{id}_{\mathcal{H}},$$

i.e.,

$$\|h\|^2=\left\|\psi_T^{\mathrm{rad}}h\right\|^2+\langle T_\infty h,h\rangle$$

holds for each $h\in\mathcal{H}$. Furthermore, we have that

$$\Sigma^{\mathrm{rad}}(T)=T_\infty.$$ □

3.36 Definition. We say that a row contraction $T\in B(\mathcal{H})^d$ belongs to the class $C_{\cdot 0}$ if $T_\infty=0$.

3.37 Corollary. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Property 3.6 holds. Furthermore, let $T\in B(\mathcal{H})^d$ be a row contraction and a radial K-hypercontraction. Consider the following statements:*

(i) T belongs to the class $C_{\cdot 0}$,

(ii) T is radial K-pure,

(iii) ψ_T^{rad} is an isometry.

The implications (i) $\Leftarrow$ (ii) $\iff$ (iii) hold. If in addition Property 3.31 holds, all statements are equivalent.

Proof. The first part follows from Lemma 3.29 and Proposition 3.30. The latter is an application of Proposition 3.35. □

For the rest of this section, we are interested in the connection between strong K-contractions and radial K-hypercontractions under suitable additional conditions.

3.38 Lemma. *Let $T \in B(\mathcal{H})^d$ be a row contraction. Then*

$$\tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \sigma_T^N\left(\frac{1}{K_r}(T,T^*)\right) = \frac{1}{k(r)}T_\infty$$

for all $0 < r < 1$.

Proof. Since

$$\left\|c_n r^n \sigma_T^{N+n}(\mathrm{id}_{\mathcal{H}})h\right\| \leq |c_n|\, r^n\, \|h\|$$

for all $k, N \in \mathbb{N}$ and $\sum_{n=0}^{\infty} c_n r^n$ is absolutely convergent, we have that

$$\begin{aligned} \lim_{N\to\infty} \sigma_T^N\left(\frac{1}{K_r}(T,T^*)\right) h &= \lim_{N\to\infty} \sum_{n=0}^{\infty} c_n r^n \sigma_T^{N+n}(\mathrm{id}_{\mathcal{H}})h \\ &= \sum_{n=0}^{\infty} c_n r^n \lim_{N\to\infty} \sigma_T^N(\mathrm{id}_{\mathcal{H}})h \\ &= \frac{1}{k(r)} \lim_{N\to\infty} \sigma_T^N(\mathrm{id}_{\mathcal{H}})h \end{aligned}$$

for all $h \in \mathcal{H}$. □

The set

$$A^+(\mathbb{D}) = \left\{ f = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{O}(\mathbb{D})\ ;\ \|f\|_{A^+(\mathbb{D})} = \sum_{n=0}^{\infty} |f_n| < \infty \right\}.$$

equipped with the usual addition and multiplication of analytic functions is an abelian Banach algebra.

3.39 Property. Let $1/k \in A^+(\mathbb{D})$, i.e., suppose that $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

3.40 Proposition. *Suppose that Property 3.39 holds and let $T \in B(\mathcal{H})^d$ be a row contraction. Then*

$$\frac{1}{K}_{\mathrm{rad}}(T, T^*) = \frac{1}{K}(T, T^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}).$$

Proof. Since $T \in B(\mathcal{H})^d$ is a row contraction, we have by Russo-Dye's theorem (cf. [57, Corollary 2.9])

$$\|\sigma_T\| = \|\sigma_T(\mathrm{id}_{\mathcal{H}})\| \leq 1.$$

Since $\sum_{n=0}^{\infty} c_n$ is absolutely convergent, we obtain, by the dominated convergence theorem,

$$\begin{aligned} \frac{1}{K}(T, T^*) &= \sum_{n=0}^{\infty} c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) = \sum_{n=0}^{\infty} \lim_{r \to 1} c_n r^n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) \\ &= \lim_{r \to 1} \sum_{n=0}^{\infty} c_n r^n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) = \frac{1}{K}_{\mathrm{rad}}(T, T^*), \end{aligned}$$

where the series are norm convergent and all limits are formed with respect to the operator norm. □

3.41 Corollary. *Suppose that Property 3.39 holds. Let $T \in B(\mathcal{H})^d$ be a row contraction. Then T is a K-contraction if and only if T is a radial K-hypercontraction. In this case, we have*

$$\frac{1}{K}_{\mathrm{rad}}(T, T^*) = \frac{1}{K}(T, T^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} c_n \sigma_T^n(\mathrm{id}_{\mathcal{H}}) \geq 0.$$

3.42 Property. Suppose that Property 3.39 holds and that the functions k_r/k $(0 < r < 1)$ form a norm-bounded family in the Banach algebra $A^+(\mathbb{D})$.

3.43 Remark. Since

$$a_n(r, s) = a_n\left(\frac{r}{s}, 1\right) s^n$$

holds for all $0 < r < s < 1$ and $n \in \mathbb{N}$, Property 3.42 implies that

$$\sup_{0<r<s<1} \sum_{n=0}^{\infty} |a_n(r, s)| < \infty.$$

The following result by Olofsson [56, Proposition 5.4] gives us a class of reproducing kernel Hilbert spaces satisfying Property 3.42.

3.44 Proposition. *If K is a finite product of reproducing kernels of irreducible unitarily invariant complete Nevanlinna-Pick spaces, then K fulfills Properties 3.6 and 3.42.*

In particular, if K is the reproducing kernel of a space mentioned in Sections 1.5.1 and 1.5.2, then K fulfills Properties 3.6 and 3.42.

Since convergence in the weak* topology τ_{w^*} on $\ell^1(\mathbb{N}) = c_0'$ coincide with norm boundedness and pointwise convergence, we obtain the following result.

3.45 Lemma. *Suppose that Property 3.42 holds. Then*

$$\tau_{w^*}\text{-}\lim_{t\to 1} (a_n(s,t))_{n\in\mathbb{N}} = (a_n(s,1))_{n\in\mathbb{N}} \quad \text{in } \ell^1(\mathbb{N}) = c_0'$$

for all $0 < s < 1$ *and*

$$\tau_{w^*}\text{-}\lim_{s\to 1} (a_n(s,1))_{n\in\mathbb{N}} = (1, 0, \ldots) \quad \text{in } \ell^1(\mathbb{N}) = c_0'.$$

3.46 Lemma. *Suppose that Properties 3.6 and 3.42 hold. Let $T \in B(\mathcal{H})^d$ be a row contraction and a K-contraction. Then, for all $0 < r < 1$, we have*

$$\frac{1}{K_r}(T,T^*) \geq \tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} a_n(1,r)\sigma_T^n\left(\frac{1}{K}(T,T^*)\right) + \frac{1}{k(r)}T_\infty$$

Proof. Let $0 < r < s < t < 1$ and $h \in \mathcal{H}$. Since

$$\frac{k_s(z)}{k_t(z)}\frac{1}{k_r(z)} = \frac{k_s(z)}{k_r(z)}\frac{1}{k_t(z)} \qquad (z \in \mathbb{D}),$$

Taylor's analytic functional calculus gives us

$$\sum_{n=0}^{\infty} a_n(s,t)\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h, h\right\rangle = \sum_{n=0}^{\infty} a_n(s,r)\left\langle \sigma_T^n\left(\frac{1}{K_t}(T,T^*)\right)h, h\right\rangle.$$

By the proof of Proposition 3.40, we know that

$$\lim_{t\to 1}\left\|\frac{1}{K_t}(T,T^*) - \frac{1}{K}(T,T^*)\right\| = 0.$$

Hence, the dominated convergence theorem implies that

$$\lim_{t\to 1}\sum_{n=0}^{\infty} a_n(s,r)\sigma_T^n\left(\frac{1}{K_t}(T,T^*)\right) = \sum_{n=0}^{\infty} a_n(s,r)\sigma_T^n\left(\frac{1}{K}(T,T^*)\right).$$

in the operator norm. Set

$$L_{r,h} = \frac{1}{k(r)} \lim_{N\to\infty} \left\langle \sigma_T^N(\mathrm{id}_{\mathcal{H}})h, h\right\rangle,$$

which exists since T is a row contraction. Since

$$\lim_{t\to 1}\sum_{n=0}^{\infty} a_n(s,t) = \lim_{t\to 1}\frac{k(s)}{k(t)} = k(s)\frac{1}{\sum_{n=0}^{\infty} a_n} = 0$$

and $\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right)_{n\in\mathbb{N}} \in c_0$ by Lemma 3.38, we conclude that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n(s,t)\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle\\
=&\sum_{n=0}^{\infty} a_n(s,t)\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right) + L_{r,h}\sum_{n=0}^{\infty} a_n(s,t)\\
=&\left\langle\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right)_{n\in\mathbb{N}}, (a_n(s,t))_{n\in\mathbb{N}}\right\rangle_{c_0,\ell^1(\mathbb{N})}\\
&+ L_{r,h}\sum_{n=0}^{\infty} a_n(s,t)\\
\to&\left\langle\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right)_{n\in\mathbb{N}}, (a_n(s,1))_{n\in\mathbb{N}}\right\rangle_{c_0,\ell^1(\mathbb{N})}\\
=&\sum_{n=0}^{\infty} a_n(s,1)\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right)
\end{aligned}$$

as $t\to 1$, where we have used Lemma 3.45. It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n(s,r)\left\langle \sigma_T^n\left(\frac{1}{K}(T,T^*)\right)h,h\right\rangle\\
=&\lim_{t\to 1}\sum_{n=0}^{\infty} a_n(s,r)\left\langle \sigma_T^n\left(\frac{1}{K_t}(T,T^*)\right)h,h\right\rangle\\
=&\lim_{t\to 1}\sum_{n=0}^{\infty} a_n(s,t)\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle\\
=&\sum_{n=0}^{\infty} a_n(s,1)\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right).
\end{aligned}$$

Again by Lemma 3.45, we see that

$$\lim_{s\to 1}\sum_{n=0}^{\infty} a_n(s,1)\left(\left\langle \sigma_T^n\left(\frac{1}{K_r}(T,T^*)\right)h,h\right\rangle - L_{r,h}\right) = \left\langle \frac{1}{K_r}(T,T^*)h,h\right\rangle - L_{r,h}.$$

By Remark 3.7 and the lemma of Fatou, we have that

$$\lim_{s\to 1}\sum_{n=0}^{\infty} a_n(s,r)\left\langle \sigma_T^n\left(\frac{1}{K}(T,T^*)\right)h,h\right\rangle \geq \sum_{n=0}^{\infty} a_n(1,r)\left\langle \sigma_T^n\left(\frac{1}{K}(T,T^*)\right)h,h\right\rangle.$$

Thus, the result follows. □

3.47 Theorem. *Suppose that Properties 3.6 and 3.42 hold. Let $T \in B(\mathcal{H})^d$ be a row contraction and a K-contraction. Then T is a radial K-hypercontraction.*

Proof. This follows from Lemma 3.46. □

3.48 Lemma. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Properties 3.6 and 3.31 hold. Let $T \in B(\mathcal{H})^d$ be a row contraction and a radial K-hypercontraction such that $1/K(T,T^*)$ exists. Then T is a strong K-contraction and $\Sigma(T) = \Sigma^{\mathrm{rad}}(T) = T_\infty$.*

Proof. By Abel's theorem, it follows that

$$\left\langle \frac{1}{K}(T,T^*)h,h\right\rangle = \sum_{n=0}^{\infty} c_n \left\langle \sigma_T^n(\mathrm{id}_{\mathcal{H}})h,h\right\rangle = \lim_{r\to 1}\sum_{n=0}^{\infty} c_n r^n \left\langle \sigma_T^n(\mathrm{id}_{\mathcal{H}})h,h\right\rangle$$

for all $h \in \mathcal{H}$. Hence,

$$\frac{1}{K}(T,T^*) = \tau_{\mathrm{WOT}}\text{-}\lim_{r\to 1}\frac{1}{K_r}(T,T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{r\to 1}\frac{1}{K_r}(T,T^*) = \frac{1}{K_{\mathrm{rad}}}(T,T^*) \geq 0$$

since $1/K_{\mathrm{rad}}(T,T^*)$ exists and defines a positive operator by Proposition 3.13. By Proposition 3.35, we have that

$$\Sigma(T) = \Sigma^{\mathrm{rad}}(T) = T_\infty.$$

Thus, the result follows. □

3.49 Theorem. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is increasing and that Properties 3.6 and 3.42 hold. Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

(i) T is a row contraction and a radial K-hypercontraction,

(ii) T *is a row contraction and a* K*-contraction,*

(iii) T *is a strong* K*-contraction,*

(iv) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$*, a spherical unitary* $U \in B(\mathcal{K})^d$*, and an isometry* $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ *such that*

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, d$*.*

In this case, we have

$$\Sigma^{\mathrm{rad}}(T) = \Sigma(T) = T_\infty$$

and

$$\tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} a_n \sigma_T^n \left(\frac{1}{K}(T, T^*) \right) + T_\infty = \mathrm{id}_{\mathcal{H}}\,.$$

If in addition H_K *is regular, then the above are also equivalent to*

(v) there exists a unital, completely contractive linear map

$$\varphi\colon\ \mathrm{span}\left\{ \mathrm{id}_{H_K}, M_{z_i}, M_{z_i} M_{z_i}^*\ ;\ i = 1, \ldots, d \right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \text{and} \quad \varphi(M_{z_i} M_{z_i}^*) = T_i T_i^*$$

for all $i = 1, \ldots, d$*.*

Proof. (i) $\implies$ (ii): This is clear.
(ii) $\implies$ (i): This follows from Theorem 3.47.
(i) $\implies$ (iii): This follows from Lemma 3.48 and Corollary 3.41.
(iii) $\iff$ (iv): This is clear by Theorem 2.25 and Proposition 3.15.
(iii) & (iv) $\implies$ (ii): This is clear.
The extra follows from Theorem 2.30. □

3.50 Definition. Let $\nu \geq 1$ be a real number. We call a commuting tuple $T \in B(\mathcal{H})^d$ an *ν-hypercontraction* if

$$\frac{1}{K^{(\mu)}}(T, T^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} c_n^{(\mu)} \sigma_T^n(\mathrm{id}_{\mathcal{H}}) \geq 0$$

for all $1 \leq \mu \leq \nu$.

The next result shows that the definition above coincides with the notion of m-hypercontractions if $\nu = m \in \mathbb{N}^*$ (cf. Remark 2.2).

3.51 Theorem. *Let $\nu \geq 1$ and let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following assertions are equivalent:*

(i) T is a row contraction and a radial $K^{(\nu)}$-hypercontraction,

(ii) T is an ν-hypercontraction,

(iii) T is row contraction and a $K^{(\nu)}$-contraction,

(iv) T is a strong $K^{(\nu)}$-contraction,

(v) there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi\colon \mathcal{H} \to (H_{K^{(\nu)}} \otimes \mathcal{D}) \oplus \mathcal{K}$ such that

$$\Pi T_i^* = ((M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \dots, d$,

(vi) there exists a unital, completely contractive linear map

$$\varphi\colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \; ; \; i = 1, \dots, d\right\} \to B(\mathcal{H})$$

with

$$\varphi(M_{z_i}) = T_i \quad \text{and} \quad \varphi(M_{z_i}M_{z_i}^*) = T_i T_i^*$$

for all $i = 1, \dots, d$.

In this case, we have

$$\Sigma^{\mathrm{rad}}(T) = \Sigma(T) = T_\infty$$

and

$$\tau_{\mathrm{SOT}}\text{-}\sum_{n=0}^{\infty} a_n^{(\nu)} \sigma_T^n \left(\frac{1}{K^{(\nu)}}(T, T^*)\right) + T_\infty = \mathrm{id}_{\mathcal{H}}\,.$$

Proof. Recall Proposition 3.44. By Theorem 3.49, we only have to show that (i) $\implies$ (ii). To this end, let $1 \leq \mu \leq \nu$ and $0 < r < 1$. By Taylor's analytic functional calculus, the identity

$$\frac{1}{k_r^{(\mu)}(z)} = k_r^{(\nu-\mu)}(z) \frac{1}{k_r^{(\nu)}(z)} \quad (z \in \mathbb{D})$$

yields that

$$\frac{1}{K_r^{(\mu)}}[T] = \sum_{n=0}^{\infty} a_n^{(\nu-\mu)} r^n \sigma_T^n \frac{1}{K_r^{(\nu)}}[T].$$

Let $h \in \mathcal{H}$. We obtain

$$\left\langle \frac{1}{K_r^{(\mu)}}(T,T^*)h, h \right\rangle = \sum_{n=0}^{\infty} a_n^{(\nu-\mu)} r^n \left\langle \sigma_T^n \left(\frac{1}{K_r^{(\nu)}}(T,T^*) \right) h, h \right\rangle \geq 0.$$

Thus, T is a radial $K^{(\mu)}$-hypercontraction for all $1 \leq \mu \leq \nu$. But then Lemma 3.48 implies that T is a strong $K^{(\mu)}$-contraction for all $1 \leq \mu \leq \nu$. $\square$

4. A Beurling-type theorem

In [14], Beurling studied the invariant subspaces of the shift operator on the Hardy space on the unit disc:

4.1 Theorem (Beurling)**.** *Suppose that $d = 1$ and let $M_z \in B(H_{K^{(1)}})$ be the shift operator on $H_{K^{(1)}}$. For a subspace $\mathcal{S} \subset H_{K^{(1)}}$, the following statements are equivalent:*

(i) $\mathcal{S} \in \mathrm{Lat}(M_z)$,

(ii) there exists a bounded analytic function $\theta\colon \mathbb{D} \to \mathbb{C}$ such that

$$M_\theta\colon H_{K^{(1)}} \to H_{K^{(1)}},\ f \mapsto \theta \cdot f$$

is an isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

Further progress for vector-valued Hardy spcaes was made by Lax in [53] and Halmos in [46]. McCullough and Trent obtained in [54] a similar result in the case of the Drury-Arveson space:

4.2 Theorem (McCullough, Trent)**.** *Suppose that $\mathcal{E}$ is a Hilbert space and let $M_z \in B(H_{K^{(1)}}(\mathcal{E}))^d$ be the shift operator on $H_{K^{(1)}}(\mathcal{E})$. For a subspace $\mathcal{S} \subset H_{K^{(1)}}(\mathcal{E})$, the following statements are equivalent:*

(i) $\mathcal{S} \in \mathrm{Lat}(M_z)$,

(ii) there exist a Hilbert space $\mathcal{D}$ and a bounded analytic function $\theta\colon \mathbb{B}_d \to B(\mathcal{D}, \mathcal{E})$ such that

$$M_\theta\colon H_{K^{(1)}}(\mathcal{D}) \to H_{K^{(1)}}(\mathcal{E}),\ f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

In the following, we want to obtain a Beurling-type theorem in our general setting developed in Chapter 2. Therefore, let $H_K \subset \mathcal{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space whose kernel $K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is of the form $K(z, w) = k(\langle z, w\rangle)$ $(z, w \in \mathbb{B}_d)$ with a zero-free analytic function $k\colon \mathbb{D} \to \mathbb{C},\ z \mapsto$

$\sum_{n=0}^{\infty} a_n z^n$ such that $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$. Furthermore, we suppose that $1/K(M_z, M_z^*)$ exists.

Before we start looking at the K-shift, we state a preliminary result for general K-pure commuting tuples. Our approach is inspired by [52, Section 3.2].

4.3 Proposition. *Let $\mathcal{H}$ be a Hilbert space, $T \in B(\mathcal{H})^d$ be K-pure and $\mathcal{S} \subset \mathcal{H}$. The following statements are equivalent:*

(i) $\mathcal{S} \in \mathrm{Lat}(T)$ and $T|_{\mathcal{S}}$ is K-pure,

(ii) there exist a Hilbert space $\mathcal{D}$, and a partial isometry $\pi\colon H_K \otimes \mathcal{D} \to \mathcal{H}$ with

$$T_i\pi = \pi(M_{z_i} \otimes \mathrm{id}_{\mathcal{D}})$$

for all $i = 1, \ldots, d$ and $\mathrm{Im}(\pi) = \mathcal{S}$.

Proof. Suppose that (i) holds. Since $T|_{\mathcal{S}}$ is K-pure, by Theorem 2.15, there exist a Hilbert space $\mathcal{D}$ and an isometry $\Pi\colon \mathcal{S} \to H_K \otimes \mathcal{D}$ such that

$$\Pi(T|_{\mathcal{S}})_i^* = (M_{z_i} \otimes \mathrm{id}_{\mathcal{D}})^*\Pi$$

for all $i = 1, \ldots, d$. Hence, Π^* is surjective. Denoting the inclusion map by $\iota\colon \mathcal{S} \to \mathcal{H}$, we set

$$\pi = \iota \circ \Pi^*.$$

Then, $\pi\colon H_K \otimes \mathcal{D} \to \mathcal{H}$ is a partial isometry with $\mathrm{Im}(\pi) = \mathcal{S}$ and

$$T_i\pi = T_i\iota \circ \Pi^* = \iota(\Pi(T|_{\mathcal{S}})_i^*)^* = \iota(T|_{\mathcal{S}})_i\Pi^* = \iota\Pi^*(M_{z_i} \otimes \mathrm{id}_{\mathcal{D}}) = \pi(M_{z_i} \otimes \mathrm{id}_{\mathcal{D}})$$

for all $i = 1, \ldots, d$.

Now we suppose that (ii) holds. Obviously, $\mathcal{S} = \mathrm{Im}(\pi) \in \mathrm{Lat}(T)$. The map $\kappa = \pi^*|_{\mathcal{S}}\colon \mathcal{S} \to H_K \otimes \mathcal{D}$ is an isometry, since $\mathcal{S} = \mathrm{Im}(\pi) = (\ker(\pi^*))^{\perp}$. As the adjoint of the operator $H_K \otimes \mathcal{D} \xrightarrow{\pi} \mathcal{S}$, the map κ intertwines $(T|_{\mathcal{S}})^*$ and $(M_z \otimes \mathrm{id}_{\mathcal{D}})^*$ componentwise. By Theorem 2.15, the tuple $T|_{\mathcal{S}}$ is K-pure. $\square$

We are now interested in a stronger version of the last result in the case when T is the K-shift M_z. To this end, sufficient conditions for the existence of multipliers will be elaborated.

4.4 Lemma. *Let $\mathcal{D}, \mathcal{E}$ be Hilbert spaces and $H(\mathcal{D}) \subset \mathcal{D}^{\Omega}$ and $H(\mathcal{E}) \subset \mathcal{E}^{\Omega}$ be reproducing kernel Hilbert spaces over a set $\Omega \subset \mathbb{C}^d$ such that*

(i) $M_z^{H(\mathcal{D})} \in B(H(\mathcal{D}))^d$ and $M_z^{H(\mathcal{E})} \in B(H(\mathcal{E}))^d$,

(ii) the point evaluation $\delta_\lambda^{\mathcal{D}} \colon H(\mathcal{D}) \to \mathcal{D}$ *is surjective for each* $\lambda \in \Omega$*,*

(iii) $\ker(\delta_\lambda^{\mathcal{D}}) = \overline{\sum_{i=1}^d (z_i - \lambda_i) H(\mathcal{D})}$ *for all* $\lambda \in \Omega$*.*

Then, for each operator $\pi \in B(H(\mathcal{D}), H(\mathcal{E}))$ *such that*

$$\pi M_{z_i}^{H(\mathcal{D})} = M_{z_i}^{H(\mathcal{E})} \pi$$

for all $i = 1, \dots, d$*, there exists a multiplier* $\theta \in \mathcal{M}(H(\mathcal{D}), H(\mathcal{E}))$ *such that* $\pi = M_\theta$*.*

If in addition Ω *is open,* $\mathcal{D} \subset H(\mathcal{D})$*, and* $H(\mathcal{E}) \subset \mathcal{O}(\Omega, \mathcal{E})$*, then* θ *is analytic.*

Proof. Let $f \in \ker(\delta_\lambda^{\mathcal{D}})$ and let $\pi \in B(H(\mathcal{D}), H(\mathcal{E}))$ such that

$$\pi M_{z_i}^{H(\mathcal{D})} = M_{z_i}^{H(\mathcal{E})} \pi$$

for all $i = 1, \dots, d$. By (iii), there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $\sum_{i=1}^d (z_i - \lambda_i) H(\mathcal{D})$ with $\tau_{\|\cdot\|_{H(\mathcal{D})}}$-limit f and hence, for all $i = 1, \dots, d$ and $n \in \mathbb{N}$, there exist $f_{n,i} \in H(\mathcal{D})$ such that

$$f_n = \sum_{i=1}^d (z_i - \lambda_i) f_{n,i}.$$

Thus,

$$\begin{aligned}
\delta_\lambda^{\mathcal{E}}(\pi f) &= \lim_{n \to \infty} \delta_\lambda^{\mathcal{E}}(\pi f_n) \\
&= \lim_{n \to \infty} \delta_\lambda^{\mathcal{E}} \left(\pi \sum_{i=1}^d (z_i - \lambda_i) f_{n,i} \right) \\
&= \lim_{n \to \infty} \delta_\lambda^{\mathcal{E}} \left(\sum_{i=1}^d (z_i - \lambda_i)(\pi f_{n,i}) \right) \\
&= 0.
\end{aligned}$$

Then, by Proposition 1.22, there exists a map $\theta \colon \Omega \to B(\mathcal{D}, \mathcal{E})$ such that

$$\pi f = \theta f$$

for all $f \in H(\mathcal{D})$.

If in addition Ω is open, $\mathcal{D} \subset H(\mathcal{D})$, and $H(\mathcal{E}) \subset \mathcal{O}(\Omega, \mathcal{E})$, then

$$\theta(\cdot)(x) = (\theta x)(\cdot) = (\pi x)(\cdot)$$

is analytic for all $x \in \mathcal{D}$. Hence, θ is analytic. $\square$

4.5 Proposition. *Let $\mathcal{D}, \mathcal{E}$ be Hilbert spaces and let $H(\mathcal{D}) \subset \mathcal{D}^{\mathbb{B}_d}$ and $H(\mathcal{E}) \subset \mathcal{E}^{\mathbb{B}_d}$ be reproducing kernel Hilbert spaces such that*

(i) $H(\mathcal{D}) = \overline{\mathcal{D}[z]}$ *and* $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_d, \mathcal{E})$,

(ii) $M_z^{H(\mathcal{D})} \in B(H(\mathcal{D}))^d$ *and* $M_z^{H(\mathcal{E})} \in B(H(\mathcal{E}))^d$.

Then, for each operator $\pi \in B(H(\mathcal{D}), H(\mathcal{E}))$ *with*

$$\pi M_{z_i}^{H(\mathcal{D})} = M_{z_i}^{H(\mathcal{E})} \pi$$

for all $i = 1, \ldots, d$, *there exists an analytic multiplier* $\theta \in \mathcal{M}(H(\mathcal{D}), H(\mathcal{E}))$ *such that* $\pi = M_\theta$. *Furthermore, if we suppose that* $H(\mathcal{D})$ *is non-degenerate, then* θ *is also bounded.*

Proof. Let $\lambda \in \mathbb{B}_d$ and $f \in \ker(\delta_\lambda^{\mathcal{D}})$. Then there exists a sequence $(\tilde{p}_n)_{n\in\mathbb{N}}$ in $\mathcal{D}[z]$ with $\tau_{\|\cdot\|_{H(\mathcal{D})}}$-limit f. By the closed graph theorem, we see that the inclusion map $\mathcal{D} \to H(\mathcal{D})$ is continuous and hence

$$p_n = \tilde{p}_n - \tilde{p}_n(\lambda) \to f$$

in $H(\mathcal{D})$ as $n \to \infty$. Furthermore, we have

$$p_n \in \sum_{i=1}^{d} (z_i - \lambda_i)\mathcal{D}[z] \subset \sum_{i=1}^{d} (z_i - \lambda_i) H(\mathcal{D})$$

for all $n \in \mathbb{N}$ and thus,

$$f \in \overline{\sum_{i=1}^{d} (z_i - \lambda_i) H(\mathcal{D})}.$$

Since $\mathcal{D} \subset \mathcal{D}[z] \subset H(\mathcal{D})$, the conditions (ii) and (iii) in Lemma 4.4 are satisfied and the result follows.

The remaining assertion follows from Proposition 1.21. □

With these preparations, we are now able to proof a Beurling-type theorem in our general setting.

4.6 Theorem. *Let $\mathcal{E}$ be a Hilbert space, $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_d, \mathcal{E})$ a reproducing kernel Hilbert space, and let $M_z \in B(H(\mathcal{E}))^d$ be K-pure. For $\mathcal{S} \subset H(\mathcal{E})$, the following statements are equivalent:*

(i) $\mathcal{S} \in \operatorname{Lat}(M_z)$ *and* $M_z|_{\mathcal{S}}$ *is K-pure,*

(ii) there exist a Hilbert space $\mathcal{D}$ and a bounded analytic inner multiplier $\theta \in \mathcal{M}(H_K(\mathcal{D}), H(\mathcal{E}))$ such that $\operatorname{Im}(M_\theta) = \mathcal{S}$*.*

Proof. By Propositions 1.27 and 4.3 and Remark 1.40, the implication (i) $\Longrightarrow$ (ii) follows from Proposition 4.5. The other direction is clear by Proposition 4.3. □

For the rest of this chapter, we want to focus on the case of weighted Bergman spaces. For this purpose, we first state the following easy observations.

4.7 Remark. Let $\mathcal{H}$ be a Hilbert space, $T \in B(\mathcal{H})^d$ be a commuting tuple and $\mathcal{S} \in \operatorname{Lat}(T)$.

(i) If T is $C_{\cdot 0}$, then $T|_{\mathcal{S}}$ is also $C_{\cdot 0}$.

(ii) If T is a $K^{(1)}$-contraction, then $T|_{\mathcal{S}}$ is a $K^{(1)}$-contraction.

(iii) If T is $K^{(1)}$-pure, then $T|_{\mathcal{S}}$ is $K^{(1)}$-pure.

Proof. (i) For $x \in \mathcal{S} \subset \mathcal{H}$ and $N \in \mathbb{N}$, we have

$$\begin{aligned}\left\langle \sigma_T^N(\mathrm{id}_{\mathcal{H}})x, x\right\rangle &= \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} \langle T^\alpha T^{*\alpha} x, x\rangle \\ &= \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} \langle T^{*\alpha} x, T^{*\alpha} x\rangle \\ &\geq \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} \langle P_{\mathcal{S}} T^{*\alpha} x, P_{\mathcal{S}} T^{*\alpha} x\rangle \\ &= \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} \langle T|_{\mathcal{S}}{}^\alpha T|_{\mathcal{S}}^{*\,\alpha} x, x\rangle \\ &= \left\langle \sigma_{T|_{\mathcal{S}}}^N(\mathrm{id}_{\mathcal{S}})x, x\right\rangle.\end{aligned}$$

(ii) This follows from the calculation above with $N = 1$.

(iii) By Remark 2.5, the result follows from a combination of (i) and (ii). □

A special case of the following result has been proven by Klauk in [52, Korollar 3.2.3].

4.8 Theorem. *Let $\nu \geq 1$, $\mathcal{E}$ be a Hilbert space, and $\mathcal{S} \subset H_{K^{(\nu)}}(\mathcal{E})$ be a subspace. For $M_z \in B(H_{K^{(\nu)}}(\mathcal{E}))^d$ and $1 \leq \mu \leq \nu$, the following statements are equivalent:*

(i) $\mathcal{S} \in \mathrm{Lat}\,(M_z)$ *and* $M_z|_{\mathcal{S}}$ *is a* $K^{(\mu)}$*-contraction,*

(ii) $\mathcal{S} \in \mathrm{Lat}\,(M_z)$ *and* $M_z|_{\mathcal{S}}$ *is a* μ*-hypercontraction,*

(iii) there exist a Hilbert space $\mathcal{D}$ *and a bounded analytic inner multiplier* $\theta \in \mathcal{M}(H_{K^{(\mu)}}(\mathcal{D}), H_{K^{(\nu)}}(\mathcal{E}))$ *such that* $\mathrm{Im}(M_\theta) = \mathcal{S}$*.*

Proof. (i) $\implies$ (ii): By Proposition 3.24, $M_z \in B(H_{K^{(\nu)}}(\mathcal{E}))^d$ is a $K^{(1)}$-contraction, and hence, Remark 4.7 implies that $M_z|_{\mathcal{S}}$ is also a $K^{(1)}$-contraction. The result follows now from Theorem 3.51.

(ii) $\implies$ (i): This is clear.

(ii) $\iff$ (iii): We only have to show that the statement (ii) implies condition (i) of Theorem 4.6. By Lemma 3.48, we see that $K^{(\mu)}$-pureness coincides with the membership in $C_{\cdot 0}$. Hence if we suppose that (ii) holds, by Proposition 2.12 and Remark 4.7, $M_z|_{\mathcal{S}}$ is $K^{(\mu)}$-pure. $\square$

The second condition of (i) in the last theorem is not always fulfilled for $\nu > 1$, as the following example (cf. [12, Example 3.3.3 (c)] and [52, Bemerkung 3.2.4]) shows.

Let $\mathcal{E}$ be a Hilbert space and $\nu > 1$. Consider for $M_z \in B\left(H_{K^{(\nu)}}(\mathcal{E})\right)^d$ the space

$$\mathcal{S} = \{f \in H_{K^{(\nu)}}(\mathcal{E}) \ ; \ f(0) = 0\} \in \mathrm{Lat}\,(M_z)\,.$$

For $1 \le \mu \le \nu$ and $\eta \in \mathcal{E}$ with $\|\eta\| = 1$, an easy calculation shows that

$$\begin{aligned}\left\langle \frac{1}{K^{(\mu)}}(M_z|_{\mathcal{S}}, M_z|_{\mathcal{S}}^*)\eta z_1^2, \eta z_1^2 \right\rangle &= \left(c_0^{(\mu)} + c_1^{(\mu)} \frac{\gamma^{(\nu)}_{(1,0,\ldots,0)}}{\gamma^{(\nu)}_{(2,0,\ldots,0)}} \right) \langle \eta z_1^2, \eta z_1^2 \rangle \\ &= \frac{2}{\nu(\nu+1)^2}(\nu + 1 - 2\mu).\end{aligned}$$

Hence, at least for $\mu > (\nu+1)/2$, $M_z|_{\mathcal{S}}$ is a not a $K^{(\mu)}$-contraction.

For $\nu = \mu > d$ and $\mathcal{E} = \mathbb{C}$, the only closed invariant subspaces fulfilling this additional property are the trivial ones, as the following proposition shows. The case $\nu = d+1$ has first been proven by Guo in [43, Proposition 4.1], and the case $\nu \geq d+1$ with $\nu \in \mathbb{N}$ originates from [52, Satz 3.2.5].

4.9 Proposition. *Let* $\nu > d$*,* $M_z \in B(H_{K^{(\nu)}})^d$*, and* $\mathcal{S} \in \mathrm{Lat}(M_z)$*. Then* $M_z|_{\mathcal{S}}$ *is a* $K^{(\nu)}$*-contraction if and only if* $\mathcal{S}$ *is a trivial closed invariant subspace, i.e.,* $\mathcal{S} = \{0\}$ *or* $\mathcal{S} = H_{K^{(\nu)}}$*.*

Proof. The if-part is clear.

For the only if-part, suppose that $\{0\} \neq \mathcal{S} \in \operatorname{Lat}(M_z)$ such that $M_z|_{\mathcal{S}}$ is a $K^{(\nu)}$-contraction. By [12, Example 3.3.3], the space

$$\mathcal{T} = \mathcal{S} \ominus \sum_{i=1}^{d} M_{z_i}\mathcal{S} \neq \{0\}$$

consists of eigenvectors of $1/K^{(\nu)}(M_z|_{\mathcal{S}}, M_z|_{\mathcal{S}}^*)$ to the eigenvalue 1. An easy calculation shows that

$$\left\langle \frac{1}{K^{(\nu)}}(M_z|_{\mathcal{S}}, M_z|_{\mathcal{S}}^*)K^{(\nu)}(\cdot, w), K^{(\nu)}(\cdot, w) \right\rangle = \left\| P_{\mathcal{S}} \frac{K^{(\nu)}(\cdot, w)}{\|K^{(\nu)}(\cdot, w)\|} \right\|^2$$

for all $w \in \mathbb{B}_d$. For $g \in \mathcal{T}$ with $\|g\| = 1$, we have

$$\begin{aligned} |g(w)|^2 &= \left\langle (g \otimes g)K^{(\nu)}(\cdot, w), K^{(\nu)}(\cdot, w) \right\rangle \\ &\leq \left\langle \frac{1}{K^{(\nu)}}(M_z|_{\mathcal{S}}, M_z|_{\mathcal{S}}^*)K^{(\nu)}(\cdot, w), K^{(\nu)}(\cdot, w) \right\rangle \\ &= \left\| P_{\mathcal{S}} \frac{K^{(\nu)}(\cdot, w)}{\|K^{(\nu)}(\cdot, w)\|} \right\|^2 \end{aligned}$$

for all $w \in \mathbb{B}_d$, where

$$g \otimes g \colon H_{K^{(\nu)}} \to H_{K^{(\nu)}}, \; f \mapsto \langle f, g \rangle \, g,$$

and hence,

$$1 = \langle g, g \rangle = \int_{\mathbb{B}_d} |g(w)|^2 \, \mathrm{d}v_\nu(w) \leq \int_{\mathbb{B}_d} \left\| P_{\mathcal{S}} \frac{K^{(\nu)}(\cdot, w)}{\|K^{(\nu)}(\cdot, w)\|} \right\|^2 \, \mathrm{d}v_\nu(w) \leq 1.$$

Thus,

$$\left\| P_{\mathcal{S}} \frac{K^{(\nu)}(\cdot, w)}{\|K^{(\nu)}(\cdot, w)\|} \right\| = 1$$

for v_ν-almost all $w \in \mathbb{B}_d$ which implies that

$$K^{(\nu)}(\cdot, w) \in \mathcal{S}$$

for v_ν-almost all $w \in \mathbb{B}_d$. If we can show that

$$\left(\bigvee \left\{ K^{(\nu)}(\cdot, w) \; ; \; w \in \mathbb{B}_d \setminus N \right\} \right)^{\perp} = \{0\}$$

for all v_ν-null sets N, the proof is complete.

To this end, let N be we a v_ν-null set and observe that

$$\left(\bigvee \left\{K^{(\nu)}(\cdot, w) \;;\; w \in \mathbb{B}_d \setminus N\right\}\right)^{\perp} \subset \left\{K^{(\nu)}(\cdot, w) \;;\; w \in \mathbb{B}_d \setminus N\right\}^{\perp}.$$

Let $f \in \left\{K^{(\nu)}(\cdot, w) \;;\; w \in \mathbb{B}_d \setminus N\right\}^{\perp}$. Then

$$0 = \langle f, K(\cdot, w)\rangle = f(w)$$

for all $w \in \mathbb{B}_d \setminus N$. Since N has no inner points and f is continuous, we obtain that $f = 0$, i.e.,

$$\left\{K^{(\nu)}(\cdot, w) \;;\; w \in \mathbb{B}_d \setminus N\right\}^{\perp} = \{0\}.$$

Hence, $\mathcal{S} = H_{K^{(\nu)}}$. □

4.10 Remark. If $\nu = d$, then, by [12, Proposition 5.1.3], the multipliers θ in Theorem 4.8 (iii) such that $\{0\} \neq \operatorname{Im}(M_\theta) \neq H_{K^{(\nu)}}(\mathcal{E})$ coincide with the non-constant inner functions (for a definition of inner functions, see Section 6.1).

By Remark 4.7 (iii), it is clear that, for $\nu = 1$ and a Hilbert space $\mathcal{E}$, $M_z \in B(H_{K^{(1)}}(\mathcal{E}))^d$ restricted to a closed invariant subspace is always a $K^{(1)}$-contraction. The next two results will help us to obtain a stronger corollary of Theorem 4.6 in the case $\nu = 1$, where the assumptions are slightly weaker.

4.11 Lemma. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})^d$ be a row contraction. Then we have*

$$\left\| \sum_{i=1}^{d} \lambda_i T_i \right\| \leq |\lambda|$$

for all $\lambda \in \mathbb{C}^d$.

Proof. Let $\lambda \in \mathbb{C}^d$. Since $T \in B(\mathcal{H})^d$ is a row contraction if and only if

$$\varphi_T \colon \mathcal{H}^d \to \mathcal{H},\ (x_i)_{i=1}^d \mapsto \sum_{i=1}^{d} T_i x_i$$

is a contraction, we have with

$$\Lambda = \operatorname{diag}(\lambda_1 \operatorname{id}_{\mathcal{H}}, \dots, \lambda_d \operatorname{id}_{\mathcal{H}}) \in B(\mathcal{H}^d)$$

that

$$\left\| \sum_{i=1}^{d} \lambda_i T_i x_i \right\| = \|\varphi_T(\Lambda x)\| \leq \|\Lambda x\| \leq \|\Lambda\| \, \|x\| \leq |\lambda| \, \|x\|$$

for all $x = (x_i)_{i=1}^d \in \mathcal{H}^d$. □

4.12 Lemma. *Let $\mathcal{E}$ be a Hilbert space and let $H(\mathcal{E}) \subset \mathcal{E}^\Omega$ be a reproducing kernel Hilbert space over a set $\Omega \subset \mathbb{C}^d$ with reproducing kernel K such that*

(i) $M_z \in B(H(\mathcal{E}))^d$ *is a row contraction and*

(ii) *the set* $\{K(\cdot,\lambda)x \,;\, \lambda \in \Omega \cap \mathbb{B}_d, x \in \mathcal{E}\}$ *is total.*

Then $M_z \in B(H(\mathcal{E}))^d$ *is* $K^{(1)}$*-pure.*

Proof. Let $\lambda \in \Omega \cap \mathbb{B}_d$ and $x \in \mathcal{E}$. Then, by Lemma 1.20,

$$\begin{aligned}\sigma_{M_z}^N(\mathrm{id}_{H(\mathcal{E})})K(\cdot,\lambda)x &= \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} M_z^\alpha M_z^{*\alpha} K(\cdot,\lambda)x \\ &= \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} \overline{\lambda}^\alpha M_z^\alpha K(\cdot,\lambda)x \\ &= \left(\sum_{i=1}^d \overline{\lambda_i} M_{z_i}\right)^N K(\cdot,\lambda)x \\ &\to 0\end{aligned}$$

as $N \to \infty$ by Lemma 4.11. Since $(\sigma_{M_z}^N(\mathrm{id}_{H(\mathcal{E})}))_{N\in\mathbb{N}}$ is a monotone decreasing sequence of positive operators and hence bounded (cf. Lemma 2.3), we obtain that $\Sigma(M_z) = 0$ by condition (ii) and Remark 2.5. □

With these preparations, we obtain [61, Theorem 4.4].

4.13 Theorem (Sarkar)**.** *Let $\mathcal{E}$ be a Hilbert space and let $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_d, \mathcal{E})$ be a reproducing kernel Hilbert space of analytic functions such that $M_z \in B(H(\mathcal{E}))^d$ is a row contraction as well as $\mathcal{S} \subset H(\mathcal{E})$. Then the following statements are equivalent:*

(i) $\mathcal{S} \in \mathrm{Lat}\,(M_z)$,

(ii) *there exist a Hilbert space $\mathcal{D}$ and a bounded analytic inner multiplier $\theta \in \mathcal{M}(H_{K^{(1)}}(\mathcal{D}), H(\mathcal{E}))$ such that* $\mathrm{Im}(M_\theta) = \mathcal{S}$.

5. Minimal coextensions

In Remark 2.24 we have seen that a strong K-contraction $T \in B(\mathcal{H})^d$ on a Hilbert space $\mathcal{H}$ has a canonical coextension of the type $(M_z \otimes \mathrm{id}_{\mathcal{D}_T}) \oplus W$, where $\mathcal{D}_T$ is the defect space and $W \in B(\mathcal{L})^d$ is a spherical unitary on a Hilbert space $\mathcal{L}$. In this chapter, we want to answer the question under which conditions this coextension is in some sense unique (cf. Theorem 5.16 and Corollary 5.17). Here, the presentation is influenced by the works of Arveson [9] and Bhattacharjee et al [15]. This chapter is split in two section: the first one will be concerned with the so-called Toeplitz algebra, i.e., the C^* algebra generated by M_z, and the second one will contain the study of (minimal) coextensions and factorizations thereof. As in the preceding chapters, let $H_K \subset \mathcal{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space whose kernel $K \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is of the form $K(z,w) = k(\langle z,w \rangle)$ $(z,w \in \mathbb{B}_d)$ with a zero-free analytic function $k \colon \mathbb{D} \to \mathbb{C},\ z \mapsto \sum_{n=0}^\infty a_n z^n$ such that $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$.

5.1. The Toeplitz algebra

The main goal of this section is to obtain an explicit representation of the C^* algebra generated by the K-shift. For this purpose, we have to suppose that the following holds.

5.1 Assumption. Let $M_z \in B(H_K)^d$ be essentially normal and

$$P_{\mathbb{C}} \in \bigvee \left\{ M_z^\alpha M_z^{*\beta}\ ;\ \alpha, \beta \in \mathbb{N}^d \right\}.$$

5.2 Example. Let $\nu \geq 1$. Then, by Remark 1.36 and Section 1.5.2, $M_z \in B\left(H_{K^{(\nu)}}\right)^d$ is essentially normal, and, by Theorem 3.51 and Proposition 3.15, we have that

$$P_{\mathbb{C}} = \frac{1}{K^{(\nu)}}(M_z, M_z^*) \in \bigvee \left\{ M_z^\alpha M_z^{*\beta}\ ;\ \alpha, \beta \in \mathbb{N}^d \right\}.$$

For $u, v \in \mathcal{H}$, we use the notation

$$u \otimes v \colon \mathcal{H} \to \mathcal{H},\ h \mapsto \langle h, v \rangle\, u$$

and observe that, since

$$\|u \otimes v\| \leq \|u\| \, \|v\|$$

for all $u, v \in \mathcal{H}$, the map

$$\Phi \colon \mathcal{H} \times \mathcal{H} \to B(\mathcal{H}), \; (u, v) \mapsto u \otimes v$$

is continuous.

5.3 Lemma. *Let $A \in B(\mathcal{H})$ be a rank-one operator. Then there exists $h \in \mathcal{H}$ such that*

$$A = h \otimes A^* h.$$

Proof. Since the image of A is one-dimensional, there exist $h \in \mathcal{H}$ with $\|h\| = 1$ such that, for all $g \in \mathcal{H}$, there exists $\lambda_g \in \mathbb{C}$ with

$$Ag = \lambda_g h.$$

Then

$$Ag = \lambda_g h = \lambda_g \|h\|^2 h = \langle \lambda_g h, h \rangle h = \langle Ag, h \rangle h = \langle g, A^* h \rangle h = (h \otimes A^* h)(g)$$

for all $g \in \mathcal{H}$. □

5.4 Lemma. *The inclusion*

$$K(H_K) \subset \bigvee \left\{ M_z^\alpha M_z^{*\beta} \; ; \; \alpha, \beta \in \mathbb{N}^d \right\}$$

holds.

Proof. Define $M = \bigvee \left\{ M_z^\alpha M_z^{*\beta} \; ; \; \alpha, \beta \in \mathbb{N}^d \right\}$. It is enough to show that all rank one operators on H_K belong to M.

To this end, let $A \in B(H_K)$ be a rank-one operator. By Lemma 5.3, there exist $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha, g = \sum_{\alpha \in \mathbb{N}^d} g_\alpha z^\alpha \in H_K$ such that

$$A = f \otimes g.$$

We set

$$A_n = \sum_{|\alpha|, |\beta| \leq n} f_\alpha \overline{g_\beta} M_z^\alpha P_{\mathbb{C}} M_z^{*\beta} \in B(H_K)$$

for all $n \in \mathbb{N}$. Since $P_{\mathbb{C}} \in M$, we see that $A_n \in M$ for all $n \in \mathbb{N}$. For $\alpha, \beta \in \mathbb{N}^d$ and $z, w \in \mathbb{B}_d$, we have

$$(M_z^\alpha P_{\mathbb{C}} M_z^{*\beta} K(\cdot, w))(z) = (M_z^\alpha P_{\mathbb{C}} \overline{w}^\beta K(\cdot, w))(z) = (\overline{w}^\beta M_z^\alpha 1)(z) = \overline{w}^\beta z^\alpha$$

and hence,

$$\begin{aligned}
\langle A_n K(\cdot, w), K(\cdot, z)\rangle = (A_n K(\cdot, w))(z) &= \sum_{|\alpha|,|\beta|\le n} f_\alpha z^\alpha \overline{g_\beta w^\beta} \\
&= \left(\sum_{|\alpha|\le n} f_\alpha z^\alpha\right) \overline{\left(\sum_{|\beta|\le n} g_\beta w^\beta\right)} \\
&= \left\langle K(\cdot, w), \sum_{|\beta|\le n} g_\beta z^\beta \right\rangle \left\langle \sum_{|\alpha|\le n} f_\alpha z^\alpha, K(\cdot, z)\right\rangle \\
&= \left\langle \left\langle K(\cdot, w), \sum_{|\beta|\le n} g_\beta z^\beta \right\rangle \sum_{|\alpha|\le n} f_\alpha z^\alpha, K(\cdot, z)\right\rangle \\
&= \left\langle \left(\sum_{|\alpha|\le n} f_\alpha z^\alpha \otimes \sum_{|\beta|\le n} g_\beta z^\beta\right) K(\cdot, w), K(\cdot, z)\right\rangle
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{K(\cdot, w)\ ;\ w \in \mathbb{B}_d\} \subset H_K$ is a total subset by Proposition 1.9, we conclude that

$$A_n = \sum_{|\alpha|\le n} f_\alpha z^\alpha \otimes \sum_{|\beta|\le n} g_\beta z^\beta$$

for all $n \in \mathbb{N}$. Since the map Φ from the beginning of this section is continuous, we conclude that

$$A_n = \sum_{|\alpha|\le n} f_\alpha z^\alpha \otimes \sum_{|\beta|\le n} g_\beta z^\beta \to \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \otimes \sum_{\beta\in\mathbb{N}^d} g_\beta z^\beta = f \otimes g = A$$

in $\tau_{\|\cdot\|}$ as $n \to \infty$. □

5.5 Theorem. *The identity*

$$C^*(M_z) = \bigvee \left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha, \beta \in \mathbb{N}^d\right\}$$

holds.

Proof. The inclusion

$$C^*(M_z) \supset \bigvee \left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha, \beta \in \mathbb{N}^d\right\}$$

is clear.

Since $\bigvee\left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha,\beta\in\mathbb{N}^d\right\}$ is a *-closed subspace, it is enough to show that $\bigvee\left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha,\beta\in\mathbb{N}^d\right\}$ is closed under multiplication. To this end, we observe that, by Lemma 1.1, there exists a compact operator K such that

$$M_z^\alpha M_z^{*\beta} M_z^\gamma M_z^{*\delta} = M_z^{\alpha+\gamma} M_z^{*\beta+\delta} + K \in \bigvee\left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha,\beta\in\mathbb{N}^d\right\}$$

for all $\alpha,\beta,\gamma,\delta\in\mathbb{N}^d$, since $M_z\in B(H_K)^d$ is essentially normal and $K(H_K)\subset \bigvee\left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha,\beta\in\mathbb{N}^d\right\}$ by Lemma 5.4. □

5.2. Factorizations of minimal coextensions

Let $\mathcal{H}$, $\mathcal{D}$, $\mathcal{K}$ be Hilbert spaces, $T\in B(\mathcal{H})^d$ be a commuting tuple, and let $U\in B(\mathcal{K})^d$ be a spherical unitary. In the following, we use the notation $M_z^{\mathcal{D}} = M_z\otimes \mathrm{id}_{\mathcal{D}}\in B(H_K\otimes\mathcal{D})^d$.

5.6 Definition. We call a pair $(M_z^{\mathcal{D}}\oplus U,\Pi)$, where $\Pi\colon \mathcal{H}\to(H_K\otimes\mathcal{D})\oplus\mathcal{K}$ is an isometry, a *coextension* of T if

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}}\oplus U_i)^*\Pi$$

for $i=1,\dots,d$.

We shall write the isometry Π in the form $\Pi=(\Pi_s,\Pi_u)\colon \mathcal{H}\to(H_K\otimes\mathcal{D})\oplus\mathcal{K}$.

Let T be a strong K-contraction, and $\Pi=\Psi_T$ from Remark 2.24. Then $(M_z^{\mathcal{D}_T}\oplus W,\Pi)$ is a coextension of T with

$$\Pi_s=\psi_T \quad\text{and}\quad \Pi_u=\Sigma(T)^{1/2}.$$

Furthermore, the identities

$$\mathcal{L}=\bigvee\left\{W^\alpha\Pi_u h\ ;\ \alpha\in\mathbb{N}^d \text{ and } h\in\mathcal{H}\right\} \quad\text{and}\quad D_T=P_{\mathcal{D}_T}\Pi_s$$

hold.

5.7 Remark. Suppose that the K-shift $M_z\in B(H_K)^d$ is essentially normal. The following statements are equivalent.

(i) $P_{\mathbb{C}}\in\bigvee\left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha,\beta\in\mathbb{N}^d\right\}$.

(ii) $P_{\mathbb{C}}\in C^*(M_z)$ and $C^*(M_z)=\bigvee\left\{M_z^\alpha M_z^{*\beta}\ ;\ \alpha,\beta\in\mathbb{N}^d\right\}$.

Proof. This follows from Section 5.1. □

Since Theorem 2.30 will play a crucial role in our factorization theorem, we make the following assumption.

5.8 Assumption. From now on, we suppose that the K-shift $M_z \in B(H_K)^d$ is essentially normal, that $\sum_{n=0}^{\infty} c_n$ converges, and that

$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \in \bigvee \left\{ M_z^{\alpha} M_z^{*\beta} \ ; \ \alpha, \beta \in \mathbb{N}^d \right\}.$$

Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. By Theorem 2.30 and its proof, there are a coextension $(M_z^{\mathcal{D}} \oplus U, \Pi)$ of T and a unital C^*-homomorphism $\pi_u \colon C^*(M_z) \to B(\mathcal{K})$ with $\pi_u(M_{z_i}) = U_i$ for $i = 1, \ldots, d$ and $\pi_u|_{K(H_K)} = 0$. By setting

$$\pi_s \colon C^*(M_z) \to B(H_K(\mathcal{D})), \ X \mapsto X \otimes \mathrm{id}_{\mathcal{D}},$$

we complete π_u to a unital C^*-homomorphism

$$\pi = (\pi_s, \pi_u) \colon C^*(M_z) \to B(H_K(\mathcal{D}) \oplus \mathcal{K}).$$

Define

$$\begin{aligned}
\mathcal{H}_{\pi_s} &= \bigvee \{ \pi_s(X)\Pi_s h \ ; \ X \in C^*(M_z), h \in \mathcal{H} \} \in \mathrm{Red}(M_z^{\mathcal{D}}) \subset H_K(\mathcal{D}), \\
\mathcal{H}_{\pi_u} &= \bigvee \{ \pi_u(X)\Pi_u h \ ; \ X \in C^*(M_z), h \in \mathcal{H} \} \in \mathrm{Red}(U) \subset \mathcal{K}, \\
\mathcal{H}_{\pi} &= \bigvee \{ \pi(X)\Pi h \ ; \ X \in C^*(M_z), h \in \mathcal{H} \} \in \mathrm{Red}(M_z^{\mathcal{D}} \oplus U) \subset \mathcal{H}_{\pi_s} \oplus \mathcal{H}_{\pi_u}.
\end{aligned}$$

Since $C^*(M_z) = \bigvee \left\{ M_z^{\alpha} M_z^{*\beta} \ ; \ \alpha, \beta \in \mathbb{N}^d \right\}$ and $M_z^{\mathcal{D}*\alpha}\Pi_s = \Pi_s T^{*\alpha}$ for all $\alpha \in \mathbb{N}^d$, we obtain that

$$\mathcal{H}_{\pi_s} = \bigvee \left\{ M_z^{\mathcal{D}\alpha}\Pi_s h \ ; \ \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\}.$$

The following lemma is an adaption of [52, Lemma 4.1.6].

5.9 Lemma. *If $M \subset H_K(\mathcal{D})$ is a reducing subspace for $M_z^{\mathcal{D}}$, then*

$$M = \bigvee \left\{ z^{\alpha}(M \cap \mathcal{D}) \ ; \ \alpha \in \mathbb{N}^d \right\} = \bigvee \left\{ z^{\alpha} P_{\mathcal{D}} M \ ; \ \alpha \in \mathbb{N}^d \right\}.$$

Proof. Since M is reducing for $M_z^{\mathcal{D}}$, M is invariant under $P_{\mathcal{D}}$. Hence, $M \cap \mathcal{D} = P_{\mathcal{D}} M$ and the second equality holds.

The inclusion $M \supset \bigvee \left\{ z^{\alpha}(M \cap \mathcal{D}) \ ; \ \alpha \in \mathbb{N}^d \right\}$ is clear.

To establish $M \subset \bigvee \left\{ z^{\alpha}(M \cap \mathcal{D}) \ ; \ \alpha \in \mathbb{N}^d \right\}$, we conclude with Lemma 1.29 that

$$P_{\mathcal{D}} M_z^{\mathcal{D}*\beta} f = \frac{1}{\gamma_{\beta}} f_{\beta}$$

for all $\beta \in \mathbb{N}^d$ and $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in H_K(\mathcal{D})$. Let $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in M$. Since $M_z^{\mathcal{D}*\beta} f \in M$ for all $\beta \in \mathbb{N}^d$, we obtain that $f_\beta \in M \cap \mathcal{D}$ for all $\beta \in \mathbb{N}^d$ and hence,

$$f \in \bigvee \left\{ z^\alpha (M \cap \mathcal{D}) \; ; \; \alpha \in \mathbb{N}^d \right\}. \qquad \square$$

By Lemma 5.9, we have that

$$\begin{aligned} \mathcal{H}_{\pi_s} &= \bigvee \left\{ z^\alpha P_{\mathcal{D}} \mathcal{H}_{\pi_s} \; ; \; \alpha \in \mathbb{N}^d \right\} \\ &= \bigvee \left\{ z^\alpha P_{\mathcal{D}} M_z^{\mathcal{D}\beta} \Pi_s h \; ; \; \alpha, \beta \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\}. \end{aligned}$$

Since

$$\pi(M_z^\alpha P_{\mathbb{C}} M_z^\beta) \Pi h = (M_z^{\mathcal{D}\alpha} P_{\mathcal{D}} M_z^{\mathcal{D}\beta} \Pi_s h) \oplus 0$$

for all $\alpha, \beta \in \mathbb{N}^d$ and $h \in \mathcal{H}$, we find that

$$\mathcal{H}_{\pi_s} \oplus \{0\} \subset \mathcal{H}_\pi.$$

Since

$$0 \oplus (\pi_u(X) \Pi_u h) = \pi(X) \Pi h - (\pi_s(X) \Pi_s h) \oplus 0 \in \mathcal{H}_\pi$$

for $X \in C^*(M_z)$ and $h \in \mathcal{H}$, it follows that

$$\mathcal{H}_\pi = \mathcal{H}_{\pi_s} \oplus \mathcal{H}_{\pi_u}.$$

Furthermore, the smallest reducing subspace for $M_z^{\mathcal{D}} \oplus U \in B(H_K(\mathcal{D}) \oplus \mathcal{K})^d$ containing $\Pi \mathcal{H}$ is $\mathcal{H}_\pi$.

5.10 Definition. We call the coextension $(M_z^{\mathcal{D}} \oplus U, \Pi)$ of T *minimal* if the only reducing subspace for $M_z^{\mathcal{D}} \oplus U$ which contains $\Pi \mathcal{H}$ is $H_K(\mathcal{D}) \oplus \mathcal{K}$.

5.11 Proposition. *With the notations from above, the following assertions are equivalent:*

(i) Π *is minimal,*

(ii) $\mathcal{H}_\pi = H_K(\mathcal{D}) \oplus \mathcal{K}$,

(iii) $\mathcal{H}_{\pi_s} = H_K(\mathcal{D})$ *and* $\mathcal{H}_{\pi_u} = \mathcal{K}$.

5.12 Proposition. *Let* $(M_z^{\mathcal{D}_T} \oplus W, \Psi_T)$ *with* $\Psi_T \colon \mathcal{H} \to (H_K(\mathcal{D}_T)) \oplus \mathcal{L}$ *be a coextension of* T *as in Remark 2.24 such that*

$$\mathcal{L} = \bigvee \left\{ W^\alpha \Sigma(T)^{1/2} h \; ; \; \alpha \in \mathbb{N}^d, h \in \mathcal{H} \right\}.$$

Then $(M_z^{\mathcal{D}_T} \oplus W, \Psi_T)$ *is minimal.*

Proof. Since

$$\mathcal{L} = \bigvee \left\{ W^{\alpha}\Pi_u h \; ; \; \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\} = \mathcal{H}_{\pi_u}$$

we only have to show that $\mathcal{H}_{\pi_s} = H_K(\mathcal{D}_T)$.

To this end, we observe that

$$D_T h = P_{\mathcal{D}_T}\Pi_s h \in \mathcal{H}_{\pi_s} \cap \mathcal{D}_T$$

for all $h \in \mathcal{H}$ and hence,

$$\mathcal{D}_T = \overline{D_T\mathcal{H}} = \mathcal{H}_{\pi_s} \cap \mathcal{D}_T,$$

i.e.,

$$\mathcal{H}_{\pi_s} = H_K(\mathcal{H}_{\pi_s} \cap \mathcal{D}_T) = H_K(\mathcal{D}_T). \qquad \square$$

Let $\mathcal{B}$ be a unital C^*-algebra and $\varphi\colon \mathcal{B} \to B(\mathcal{H})$ be completely positive. Furthermore, for $i = 1, 2$, let $(\pi_i, \Pi_i, \mathcal{L}_i)$ be a minimal Stinespring representation for φ. Then, by [57, Proposition 4.2], there exists a unitary operator $V\colon \mathcal{L}_1 \to \mathcal{L}_2$ such that $V\Pi_1 = \Pi_2$ and $V\pi_1 = \pi_2 V$.

5.13 Definition. Let $\mathcal{B}$ be a unital C^*-algebra with unit $1_{\mathcal{B}}$ and let $\mathcal{A} \subset \mathcal{B}$ be a (not necessarily closed) subalgebra. We call a completely positive map

$$\varphi\colon \mathcal{B} \to B(\mathcal{H})$$

an $\mathcal{A}$-*morphism* if

(i) $\varphi(1_{\mathcal{B}}) = \mathrm{id}_{\mathcal{H}}$,

(ii) $\varphi(AX) = \varphi(A)\varphi(X)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $(M_z^{\mathcal{D}} \oplus U, \Pi)$ be a coextension of T and $\pi\colon C^*(M_z) \to B(H_K(\mathcal{D}) \oplus \mathcal{K})$ a unital C^*-homomorphism as constructed in the section following Assumption 5.8. Define $\mathcal{A} = \{p(M_z) \; ; \; p \in \mathbb{C}[z]\} \subset B(H_K)$. Then the map

$$\varphi\colon C^*(M_z) \to B(\mathcal{H}), \; X \mapsto \Pi^*\pi(X)\Pi$$

is an $\mathcal{A}$-morphism and the triple $(\pi, \Pi, H_K(\mathcal{D}) \oplus \mathcal{K})$ is a Stinespring representation for the completely positive map φ.

5.14 Remark. With the notations from above, the following assertions are equivalent:

(i) Π is minimal,

(ii) $\mathcal{H}_\pi = H_K(\mathcal{D}) \oplus \mathcal{K}$,

(iii) $\mathcal{H}_{\pi_s} = H_K(\mathcal{D})$ and $\mathcal{H}_{\pi_u} = \mathcal{K}$,

(iv) $(\pi, \Pi, H_K(\mathcal{D}) \oplus \mathcal{K})$ is the minimal Stinespring representation of φ.

A proof of the following lemma can be found in [9, Lemma 8.6].

5.15 Lemma. *Let $\mathcal{B}$ be a unital C^*-algebra and $\mathcal{A}$ a subalgebra of $\mathcal{B}$ such that $\mathcal{B} = \bigvee \mathcal{A}\mathcal{A}^*$. For $i = 1, 2$, let $\mathcal{H}_i$ be a Hilbert space, $\varphi_i \colon \mathcal{B} \to B(\mathcal{H}_i)$ an $\mathcal{A}$-morphism, and $V \colon \mathcal{H}_1 \to \mathcal{H}_2$ a unitary operator such that*

$$V\varphi_1(a) = \varphi_2(a)V$$

for all $a \in \mathcal{A}$. Furthermore, for $i = 1, 2$, let $(\pi_i, \Pi_i, \mathcal{L}_i)$ be the minimal Stinespring representation of φ_i. Then there exists a unique unitary operator $\tilde{V} \colon \mathcal{L}_1 \to \mathcal{L}_2$ such that

(i) $\tilde{V}\pi_1(x) = \pi_2(x)\tilde{V}$ for all $x \in \mathcal{B}$,

(ii) $\tilde{V}\Pi_1 = \Pi_2 V$.

If $\mathcal{A} \subset C^*(M_z)$ is the unital subalgebra consisting of all polynomials in $M_{z_1}, \ldots, M_{z_d}$, then $\mathcal{B} = C^*(M_z) = \bigvee \mathcal{A}\mathcal{A}^*$ by hypothesis.

We are now able to prove an analogue of [15, Theorem 3.1] in our setting.

5.16 Theorem. *For $i = 1, 2$, let $(M_z^{\mathcal{D}_i} \oplus U_i, \Pi_i)$ with $\Pi_i \colon \mathcal{H} \to H_K(\mathcal{D}_i) \oplus \mathcal{K}_i$ be a minimal coextension of T. Then there exist unitary operators $V_s \in B(\mathcal{D}_1, \mathcal{D}_2)$ and $V_u \in B(\mathcal{K}_1, \mathcal{K}_2)$ such that the diagram*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Pi_2"] \arrow[d, "\Pi_1"'] & H_K(\mathcal{D}_2) \oplus \mathcal{K}_2 \\ H_K(\mathcal{D}_1) \oplus \mathcal{K}_1 \arrow[ur, "(\mathrm{id}_{H_K} \otimes V_s) \oplus V_u"'] & \end{tikzcd}$$

commutes.

Proof. For $i = 1, 2$, let $\pi_i \colon C^*(M_z) \to B(H_K(\mathcal{D}_i) \oplus \mathcal{K}_i)$ be a unital C^*-algebra homomorphism as constructed in the section following Assumption 5.8, and denote by $\varphi_i \colon C^*(M_z) \to B(\mathcal{H}),\ X \mapsto \Pi_i^* \pi_i(X) \Pi_i$ the induced $\mathcal{A}$-morphism, where $\mathcal{A} \subset C^*(M_z)$ is the unital subalgebra consisting of all polynomials in $(M_{z_1}, \ldots, M_{z_d})$. Since

$$\varphi_1(p(M_z)) = p(T) = \varphi_2(p(M_z))$$

for every polynomial $p \in \mathbb{C}[z]$, i.e., $\varphi_1 = \varphi_2$ on $\mathcal{A}$, by Lemma 5.15 (with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $V = \mathrm{id}_{\mathcal{H}}$), there exists a unitary operator $\tilde{V} \colon H_K(\mathcal{D}_1) \oplus \mathcal{K}_1 \to H_K(\mathcal{D}_2) \oplus \mathcal{K}_2$ such that

$$\tilde{V}\pi_1(X) = \pi_2(X)\tilde{V}$$

for all $X \in \mathbb{C}^*(M_z)$ and

$$\tilde{V}\Pi_1 = \Pi_2$$

We first show that $\tilde{V}(H_K(\mathcal{D}_1) \oplus \{0\}) \subset H_K(\mathcal{D}_2) \oplus \{0\}$ and $\tilde{V}(\{0\} \oplus \mathcal{K}_1) \subset \{0\} \oplus \mathcal{K}_2$.

Since π_1 and π_2 are minimal, we have that

$$H_K(\mathcal{D}_i) = \bigvee \left\{ M_z^{\mathcal{D}\alpha} P_{\mathcal{D}} M_z^{\mathcal{D}\beta} \Pi_{is} h \; ; \; \alpha, \beta \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\}$$

and

$$\pi_i(M_z^\alpha P_{\mathbb{C}} M_z^\beta)\Pi_i h = (M_z^{\mathcal{D}_i\alpha} P_{\mathcal{D}_i} M_z^{\mathcal{D}_i\beta} \Pi_{is} h) \oplus 0$$

for $i = 1, 2$ and all $\alpha, \beta \in \mathbb{N}^d$ and $h \in \mathcal{H}$. Hence,

$$\begin{aligned} \tilde{V}\left((M_z^{\mathcal{D}_1\alpha} P_{\mathcal{D}_1} M_z^{\mathcal{D}_1\beta} \Pi_{1s} h) \oplus 0 \right) &= \tilde{V}\pi_1(M_z^\alpha P_{\mathbb{C}} M_z^\beta)\Pi_1 h \\ &= \pi_2(M_z^\alpha P_{\mathbb{C}} M_z^\beta)\Pi_2 h \\ &= (M_z^{\mathcal{D}_2\alpha} P_{\mathcal{D}_2} M_z^{\mathcal{D}_2\beta} \Pi_{2s} h) \oplus 0 \in H_K(\mathcal{D}_2) \oplus \{0\} \end{aligned}$$

for all $\alpha, \beta \in \mathbb{N}^d$ and $h \in \mathcal{H}$, i.e., $\tilde{V}(H_K(\mathcal{D}_1) \oplus \{0\}) \subset H_K(\mathcal{D}_2) \oplus \{0\}$. Furthermore, we see that

$$\begin{aligned} \tilde{V}\left(0 \oplus \pi_{1u}(X)\Pi_{1u} h\right) &= \tilde{V}\left(\pi_1(X)\Pi_1 h - \pi_{1s}(X)\Pi_{1s} h \oplus 0\right) \\ &= \pi_2(X)\Pi_2 h - \pi_{2s}(X)\Pi_{2s} h \oplus 0 \\ &= 0 \oplus \pi_{2u}(X)\Pi_{2u} h \end{aligned}$$

for all $X \in C^*(M_z)$ and $h \in \mathcal{H}$. Thus, $\tilde{V}(\{0\} \oplus \mathcal{K}_1) \subset \{0\} \oplus \mathcal{K}_2$. Therefore, we write $\tilde{V} = \tilde{V}_s \oplus \tilde{V}_u$.

Since $\tilde{V}_s$ and $\tilde{V}_s^*$ both intertwine $M_z^{\mathcal{D}_1}$ with $M_z^{\mathcal{D}_2}$, Proposition 4.5 implies that there exist bounded analytic multipliers $\theta \in \mathcal{M}(H_K(\mathcal{D}_1), H_K(\mathcal{D}_2))$ and $\psi \in \mathcal{M}(H_K(\mathcal{D}_2), H_K(\mathcal{D}_1))$ such that $\tilde{V}_s = M_\theta$ and $\tilde{V}_s^* = M_\psi$. Since $\tilde{V}_s$ is unitary, we obtain that

$$M_{\theta\psi} = M_\theta M_\psi = \mathrm{id}_{H_K(\mathcal{D}_2)} \quad \text{and} \quad M_{\psi\theta} = M_\psi M_\theta = \mathrm{id}_{H_K(\mathcal{D}_1)}.$$

Hence, for every $z \in \mathbb{B}_d$, we have $\theta(z) = \psi(z)^{-1}$. By Lemma 1.20, we have that

$$\begin{aligned} K(z,w)\eta &= (K(\cdot,w)\eta)(z) \\ &= (M_\theta M_\theta^* K(\cdot,w)\eta)(z) \\ &= (M_\theta K(\cdot,w)\theta(w)^*\eta)(z) \\ &= \theta(z)K(z,w)\theta(w)^*\eta \\ &= \theta(z)\theta(w)^*K(z,w)\eta \end{aligned}$$

for all $z,w \in \mathbb{B}_d$ and $\eta \in \mathcal{D}_2$. Thus, $\theta(z)\theta(w)^* = 1$ for all $z,w \in \mathbb{B}_d$ which implies that, for all $z \in \mathbb{B}_d$, the operator $\theta(z)$ is a unitary operator, and we have that $\theta(z) = \theta(w)$ for all $z,w \in \mathbb{B}_d$. Finally, there exists a unitary operator $V_s \in B(\mathcal{D}_1, \mathcal{D}_2)$ such that $\theta(z) = V_s$ for all $z \in \mathbb{B}_d$. If we set $V_u = \tilde{V}_u$, we obtain

$$\Pi_2 = ((\mathrm{id}_{H_K} \otimes V_s) \oplus V_u)\Pi_1. \qquad \square$$

5.17 Corollary. *Let $(M_z^{\mathcal{D}} \oplus U, \Pi)$ be a coextension of T and recall the notation from Remark 2.24. Then there exist isometries $V_s \in B(\mathcal{D}_T, \mathcal{D})$ and $V_u \in B(\mathcal{L}, \mathcal{K})$ such that the diagram*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Pi"] \arrow[d, "\Psi_T"'] & H_K(\mathcal{D}) \oplus \mathcal{K} \\ H_K(\mathcal{D}_T) \oplus \mathcal{L} \arrow[ur, "(\mathrm{id}_{H_K} \otimes V_s) \oplus V_u"'] & \end{tikzcd}$$

commutes.

Proof. Since $\mathcal{H}_\pi$ is the smallest reducing subspace for $M_z^{\mathcal{D}} \oplus U$ containing $\Pi\mathcal{H}$, we see that

$$\tilde{\Pi}\colon \mathcal{H} \to \mathcal{H}_\pi = H_K(\mathcal{H}_{\pi_s} \cap \mathcal{D}) \oplus \mathcal{H}_{\pi_u},\ h \mapsto \Pi(h)$$

defines a minimal coextension of T. By Proposition 5.12 and Theorem 5.16, there exist unitary operators $\tilde{V}_s \in B(\mathcal{D}_T, \mathcal{H}_{\pi_s} \cap \mathcal{D})$ and $\tilde{V}_u \in B(\mathcal{L}, \mathcal{H}_{\pi_u})$ such that

$$\Pi = ((\mathrm{id}_{H_K} \otimes \tilde{V}_s) \oplus \tilde{V}_u)\Psi_T.$$

Denoting by $\iota_{\mathcal{D}}\colon \mathcal{H}_{\pi_s} \cap \mathcal{D} \to \mathcal{D}$ and $\iota_{\mathcal{K}}\colon \mathcal{H}_{\pi_u} \to \mathcal{K}$ the inclusion maps, the operators

$$V_s = \iota_{\mathcal{D}} \circ \tilde{V}_s \quad \text{and} \quad V_u = \iota_{\mathcal{K}} \circ \tilde{V}_u$$

are the required isometries. $\square$

One should note that the following Corollary (see [15, Corollary 3.3] for the Drury-Arveson space case) does not need the assumption that $\sum_{n=0}^{\infty} c_n$ exists.

5.18 Corollary. *Let $T \in B(\mathcal{H})^d$ be a K-pure commuting tuple and $(M_z^{\mathcal{D}}, \Pi)$ be a coextension of T. Then there exists an isometry $V \in B(\mathcal{D}_T, \mathcal{D})$ such that the diagram*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Pi"] \arrow[d, "\Psi_T"'] & H_K(\mathcal{D}) \\ H_K(\mathcal{D}_T) \arrow[ur, "\mathrm{id}_{H_K} \otimes V"'] & \end{tikzcd}$$

commutes.

Part II.

Perturbations of Toeplitz operators

6. Preliminaries

A Result of J. Xia from [64], answering a question of R. Douglas [32], shows that a given operator $X \in B(H^2(\mathbb{D}))$ on the Hardy space of the unit disc $\mathbb{D}$ is a compact perturbation of a Toeplitz operator if and only if $T_\theta^* X T_\theta - X$ is compact for every inner function θ. Whether the corresponding result holds true in higher dimensions on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ is still an open question. In this part we show that the corresponding characterization of Schatten-p-class perturbations of Toeplitz operators on $H^2(\mathbb{B}_d)$ holds true also in the multidimensional case. We work in a much more general setting which applies at the same time to Toeplitz operators on all smooth strictly pseudoconvex domains and all bounded symmetric domains $D \subset \mathbb{C}^d$. The results of this part have been published in a joint paper [30] with M. Didas and J. Eschmeier.

Throughout Part II, all Hilbert spaces are supposed to be complex and separable, and d is again a positive integer.

6.1. Schatten-classes

Let $\mathcal{H}$ be a Hilbert space. We first recall the definition of the Schatten-classes.

6.1 Definition. For $p \in [1, \infty)$ and a Hilbert space $\mathcal{K}$, we denote by

$$\mathcal{S}_p(\mathcal{H}, \mathcal{K}) = \Big\{ X \in B(\mathcal{H}, \mathcal{K}) \ ; \ \|X\|_p = \operatorname{tr}(|X|^p)^{1/p} < \infty \Big\}$$

the *Schatten-p-class*. Furthermore, we write $\mathcal{S}_0(\mathcal{H}, \mathcal{K})$ and $\mathcal{S}_\infty(\mathcal{H}, \mathcal{K})$ for the set of finite-rank and compact operators from $\mathcal{H}$ to $\mathcal{K}$ equipped with the operator norm, respectively. If $\mathcal{K} = \mathcal{H}$, then we shorten the notation to $\mathcal{S}_p(\mathcal{H}) = \mathcal{S}_p(\mathcal{H}, \mathcal{H})$ for $p \in \{0\} \cup [1, \infty]$.

For $1 \leq p < q < \infty$ and a Hilbert space $\mathcal{K}$, we have the chain of inclusions

$$\mathcal{S}_0(\mathcal{H}, \mathcal{K}) \subset \mathcal{S}_p(\mathcal{H}, \mathcal{K}) \subset \mathcal{S}_q(\mathcal{H}, \mathcal{K}) \subset \mathcal{S}_\infty(\mathcal{H}, \mathcal{K}).$$

The following lemma and corollary are technical results which will be helpful in the upcoming proposition.

6.2 Lemma. *Let $(a_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Then there exist $(b_n)_{n\in\mathbb{N}} \in c_0$ and $(c_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ with*

$$a_n = b_n c_n$$

for all $n \in \mathbb{N}$.

If in addition $a_n \geq 0$ for all $n \in \mathbb{N}$, then we can choose $c_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$.

Proof. Let $(a_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$.

Define $N_1 = 0$ and

$$N_m = \min\left\{ i \in \mathbb{N} \ ; \ i > N_{m-1} \text{ and } \sum_{j=i}^{\infty} |a_n| < \frac{1}{m^3} \right\}$$

for $m \geq 2$. Then $(N_m)_{m\geq 1}$ is a strictly increasing sequence in $\mathbb{N}$.

Let $n \in \mathbb{N}$. Then there exists exactly one $k_n \in \mathbb{N}^*$ with

$$N_{k_n} \leq n < N_{k_n+1}.$$

The sequence $(k_n)_{n\in\mathbb{N}}$ is increasing and unbounded.

Define

$$b_n = \frac{1}{k_n} \quad \text{and} \quad c_n = k_n a_n$$

for all $n \in \mathbb{N}$. Then $(b_n)_{n\in\mathbb{N}} \in c_0$. The estimates

$$\sum_{m=0}^{\infty} |c_m| = \sum_{m=1}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} |c_n| = \sum_{m=1}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} m\,|a_n| \leq \sum_{m=1}^{\infty} m \sum_{n=N_m}^{\infty} |a_n| < \sum_{m=1}^{\infty} \frac{1}{m^2}$$

show that $(c_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$.

The second statement follows immediately from the construction above. $\square$

6.3 Corollary. *Let $1 \leq p < \infty$ and $(a_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Then there exist $(b_n)_{n\in\mathbb{N}} \in c_0$ and $(c_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$ with*

$$a_n = b_n c_n$$

for all $n \in \mathbb{N}$.

Proof. Let $1 \leq p < \infty$ and $(a_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Then $(|a_n|^p)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and, by Lemma 6.2, there exist non-negative sequences $(\tilde{b}_n)_{n\in\mathbb{N}} \in c_0$ and $(\tilde{c}_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ with

$$|a_n|^p = \tilde{b}_n \tilde{c}_n$$

for all $n \in \mathbb{N}$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{T}$ such that

$$a_n = |a_n|\,\tau_n$$

for all $n \in \mathbb{N}$. Define

$$b_n = \tilde{b}_n^{1/p} \quad \text{and} \quad c_n = \tau_n \tilde{c}_n^{1/p}$$

for all $n \in \mathbb{N}$. We obtain $(b_n)_{n\in\mathbb{N}} \in c_0$, $(c_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and

$$a_n = |a_n|\,\tau_n = \tilde{b}_n^{1/p}\tau_n\tilde{c}_n^{1/p} = b_n c_n$$

for all $n \in \mathbb{N}$. □

6.4 Proposition. *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence in $B(\mathcal{H})$ with*

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} X_k = 0.$$

Then, for $p \in \{0\} \cup [1,\infty]$ and $S \in \mathcal{S}_p(\mathcal{H})$, we have

$$\tau_{\|\cdot\|_p}\text{-}\lim_{k\to\infty} X_k S = 0 = \tau_{\|\cdot\|_p}\text{-}\lim_{k\to\infty} S X_k^*.$$

Proof. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence in $B(\mathcal{H})$ with

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} X_k = 0.$$

We start with the case $p = 0$. Since every finite-rank operator is a linear combination of rank-one operators, we only have to show the claim for rank-one operators. To this end, let $S \in B(\mathcal{H})$ be a rank-one operator. By Lemma 5.3, there exist $u, v \in \mathcal{H}$ such that $S = u \otimes v$. Hence,

$$\|X_k S h\| = \|X_k \langle h, v\rangle u\| \le \|h\|\,\|v\|\,\|X_k u\| \to 0$$

as $k \to \infty$. Since S^* is also a rank-one operator and $\|S X_k^*\| = \|X_k S^*\|$ holds, the second equality holds.

Now let $p = \infty$ and let $S \in B(\mathcal{H})$ be compact. Then there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of finite-rank operators with $\tau_{\|\cdot\|}$-limit S. Let $\varepsilon > 0$. By the uniform boundedness principle, $\sup_{k\in\mathbb{N}}\|X_k\|$ is finite, and hence, there exists a natural number $N \in \mathbb{N}$ such that

$$\|S - S_N\| < \frac{\varepsilon}{\sup_{k\in\mathbb{N}}\|X_k\| + 1}.$$

Since the assertion holds for finite-rank operators, we have

$$\|X_k S\| \leq \|X_k\| \, \|S - S_N\| + \|X_k S_N\| < \varepsilon + \|X_k S_N\| \to \varepsilon$$

as $k \to \infty$. This proofs the first equation. The second one follows as before.

Finally, let $p \in [1, \infty)$ and let $S \in \mathcal{S}_p(\mathcal{H})$. By the polar decomposition, there exists a partial isometry $U \in B(\mathcal{H})$ such that

$$S = \tilde{S}U,$$

where $\tilde{S} = \sqrt{SS^*} \in \mathcal{S}_p(\mathcal{H})$. Since $\tilde{S} \in \mathcal{S}_p(\mathcal{H}) \subset \mathcal{S}_\infty(\mathcal{H})$ is normal, we can find an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ and a sequence $(a_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$ with

$$\tilde{S}e_n = a_n e_n$$

for all $n \in \mathbb{N}$. By Corollary 6.3, there exist sequences $(b_n)_{n\in\mathbb{N}} \in c_0$ and $(c_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$ such that

$$a_n = b_n c_n$$

for all $n \in \mathbb{N}$. Let $K, S' \in B(\mathcal{H})$ be the diagonal operators defined by

$$K e_n = b_n e_n \quad \text{and} \quad S' e_n = c_n e_n$$

for all $n \in \mathbb{N}$. Then $K \in \mathcal{S}_\infty(\mathcal{H})$, $S' \in \mathcal{S}_p(\mathcal{H})$, and

$$\tilde{S} = KS'.$$

Since the assertion holds for compact operators, we obtain that

$$\|X_k S\|_p = \left\| X_k \tilde{S} U \right\|_p \leq \left\| X_k \tilde{S} \right\|_p = \|X_k K S'\|_p \leq \|X_k K\| \, \|S'\|_p \to 0$$

as $k \to \infty$. Similar as before, we have that $S^* \in \mathcal{S}_p(\mathcal{H})$ and that $\|SX_k^*\|_p = \|X_k S^*\|_p$ holds. This ends the proof. □

For later references, we state the following lemma.

6.5 Lemma. *Let $(X_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ be sequences in $B(\mathcal{H})$ such that*

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} X_k = X \quad \textit{and} \quad \tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} Y_k = Y.$$

Then

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} X_k Y_k = XY.$$

Proof. The result follows from a standard application of the uniform boundedness principle. □

6.2. A-isometries and Toeplitz operators

In this section, we introduce the notion of so-called A-isometries which are a generalization of spherical isometries in the spirit of Athavale (cf. Lemma 1.6). We use [39, Section 1.1] as a guideline.

6.6 Definition. A complex Banach algebra $\mathcal{A}$ is called a *dual algebra* if there exists a complex Banach space $\mathcal{X}$ such that $\mathcal{A}$ is isometrically isomorphic to $\mathcal{X}'$ and the maps

$$\mathcal{A} \to \mathcal{A},\ x \mapsto ax \quad \text{and} \quad \mathcal{A} \to \mathcal{A},\ x \mapsto xa$$

are τ_{w^*}-continuous for all $a \in \mathcal{A}$.

6.7 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be dual algebras. We call $\rho\colon \mathcal{A} \to \mathcal{B}$ a *dual algebra homomorphism* if ρ is a Banach algebra homomorphism that is τ_{w^*}-continuous. The map ρ is called a *dual algebra isomorphism* if it is an isometric Banach algebra isomorphism that is a τ_{w^*}-homeomorphism.

6.8 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras. We call $\rho\colon \mathcal{A} \to \mathcal{B}$ a *von Neumann algebra homomorphism* if ρ is a *-preserving dual algebra homomorphism. The map ρ is called a *von Neumann algebra isomorphism* if it is a *-preserving dual algebra isomorphism.

Let $\mathcal{H}$ be a Hilbert space. Fix a subnormal tuple $T \in B(\mathcal{H})^d$ and let $U \in B(\hat{\mathcal{H}})^d$ be a minimal normal extension of T (cf. Proposition 1.4). By $E(\cdot)$ we denote the projection-valued spectral measure of U. The von Neumann algebra $W^*(U) \subset B(\hat{\mathcal{H}})$ is abelian, and thus has a separating vector $z \in \hat{\mathcal{H}}$, i.e., $Sz \neq 0$ for all non-zero $S \in W^*(U)$. Furthermore, analogously to [20, Proposition V.17.14], we can achieve that $z \in \mathcal{H}$. The *scalar spectral measure*

$$\mu = \langle E(\cdot)z, z\rangle$$

lies in $M^+(\sigma_n(T))$, the set of all finite positive regular Borel measures on $\sigma_n(T)$, and is mutually absolutely continuous with respect to $E(\cdot)$. If we normalize z, then μ is a probability measure, denoted by $\mu \in M_1^+(\sigma_n(T))$. Using Proposition 1.4 one can show that, up to mutual absolute continuity, the measure μ does not depend on the choice of U.

The following proposition is a consequence of the spectral theorem for normal tuples (cf. [3, Appendix D]).

6.9 Proposition. *There exists a von Neumann algebra isomorphism*

$$\Psi_U\colon L^\infty(\mu) \to W^*(U) \subset B(\hat{\mathcal{H}})$$

such that

$$\Psi_U(\pi_k) = U_k$$

for all $k = 1, \ldots, d$, where

$$\pi_k \colon \mathbb{C}^d \to \mathbb{C},\ (z_1, \ldots, z_d) \mapsto z_k$$

denotes the projection map on the k-th component for $k = 1, \ldots, d$.

We call

$$\mathcal{R}_T = \{f \in L^\infty(\mu)\ ;\ \Psi_U(f)\mathcal{H} \subset \mathcal{H}\} \subset L^\infty(\mu)$$

the *restriction algebra of T*. By [39, Proposition 1.1.2], this algebra is independent of the choices of U and μ.

6.10 Proposition. *The restriction algebra is τ_{w^*}-closed.*

Proof. Let $(f_i)_{i\in I}$ be a net in $\mathcal{R}_T$ with τ_{w^*}-limit $f \in L^\infty(\mu)$. Since Ψ_U is τ_{w^*}-continuous and $\mathcal{H} \subset \hat{\mathcal{H}}$ is τ_{w^*}-closed, we obtain

$$\Psi_U(f)h = \tau_{w^*}\text{-}\lim_{i\in I} \Psi_U(f_i)h \in \mathcal{H}$$

for all $h \in H$. $\square$

The last proposition shows that

$$\gamma_T \colon \mathcal{R}_T \to B(\mathcal{H}),\ f \mapsto \Psi_U(f)|_{\mathcal{H}}$$

is a well-defined dual algebra homomorphism. Moreover, this map is isometric (cf. [21, Proposition 1.1]).

Let

$$A(\mathbb{B}_d) = \left\{f \in C(\overline{\mathbb{B}}_d)\ ;\ f|_{\mathbb{B}_d} \in \mathcal{O}(\mathbb{B}_d)\right\} \subset C(\overline{\mathbb{B}}_d)$$

be the *ball algebra*. Then the Shilov boundary $\partial_{A(\mathbb{B}_d)}$ of $A(\mathbb{B}_d)$ coincides with the topological boundary $\partial\mathbb{B}_d$ of the open unit ball $\mathbb{B}_d$, the unit sphere $\mathbb{S}_d$. Since $A(\mathbb{B}_d)|_{\mathbb{S}_d}$ is contained in the restriction algebra of any spherical isometry, by Lemma 1.6, the spherical isometries are exactly the $A(\mathbb{B}_d)$-isometries in the sense of the next definition, which was first introduced by Eschmeier in [35].

6.11 Definition (Eschmeier)**.** Let $K \subset \mathbb{C}^d$ be a compact set and let $A \subset C(K)$ be a closed subalgebra. We call T an *A-isometry* if $\sigma_n(T) \subset \partial_A$, $\mathbb{C}[z_1, \ldots, z_d]|_K \subset A$, and $A|_{\partial_A} \subset \mathcal{R}_T$.

Here as in the following we shall regard the underlying scalar spectral measure of T via trivial extension as a Borel measure on ∂_A.

Let T be an A-isometry as in Definition 6.11. Define

$$H_A^\infty(\mu) = \overline{A|_{\partial_A}}^{\tau_{w^*}} \subset L^\infty(\mu)$$

Since $\gamma_T\colon H_A^\infty(\mu) \to B(\mathcal{H})$ is an isometric τ_{w^*}-continuous algebra homomorphism, its range

$$\mathcal{T}_{\mathrm{a}}^{(c)}(T) = \gamma_T(H_A^\infty(\mu)) \subset B(\mathcal{H})$$

is a τ_{w^*}-closed subalgebra. The induced map

$$\gamma_T\colon H_A^\infty(\mu) \to \mathcal{T}_{\mathrm{a}}^{(c)}(T),\ f \mapsto \Psi_U(f)|_{\mathcal{H}}$$

is a dual algebra isomorphism.

A special role will be played by the set

$$I_\mu = \{f \in H_A^\infty(\mu)\ ;\ |f| = 1\ \mu\text{-almost everywhere}\} \subset L^\infty(\mu),$$

whose elements will be called *μ-inner functions.* There is a one-to-one correspondence between I_μ and the set

$$I_T = \left\{J \in \mathcal{T}_{\mathrm{a}}^{(c)}(T)\ ;\ J \text{ is isometric}\right\}.$$

More precisely, one can show [28, Lemma 1.1]:

6.12 Proposition. *Let $T \in B(\mathcal{H})^d$ be an A-isometry and $\mu \in M^+(\partial_A)$ be a scalar spectral measure of T. Then*

$$I_T = \gamma_T(I_\mu).$$

In [4], Aleksandrov gave sufficient conditions under which there is a rich supply of μ-inner functions.

6.13 Definition (Aleksandrov)**.** Let $K \subset \mathbb{C}^d$ be a compact set, $A \subset C(K)$ be a closed subalgebra and let $\nu \in M^+(K)$ be a finite positive Borel measure. We call the triple (A, K, ν) *regular (in the sense of Aleksandrov)* if for every $\varphi \in C(K)$ with $\varphi > 0$, there exists a sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ in A such that $|\varphi_k| < \varphi$ on K and $\lim_{k\to\infty} |\varphi_k| = \varphi$ holds ν-almost everywhere on K.

For the upcoming examples, we introduce some notations. Let $D \subset \mathbb{C}^d$ be a bounded domain. We denote by

- $\mathcal{O}(D) \subset \mathbb{C}^D$ the set of all scalar-valued analytic functions on D,

- $H^\infty(D) \subset \mathcal{O}(D)$ the subspace of all bounded analytic functions on D,
- $A(D) = \{f \in C(\overline{D})\ ;\ f|_D \in \mathcal{O}(D)\}$ the domain algebra of D.

6.14 Examples. (i) The triple $(A(\mathbb{B}_d)|_{\mathbb{S}_d}, \mathbb{S}_d, \sigma)$, where σ is the normalized surface measure on $\mathbb{S}_d$, is regular.

(ii) The triple $(A(\mathbb{D}^d)|_{\mathbb{T}^d}, \mathbb{T}^d, \otimes_d m)$, where m is the canonical probability measure on the unit circle $\mathbb{T}$ and $\otimes_d m$ denotes the product measure of m with itself d times, is regular.

(iii) More generally, if D is a strictly pseudoconvex domain with smooth boundary or a bounded symmetric and circled domain, then the triple $(A(D)|_{\partial_{A(D)}}, \partial_{A(D)}, \nu)$, where ν is a finite positive regular Borel measure on $\partial_{A(D)}$, is regular. This follows from Proposition 2.5 and Section 5 in [27].

The measures in the first two examples also enjoy the next property.

6.15 Definition. Let K be a compact Hausdorff space. We call $\nu \in M^+(K)$ *continuous* if

$$\Delta_\nu = \{z \in K\ ;\ \nu(\{z\}) > 0\} = \emptyset.$$

The next theorem guarantees the existence of sufficiently many inner functions.

6.16 Theorem (Aleksandrov)**.** *Let (A, K, ν) be a regular triple and $\nu \in M^+(K)$ be continuous. Then the τ_{w^*}-sequential closure of the set I_ν contains all $L^\infty(\nu)$-equivalence classes of functions $f \in A$ with $\|f\|_K \leq 1$.*

This result follows from [4, Corollary 29].

A proof of the following proposition can be found in [28, Proposition 2.4 & Corollary 2.5].

6.17 Proposition. *Let (A, K, ν) be a regular triple. Then we have*

$$H_A^\infty(\nu) = \overline{\operatorname{span}}^{\tau_{w^*}}(I_\nu) \quad \textit{and} \quad L^\infty(\nu) = W^*(I_\nu) = \overline{\operatorname{span}}^{\tau_{w^*}}(\{\overline{\eta} \cdot \theta\ ;\ \eta, \theta \in I_\nu\}).$$

6.18 Definition. Let $T \in B(\mathcal{H})^d$ be an A-isometry. We call T *regular* if $(A|_{\partial_A}, \partial_A, \mu)$ is regular in the sense of Aleksandrov for some, or equivalently every, scalar spectral measure $\mu \in M^+(\partial_A)$ associated with T.

Since the measure in Theorem 6.16 has to be continuous, it is helpful to characterize those A-isometries which have a continuous spectral measure. Proposition 4.1.2 in [39] provides such a characterization in the case of regular A-isometries.

6.19 Proposition. *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry. Then*

$$\sigma_{\mathrm{p}}(T) = \left\{ z \in \mathbb{C}^d \ ; \ \bigcap_{i=1}^{d} \ker(z_i - T_i) \neq \{0\} \right\} = \Delta_\mu.$$

6.20 Corollary. *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry. Then $\sigma_{\mathrm{p}}(T) = \emptyset$ if and only if μ is continuous.*

Let $T \in B(\mathcal{H})^d$ be an A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$. We set

$$\mathcal{I}_U = \Psi_U(I_\mu).$$

If we combine Theorem 6.16, Proposition 6.17, and Corollary 6.20, we obtain the following result.

6.21 Proposition. *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Then the following statements hold:*

(i) $\mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T) = \overline{\mathrm{span}}^{\tau_{w^*}}(I_T)$ *and* $W^*(U) = \overline{\mathrm{span}}^{\tau_{w^*}}(\{J_1^* J_2 \ ; \ J_1, J_2 \in \mathcal{I}_U\})$.

(ii) If $\sigma_{\mathrm{p}}(T) = \emptyset$, then there exists a τ_{w^*}-zero sequence $(\theta_k)_{k\in\mathbb{N}}$ in I_μ, and hence, $(J_k)_{k\in\mathbb{N}} = (\gamma_T(\theta_k))_{k\in\mathbb{N}}$ is a τ_{w^*}-zero sequence in I_T.

We are now going to define the operators which play the main role in this part of the thesis. Part (iii) of the next definition is in the spirit of the Brown-Halmos condition [16] (see also [29]).

Recall from Lemma 1.1 that the inclusion $W^*(U) \subset (U)'$ holds.

6.22 Definition. Let $T \in B(\mathcal{H})^d$ be an A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M^+(\partial_A)$.

(i) We call $X \in B(\mathcal{H})$ a *generalized concrete Toeplitz operator* if there exists $Y \in (U)'$ such that

$$X = T_Y = P_{\mathcal{H}} Y|_{\mathcal{H}} \in B(\mathcal{H}).$$

The set of all generalized concrete Toeplitz operators will be denoted by $\mathcal{T}^{(\mathrm{c,g})}(T)$.

(ii) We call $X \in B(\mathcal{H})$ a *concrete Toeplitz operator* if there exists $f \in L^\infty(\mu)$ such that

$$X = T_f = T_{\Psi_U(f)} \in B(\mathcal{H}).$$

The set of all concrete Toeplitz operators will be denoted by $\mathcal{T}^{(\mathrm{c})}(T)$.

(iii) We call $X \in B(\mathcal{H})$ an *abstract Toeplitz operator* if

$$J^*XJ - X = 0$$

holds for all $J \in I_T$, and denote by $\mathcal{T}^{(\mathrm{a})}(T)$ the set of all abstract Toeplitz operators.

(iv) For $p \in \{0\} \cup [1, \infty]$, we set

$$\mathcal{T}^{(\mathrm{a},p)}(T) = \{X \in B(\mathcal{H}) \ ; \ J^*XJ - X \in \mathcal{S}_p(\mathcal{H}) \text{ for all } J \in I_T\}\,.$$

In the setting of Definition 6.22, the chain of inclusions

$$\mathcal{T}^{(\mathrm{c})}(T) \subset \mathcal{T}^{(\mathrm{c,g})}(T) \subset \mathcal{T}^{(\mathrm{a})}(T) \subset \mathcal{T}^{(\mathrm{a},p)}(T)$$

holds.

Recall that a von Neumann algebra $\mathcal{A}$ is *maximal abelian* if $\mathcal{A} = \mathcal{A}'$.

6.23 Proposition. *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$. The following statements hold:*

(i) $\mathcal{T}^{(\mathrm{a})}(T) = \mathcal{T}^{(\mathrm{c,g})}(T)$,

(ii) If $W^*(U)$ is a maximal abelian von Neumann algebra, then $\mathcal{T}^{(\mathrm{c})}(T) = \mathcal{T}^{(\mathrm{c,g})}(T) = \mathcal{T}^{(\mathrm{a})}(T)$.

A proof of the last result can be found in [29, Proposition 3.2].

From now on, let $T \in B(\mathcal{H})^d$ be an A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$.

6.24 Lemma. *Let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mu)$ and let $f \in L^\infty(\mu)$ be such that*

$$\tau_{\|\cdot\|_{L^2(\mu)}}\text{-}\lim_{k\to\infty} f_k = f.$$

Then

(i) $\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} \Psi_U(f_k) = \Psi_U(f)$ and $\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} \Psi_U(f_k)^* = \Psi_U(f)^*$,

(ii) $\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T_{f_k} = T_f$ and $\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T_{f_k}^* = T_f^*$.

Proof. Let $(f_k)_{k\in\mathbb{N}}$ and f be as in the hypothesis of the lemma. Since $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\mu)$, the sequence $(f_k - f)_{k\in\mathbb{N}}$ is also bounded in $L^\infty(\mu)$. Therefore, we can suppose that $f = 0$.

(i) We have
$$\|\Psi_U(g)x\|^2 = \int_{\partial_A} |g|^2 \,\mathrm{d}\langle E(\cdot)x, x\rangle$$
for all $g \in L^\infty(\mu)$ and $x \in \mathcal{H}$. Let $(f_{k_l})_{l\in\mathbb{N}}$ be a subsequence of $(f_k)_{k\in\mathbb{N}}$. Then there exists a subsequence $(f_{k_{l_m}})_{m\in\mathbb{N}}$ of $(f_{k_l})_{l\in\mathbb{N}}$ such that
$$f_{k_{l_m}} \to 0$$
as $m \to \infty$ μ-almost everywhere on ∂_A and hence $\langle E(\cdot)x, x\rangle$-almost everywhere for every $x \in \mathcal{H}$. By the dominant convergence theorem, we can conclude that
$$\left\|\Psi_U(f_{k_{l_m}})x\right\| \to 0 \quad (x \in \mathcal{H})$$
as $m \to \infty$.

Hence, we obtain that
$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} \Psi_U(f_k) = 0.$$
Since $\Psi_U(f_k)$ is normal for all $k \in \mathbb{N}$, we conclude that
$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} \Psi_U(f_k)^* = 0.$$

(ii) Since T_{f_k} is the compression of $\Psi_U(f_k)$ on $\mathcal{H}$ for all $k \in \mathbb{N}$, the result follows from (i). □

Since we are concerned with the weak* topology and the weak operator topology on $B(\mathcal{H})$ in the sequel, the following remark will be helpful.

6.25 Remark. By [22, Proposition 20.1], the closed norm unit ball of $B(\mathcal{H})$, $\overline{B}_1^{B(\mathcal{H})}(0)$, equipped with the relative topology of the weak* topology of $B(\mathcal{H})$ is a compact metrizable space. Furthermore, the topologies τ_{w^*} and τ_{WOT} coincide on every norm-bounded subset of $B(\mathcal{H})$.

For the rest of this section we take a closer look at the behavior of limits of Toeplitz operators. The upcoming lemmas are technical results which will be needed in the proof of Proposition 6.30.

6.26 Lemma. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\overline{B}_1^{L^\infty(\mu)}(0)$, the closed norm unit ball of $L^\infty(\mu)$. Then the following statements are equivalent:*

(i) $\tau_{w^*}\text{-}\lim_{k\to\infty} f_k = 1$ *in* $L^\infty(\mu)$,

(ii) $\lim_{k\to\infty} \int_{\partial_A} f_k \,\mathrm{d}\mu = 1$,

(iii) $\tau_{\|\cdot\|_{L^2(\mu)}}\text{-}\lim_{k\to\infty} f_k = 1$,

(iv) $\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} \Psi_U(f_k) = \mathrm{id}_{\hat{\mathcal{H}}}$,

(v) $\tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} \Psi_U(f_k) = \mathrm{id}_{\hat{\mathcal{H}}}$,

(vi) $\tau_{w^*}\text{-}\lim_{k\to\infty} \Psi_U(f_k) = \mathrm{id}_{\hat{\mathcal{H}}}$ *in* $B(\hat{\mathcal{H}})$.

In this situation, we have

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T_{f_k} = \mathrm{id}_{\mathcal{H}} \quad \textit{and} \quad \tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T_{f_k}^* = \mathrm{id}_{\mathcal{H}}$$

by Lemma 6.24.

Proof. (i) $\implies$ (ii): We have

$$1 - \int_{\partial_A} f_k \,\mathrm{d}\mu = \int_{\partial_A} 1 - f_k \,\mathrm{d}\mu = \int_{\partial_A} (1 - f_k) \cdot 1 \,\mathrm{d}\mu \to 0$$

as $k \to \infty$, since $1 \in L^1(\mu)$.

(ii) $\implies$ (iii): We have

$$\begin{aligned}
\|1 - f_k\|_{L^2(\mu)}^2 &= \langle 1 - f_k, 1 - f_k\rangle_{L^2(\mu)} \\
&= \langle 1, 1\rangle_{L^2(\mu)} - 2\,\mathrm{Re}\left(\langle f_k, 1\rangle_{L^2(\mu)}\right) + \langle f_k, f_k\rangle_{L^2(\mu)} \\
&= 1 - 2\,\mathrm{Re}\left(\int_{\partial_A} f_k \,\mathrm{d}\mu\right) + \int_{\partial_A} |f_k|^2 \,\mathrm{d}\mu \\
&\leq 2\,\mathrm{Re}\left(1 - \int_{\partial_A} f_k \,\mathrm{d}\mu\right) \\
&\to 0
\end{aligned}$$

as $k \to \infty$.

(iii) $\implies$ (iv): This follows immediately from Lemma 6.24.

(iv) $\implies$ (v): Clear.

(v) $\implies$ (vi): Since Ψ_U is isometric, the sequence $(\Psi_U(f_k))_{k\in\mathbb{N}}$ lies in $\overline{B}_1^{B(\hat{\mathcal{H}})}(0)$. The result follows now from Remark 6.25.

(vi) $\implies$ (i): This follows immediately from the fact that Ψ_U is a τ_{w^*}-homeomorphism. $\square$

6.27 Lemma. *Let* $p \in \{0\} \cup [1, \infty]$ *and let* $(f_k)_{k\in\mathbb{N}}$ *be a sequence in* I_μ *such that*

$$\tau_{w^*}\text{-}\lim_{k\to\infty} f_k = 1 \ \textit{in}\ L^\infty(\mu).$$

Then:

(i) For all $X \in B(\mathcal{H})$, we have

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T_{f_k}^* X T_{f_k} = X.$$

(ii) For all $S \in \mathcal{S}_p(\mathcal{H})$, we have

$$\tau_{\|\cdot\|_p}\text{-}\lim_{k\to\infty} T_{f_k}^* S T_{f_k} = S.$$

(iii) For all $X \in B(\mathcal{H})$ and $u \in H_A^\infty(\mu)$ with $T_u^* X T_u - X \in \mathcal{S}_p(\mathcal{H})$, it follows that

$$\lim_{k\to\infty} \left\| T_u^* \left(T_{f_k}^* X T_{f_k} - X\right) T_u - \left(T_{f_k}^* X T_{f_k} - X\right) \right\|_p = 0.$$

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in I_μ such that

$$\tau_{w^*}\text{-}\lim_{k\to\infty} f_k = 1 \text{ in } L^\infty(\mu).$$

(i) Let $X \in B(\mathcal{H})$. By Lemmas 6.5 and 6.26, we have

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T_{f_k}^* X T_{f_k} = X.$$

(ii) Let $S \in \mathcal{S}_p(\mathcal{H})$. By Lemma 6.26 and Proposition 6.4, and the fact that $H_A^\infty(\mu) \subset \mathcal{R}_T$, the result follows from the observation that

$$\begin{aligned}
\left\| T_{f_k}^* S T_{f_k} - S \right\|_p &= \left\| T_{f_k}^* (S T_{f_k} - T_{f_k} S) \right\|_p \\
&\leq \left\| S(T_{f_k} - \mathrm{id}_{\mathcal{H}}) \right\|_p + \left\| (T_{f_k} - \mathrm{id}_{\mathcal{H}}) S \right\|_p \\
&= \left\| (T_{f_k}^* - \mathrm{id}_{\mathcal{H}}) S^* \right\|_p + \left\| (T_{f_k} - \mathrm{id}_{\mathcal{H}}) S \right\|_p \\
&\to 0
\end{aligned}$$

as $k \to \infty$.

(iii) Let $X \in B(\mathcal{H})$ and $u \in H_A^\infty(\mu)$ such that $T_u^* X T_u - X \in \mathcal{S}_p(\mathcal{H})$. We have

$$\begin{aligned}
&T_u^* \left(T_{f_k}^* X T_{f_k} - X\right) T_u - \left(T_{f_k}^* X T_{f_k} - X\right) \\
=&T_u^* T_{f_k}^* X T_{f_k} T_u - T_u^* X T_u - T_{f_k}^* X T_{f_k} + X \\
=&T_{f_k}^* T_u^* X T_u T_{f_k} - T_{f_k}^* X T_{f_k} - T_u^* X T_u + X \\
=&T_{f_k}^* \left(T_u^* X T_u - X\right) T_{f_k} - \left(T_u^* X T_u - X\right)
\end{aligned}$$

for all $k \in \mathbb{N}$. The result follows from part (ii). □

6.28 Lemma. *Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in $L^\infty(\mu)$ with*

$$\tau_{w^*}\text{-}\lim_{k\to\infty} w_k = 0.$$

Then

$$\tau_{w^*}\text{-}\lim_{k\to\infty} T_{w_k} = 0.$$

Proof. Since $\Psi_U\colon L^\infty(\mu) \to W^*(U)$ and $B(\hat{\mathcal{H}}) \to B(\mathcal{H})$, $X \mapsto P_{\mathcal{H}}X|_{\mathcal{H}}$ are τ_{w^*}-continuous, the map

$$L^\infty(\mu) \to B(\mathcal{H}),\ f \mapsto T_f$$

is also τ_{w^*}-continuous. Hence, the result follows. □

6.29 Lemma. *Let $(v_k)_{k\in\mathbb{N}}$ be a sequence in $L^\infty(\mu)$ with*

$$\tau_{w^*}\text{-}\lim_{k\to\infty} v_k = v \in L^\infty(\mu).$$

Then, for all $K \in \mathcal{S}_\infty(\mathcal{H})$, we have

$$\tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{v_k}^* K T_{v_k} = T_v^* K T_v.$$

Proof. Let $f, g \in \mathcal{H}$ and $K \in \mathcal{S}_\infty(\mathcal{H})$. Define $w_k = v_k - v$ for all $k \in \mathbb{N}$. Since $K \in \mathcal{S}_\infty(\mathcal{H})$, we obtain with Lemma 6.28

$$K(T_{w_k} f) \to 0$$

as $k \to \infty$. Furthermore, since every weakly convergent sequence is norm-bounded, the sequence $(\|T_{w_k} g\|)_{k\in\mathbb{N}}$ is bounded by Lemma 6.28. We conclude that

$$\begin{aligned}
&\left|\left\langle (T_{v_k}^* K T_{v_k} - T_v^* K T_v) f, g \right\rangle\right| \\
=&\left|\left\langle (T_{v_k-v}^* K T_{v_k-v} + T_{v_k-v}^* K T_v + T_v^* K T_{v_k-v}) f, g \right\rangle\right| \\
=&\left|\langle K(T_{w_k} f), T_{w_k} g\rangle + \langle K T_v f, T_{w_k} g\rangle + \langle T_{w_k} f, K^* T_v g\rangle\right| \\
\le&\, \|K(T_{w_k} f)\|\,\|T_{w_k} g\| + |\langle K T_v f, T_{w_k} g\rangle| + |\langle T_{w_k} f, K^* T_v g\rangle| \\
\to&\, 0
\end{aligned}$$

as $k \to \infty$. □

Part (ii) of the following proposition is the starting point for characterizing perturbations of Toeplitz operators.

6.30 Proposition. *If* $(v_k)_{k\in\mathbb{N}}$ *is a sequence in* $H_A^\infty(\mu)$ *with*

$$\tau_{w^*}\text{-}\lim_{k\to\infty} v_k = \alpha \in \mathbb{C} \text{ in } L^\infty(\mu)$$

and $X \in B(\mathcal{H})$ *is an operator such that*

$$\tilde{X} = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{v_k}^* X T_{v_k} \in B(\mathcal{H})$$

exists, then:

(i) *For* $u \in H_A^\infty(\mu)$ *such that* $T_u^* X T_u - X \in \mathcal{S}_\infty(\mathcal{H})$*, it follows that*

$$T_u^* \tilde{X} T_u - \tilde{X} = |\alpha|^2 \left(T_u^* X T_u - X\right).$$

(ii) *If* $X \in \mathcal{T}^{(\mathrm{a},\infty)}(T)$ *and* $\alpha \in \mathbb{C}\setminus\mathbb{T}$*, then*

$$X - \frac{1}{1-|\alpha|^2}(X-\tilde{X}) \in \mathcal{T}^{(\mathrm{a})}(T).$$

Proof. Let $(v_k)_{k\in\mathbb{N}}$ and $X \in B(\mathcal{H})$ be as in the hypothesis of the proposition.

(i) Let $u \in H_A^\infty(\mu)$ such that $T_u^* X T_u - X \in \mathcal{S}_\infty(\mathcal{H})$. Then, by Lemma 6.29,

$$\begin{aligned} T_u^* \tilde{X} T_u - \tilde{X} &= \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_u^* T_{v_k}^* X T_{v_k} T_u - T_{v_k}^* X T_{v_k} \\ &= \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{v_k}^* (T_u^* X T_u - X) T_{v_k} \\ &= |\alpha|^2 \left(T_u^* X T_u - X\right). \end{aligned}$$

(ii) Let $X \in \mathcal{T}^{(\mathrm{a},\infty)}(T)$, $\alpha \in \mathbb{C}\setminus\mathbb{T}$, and set

$$Z = X - \frac{1}{1-|\alpha|^2}(X-\tilde{X}) = \frac{1}{1-|\alpha|^2}(\tilde{X} - |\alpha|^2 X).$$

Then, by part (i), we have

$$T_u^* Z T_u - Z = \frac{1}{1-|\alpha|^2}\Big(\Big(T_u^* \tilde{X} T_u - \tilde{X}\Big) - |\alpha|^2 \left(T_u^* X T_u - X\right)\Big) = 0$$

for all $u \in I_\mu$. Hence, $Z \in \mathcal{T}^{(\mathrm{a})}(T)$. □

6.31 Remark. In the setting of the last proposition, there is always a subsequence $(v_{k_j})_{j\in\mathbb{N}}$ of $(v_k)_{k\in\mathbb{N}}$ such that the limit $\tau_{\mathrm{WOT}}\text{-}\lim_{j\to\infty} T_{v_{k_j}}^* X T_{v_{k_j}} \in B(\mathcal{H})$ exists. This follows from Remark 6.25.

7. Analytic Toeplitz operators

In the classical setting, a necessary and sufficient condition for an operator to commute with all analytic Toeplitz operators is to be an analytic Toeplitz operator. A characterization of the commutant of the set of all analytic Toeplitz operators modulo the compact operators was first obtained by Davidson in 1977 [25] on the Hardy space $H^2(m)$. He proved that this set consists of all compact perturbations of Toeplitz operators T_f with symbol $f \in H^\infty(m) + C(\mathbb{T})$. By a classical result of Hartman [47], this symbol class consists precisely of all functions $f \in L^\infty(m)$ for which the Hankel operator H_f with symbol f is compact.

In 2006, Guo and Wang [45] characterized the commutant of all analytic Toeplitz operators modulo the finite-rank operators on $H^2(\sigma)$ and $H^2(\otimes_d m)$.

The topic of the last section is to obtain a similar result for Schatten-class perturbations of analytic Toeplitz operators. The results therein have been published in [30].

For this chapter, let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be Hilbert spaces.

7.1. Abstract analytic Toeplitz operators

Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$.

Recall that $\mathcal{T}_\mathrm{a}^{(\mathrm{c})}(T)$ is the set of all concrete analytic Toeplitz operators.

7.1 Definition. We denote by

$$\mathcal{T}_\mathrm{a}^{(\mathrm{a})}(T) = \{X \in B(\mathcal{H}) \; ; \; [X, J] = 0 \text{ for all } J \in I_T\} \subset \mathcal{T}^{(\mathrm{a})}(T)$$

the set of all *abstract analytic Toeplitz operators*, where $[X, Y]$ denotes the commutator of operators $X, Y \in B(\mathcal{H})$.

7.2 Remark. We have

$$\mathcal{T}_\mathrm{a}^{(\mathrm{c})}(T) \subset (\mathcal{T}_\mathrm{a}^{(\mathrm{c})}(T))' = \mathcal{T}_\mathrm{a}^{(\mathrm{a})}(T) \subset \mathcal{T}^{(\mathrm{a})}(T).$$

Proof. Since

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T) = \overline{\operatorname{span}}^{\tau_{w^*}}(I_T)$$

holds by Proposition 6.21 and, for $B \in B(\mathcal{H})$, the maps

$$B(\mathcal{H}) \to B(\mathcal{H}),\ A \mapsto AB \quad \text{and} \quad B(\mathcal{H}) \to B(\mathcal{H}),\ A \mapsto BA$$

are τ_{w^*}-continuous, we have

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T) \subset (\mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T))'.$$

The other inclusions are clear. □

The upcoming definition of Hankel operators is the natural generalization of the classical notion.

7.3 Definition. Let $Y \in (U)'$ and $f \in L^\infty(\mu)$. We call

$$H_Y = (\mathrm{id}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})Y|_{\mathcal{H}} \in B(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$$

the *Hankel operator with symbol* Y, and

$$H_f = H_{\Psi_U(f)} \in B(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$$

the *Hankel operator with symbol* f.

For $Y \in (U)'$ and $g \in L^\infty(\mu)$, we have

$$T_{Y\Psi_U(g)} - T_g T_Y = H_{\overline{g}}^* H_Y.$$

The following proposition is a slight extension of [39, Proposition 1.3.2] with exactly the same proof.

7.4 Proposition. *For all $Y \in (U)'$, $f \in L^\infty(\mu)$ and $g, h \in \mathcal{R}_T$, the relation*

$$P_{\mathcal{H}}(\Psi_U(\overline{g}fh)Y)|_{\mathcal{H}} = T_{\overline{g}} P_{\mathcal{H}}(\Psi_U(f)Y)|_{\mathcal{H}} T_h$$

holds. In particular, we have

$$T_{\overline{g}fh} = T_{\overline{g}} T_f T_h \quad \text{and} \quad T_{\Psi_U(\overline{g}h)Y} = T_{\overline{g}} T_Y T_h.$$

For every $\theta \in I_\mu$, the operator $\Psi_U(\theta) \in B(\hat{\mathcal{H}})$ is unitary and leaves $\mathcal{H}$ invariant.

7.5 Lemma. *For each $\theta \in I_\mu$, the Hankel operator $H_{\overline{\theta}} \in B(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$ is a partial isometry with*

$$\ker(H_{\overline{\theta}}) = \Psi_U(\theta)\mathcal{H} \quad \textit{and} \quad \operatorname{Im}(H_{\overline{\theta}}) = (\hat{\mathcal{H}} \ominus \mathcal{H}) \ominus (\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H})).$$

Proof. Let $\theta \in I_\mu$. We have

$$H_{\overline{\theta}}\Psi_U(\theta)\mathcal{H} = P_{\hat{\mathcal{H}}\ominus\mathcal{H}}\Psi_U(\overline{\theta})\Psi_U(\theta)\mathcal{H} = \{0\}$$

and

$$\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \Psi_U(\theta)\mathcal{H}) = (\Psi_U(\overline{\theta})\hat{\mathcal{H}}) \ominus \mathcal{H} = \hat{\mathcal{H}} \ominus \mathcal{H}.$$

Therefore, we obtain $H_{\overline{\theta}} = \Psi_U(\overline{\theta})$ on $\mathcal{H} \ominus \Psi_U(\theta)\mathcal{H}$. Hence, $H_{\overline{\theta}}$ is a partial isometry with $\ker(H_{\overline{\theta}}) = \Psi_U(\theta)\mathcal{H}$. Furthermore, the identity

$$\begin{aligned} H_{\overline{\theta}}(\mathcal{H} \ominus \Psi_U(\theta)\mathcal{H}) &= \Psi_U(\overline{\theta})(\mathcal{H} \ominus \Psi_U(\theta)\mathcal{H}) \\ &= (\Psi_U(\overline{\theta})\mathcal{H}) \ominus \mathcal{H} \\ &= (\Psi_U(\overline{\theta})\hat{\mathcal{H}} \ominus \mathcal{H}) \ominus (\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H})) \\ &= (\hat{\mathcal{H}} \ominus \mathcal{H}) \ominus (\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H})) \end{aligned}$$

shows that

$$\operatorname{Im}(H_{\overline{\theta}}) = (\hat{\mathcal{H}} \ominus \mathcal{H}) \ominus (\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H})). \qquad \square$$

7.6 Corollary. *For each $\theta \in I_\mu$, the operator*

$$P_\theta = H_{\overline{\theta}}H_{\overline{\theta}}^* \in B(\hat{\mathcal{H}} \ominus \mathcal{H})$$

is the orthogonal projection from $\hat{\mathcal{H}} \ominus \mathcal{H}$ onto the space

$$\hat{\mathcal{H}}_\theta = (\hat{\mathcal{H}} \ominus \mathcal{H}) \ominus (\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H})).$$

The next lemma shows that we can characterize the orthogonal complement of $\mathcal{H}$ in $\hat{\mathcal{H}}$ using the operators $\hat{\mathcal{H}}_\theta$ with $\theta \in I_\mu$.

7.7 Lemma. *With the notations from above, we have*

$$\hat{\mathcal{H}} \ominus \mathcal{H} = \bigvee(\hat{\mathcal{H}}_\theta\ ;\ \theta \in I_\mu).$$

Proof. We have

$$\begin{aligned} (\hat{\mathcal{H}} \ominus \mathcal{H}) \ominus \Big(\bigvee(\hat{\mathcal{H}}_\theta\ ;\ \theta \in I_\mu)\Big) &= \bigcap_{\theta\in I_\mu} \Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H}) \\ &= \bigcap_{\theta\in I_\mu} (\Psi_U(\overline{\theta})\hat{\mathcal{H}} \ominus \Psi_U(\overline{\theta})\mathcal{H}) \\ &= \bigcap_{\theta\in I_\mu} (\hat{\mathcal{H}} \ominus \Psi_U(\overline{\theta})\mathcal{H}) \\ &= \hat{\mathcal{H}} \ominus \bigvee_{\theta\in I_\mu} \Psi_U(\overline{\theta})\mathcal{H}. \end{aligned}$$

Since $U \in B(\hat{\mathcal{H}})^d$ is the minimal normal extension tuple of $T \in B(\mathcal{H})^d$, it is enough to show that $\bigvee_{\theta \in I_\mu} \Psi_U(\overline{\theta})\mathcal{H}$ is a reducing subspace for U. Since I_μ is closed under multiplication and since $I_\mu \subset \mathcal{R}_T$, it follows that $\bigvee_{\theta \in I_\mu} \Psi_U(\overline{\theta})\mathcal{H}$ is invariant under the von Neumann algebra

$$W^*(\Psi_U(I_\mu)) = \Psi_U(W^*(I_\mu)) = \Psi_U(L^\infty(\mu)) = W^*(U).$$

In particular, the space $\bigvee_{\theta \in I_\mu} \Psi_U(\overline{\theta})\mathcal{H}$ is reducing for U. $\square$

7.8 Remark. Let $\theta_1, \theta_2 \in I_\mu$. We write $\theta_1 \leq \theta_2$ if there exists $\theta \in I_\mu$ such that $\theta_2 = \theta_1\theta$. This defines a partial order on I_μ and $(I_\mu, \leq)$ is directed upwards. Furthermore, if $\theta_1 \leq \theta_2$ and $\theta_2 = \theta_1\theta$ as above, then

$$\Psi_U(\overline{\theta}_2)(\hat{\mathcal{H}} \ominus \mathcal{H}) = \Psi_U(\overline{\theta}_1)(\Psi_U(\overline{\theta})(\hat{\mathcal{H}} \ominus \mathcal{H})) \subset \Psi_U(\overline{\theta}_1)(\hat{\mathcal{H}} \ominus \mathcal{H}),$$

and hence, $P_{\theta_1} \leq P_{\theta_2}$.

7.9 Lemma. *We have*

$$\tau_{\mathrm{SOT}}\text{-}\lim_{\theta \in I_\mu} P_\theta = \mathrm{id}_{\hat{\mathcal{H}} \ominus \mathcal{H}}\,.$$

Proof. Define

$$M = \left\{ P_{\theta'} h \ ; \ \theta' \in I_\mu \text{ and } h \in \hat{\mathcal{H}} \ominus \mathcal{H} \right\}.$$

Since

$$P_\theta P_{\theta'} h = P_{\theta'} h$$

for all $\theta' \in I_\mu$, $h \in \hat{\mathcal{H}} \ominus \mathcal{H}$ and $\theta \in I_\mu$ with $\theta \geq \theta'$, the net $(P_\theta)_{\theta \in I_\mu}$ converges pointwise to the identity operator on M. Since $\|P_\theta\| \leq 1$ for all $\theta \in I_\mu$, it converges pointwise to the identity operator on $\overline{\mathrm{span}}^{\tau_{\|\cdot\|}}(M) = \hat{\mathcal{H}} \ominus \mathcal{H}$. $\square$

We conclude this section by characterizing abstract analytic Toeplitz operators via Hankel operators.

7.10 Proposition. *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$. Then we have*

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T) = \{T_Y \ ; \ Y \in (U)' \text{ with } H_Y = 0\}\,.$$

If in addition $W^*(U)$ is a maximal abelian von Neumann algebra, then

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T) = \{T_f \ ; \ f \in L^\infty(\mu) \text{ with } H_f = 0\} = \{T_f \ ; \ f \in \mathcal{R}_T\}\,.$$

Proof. Let $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T) \subset \mathcal{T}^{(\mathrm{a})}(T)$. By Proposition 6.23 (i), there exists $Y \in (U)'$ such that

$$X = T_Y.$$

With

$$0 = [T_Y, T_\theta] = H_{\overline{\theta}}^* H_Y$$

for all $\theta \in I_\mu$ and Lemma 7.9, we conclude that

$$H_Y = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta \in I_\mu} P_\theta H_Y = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta \in I_\mu} H_{\overline{\theta}} H_{\overline{\theta}}^* H_Y = 0.$$

The other inclusion follows from the fact that

$$[T_Y, T_u] = H_{\overline{u}}^* H_Y = 0$$

holds for all $Y \in (U)'$ with $H_Y = 0$ and $u \in I_\mu$.

The rest follows from Proposition 6.23 (ii). □

7.11 Corollary. *Let $D = \mathbb{B}_d$ or $D = \mathbb{D}^d$, $A = A(D)$ and $\mu = \sigma$ the canonical probability measure on $\partial_{A(D)}$ as well as $\mathcal{H} = H^2(\sigma) = \overline{A(D)|_{\partial_{A(D)}}}^{\tau_{\|\cdot\|_{L^2(\sigma)}}}$. Then we have*

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T_z) = \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T_z).$$

In this case, we use the abbreviation $\mathcal{T}_{\mathrm{a}}(T_z) = \mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T_z) = \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T_z)$.

Proof. The tuple $T_z \in B(H^2(\sigma))^d$ is a regular $A(D)$-isometry and σ is a scalar spectral measure of its minimal normal extension $M_z \in B(L^2(\sigma))^d$. Since $W^*(M_z)$ is maximal abelian, it follows from Proposition 7.10 that

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T_z) = \{T_f \ ; \ f \in \mathcal{R}_{T_z}\}.$$

Since

$$\mathcal{R}_{T_z} = \left\{f \in L^\infty(\sigma) \ ; \ fH^2(\sigma) \subset H^2(\sigma)\right\} = L^\infty(\sigma) \cap H^2(\sigma) = H^\infty(\sigma),$$

the assertion follows. □

7.2. Finite-rank perturbations of analytic Toeplitz operators

Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$.

For $p \in \{0\} \cup [1, \infty]$, we set

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T) = \left\{ X \in B(\mathcal{H}) \ ; \ [X, Y] \in \mathcal{S}_p(\mathcal{H}) \text{ for all } Y \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T) \right\}.$$

In this section, we want to achieve a result similar to Proposition 7.10 for $\mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T)$. We start with a general statement about limits of finite-rank operators. A proof of this result can be found in [28, Lemma 3.4].

7.12 Lemma. *(i) Let* $(F_k)_{k\in\mathbb{N}}$ *be a sequence in* $B(\mathcal{H})$ *satisfying* $\mathrm{rank}(F_k) \leq M$ *for all* $k \in \mathbb{N}$ *and some fixed* $M \in \mathbb{N}$. *If* $(F_k)_{k\in\mathbb{N}}$ *has a* τ_{WOT}*-limit* $F \in B(\mathcal{H})$, *then* $\mathrm{rank}(F) \leq M$.

(ii) Let $\mathcal{A} \subset B(\mathcal{H})$ *be a closed subspace and* $X \in B(\mathcal{H})$ *such that* $\mathrm{rank}([X, A])$ *is finite for all* $A \in \mathcal{A}$. *Then there exists a natural number* $M \in \mathbb{N}$ *such that*

$$\mathrm{rank}([X, A]) \leq M$$

for all $A \in \mathcal{A}$.

An inspection of the proof of Lemma 3.4 in [28] shows that part (i) of Lemma 7.12 remains true with sequences $(F_k)_{k\in\mathbb{N}}$ replaced by nets $(F_i)_{i\in I}$.

7.13 Proposition. *If* $\sigma_{\mathrm{p}}(T) = \emptyset$, *then*

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T) + \mathcal{S}_0(\mathcal{H}) \subset \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T) \subset \mathcal{T}^{(\mathrm{a})}(T) + \mathcal{S}_0(\mathcal{H})$$

holds.

Proof. The inclusion

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a})}(T) + \mathcal{S}_0(\mathcal{H}) \subset \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T)$$

is clear.

For the second inclusion, let $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T)$. By Proposition 6.21 (ii), there exists a τ_{w^*}-zero sequence $(u_k)_{k\in\mathbb{N}}$ in I_μ, and hence, $(T_{u_k})_{k\in\mathbb{N}}$ is a τ_{w^*}-zero sequence in I_T. By passing to a subsequence, we can suppose that

$$\tilde{X} = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{u_k}^* X T_{u_k}$$

exists. With Lemma 7.12 (ii) applied to $\mathcal{A} = \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T)$ we obtain a constant $M > 0$ such that

$$\mathrm{rank}(T_u^* X T_u - X) = \mathrm{rank}(T_u^*[X, T_u]) \leq \mathrm{rank}([X, T_u]) \leq M$$

for all $u \in I_\mu$. We conclude that

$$S = X - \tilde{X} = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty}(X - T_{u_k}^* X T_{u_k})$$

has at most rank M by using Lemma 7.12 (i). The result follows now from Proposition 6.30 (ii). $\square$

The proof of the next theorem is a slight modification of the proof of [28, Theorem 3.5], which was inspired by the proof of [45, Theorem 3.1].

7.14 Theorem. *Let $T \in B(\mathcal{H})$ be a regular A-isometry with $\sigma_{\mathrm{p}}(T) = \emptyset$, minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$. For $X \in B(\mathcal{H})$, the following statements are equivalent:*

(i) $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T)$,

(ii) $X = T_Y + F$ *for some* $F \in \mathcal{S}_0(\mathcal{H})$ *and a symbol* $Y \in (U)'$ *with* $H_Y \in \mathcal{S}_0(\mathcal{H})$.

If in addition $W^*(U)$ is a maximal abelian von Neumann algebra, then the above are also equivalent to

(iii) $X = T_f + F$ *for some* $F \in \mathcal{S}_0(\mathcal{H})$ *and* $f \in L^\infty(\mu)$ *with* $H_f \in \mathcal{S}_0(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$.

Proof. (i) $\Longrightarrow$ (ii): Let $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T)$. By Propositions 6.23 and 7.13, there exists $Y \in (U)'$ and $F \in \mathcal{S}_0(\mathcal{H})$ such that

$$X = T_Y + F.$$

Since $[T_Y, Z] = [X, Z] - [F, Z] \in \mathcal{S}_0(\mathcal{H})$ for all $Z \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T)$ and $\mathcal{A} = \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T)$ is closed, Lemma 7.12 (ii) yields a natural number $M > 0$ such that

$$\operatorname{rank}([T_Y, T_u]) \leq M$$

for all $u \in I_\mu$. Thus, using Lemma 7.9, we obtain

$$H_Y = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta\in I_\mu} P_\theta H_Y = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta\in I_\mu} H_{\overline{\theta}}(H_{\overline{\theta}}^* H_Y) = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta\in I_\mu} H_{\overline{\theta}}[T_Y, T_\theta],$$

and hence, Lemma 7.12 (i) implies that H_Y is a finite-rank operator.

(ii) $\Longrightarrow$ (i): Since

$$[T_Y, T_g] = H_{\overline{g}}^* H_Y$$

holds for all $Y \in (U)'$ and $g \in H_A^\infty(\mu)$, this implication follows immediately.

The rest follows from Proposition 6.23 (ii). $\square$

In the case $D = \mathbb{B}_d$ or $D = \mathbb{D}^d$, we get [45, Theorem 3.1] back.

7.15 Corollary (Guo, Wang)**.** *Let $D = \mathbb{B}_d$ or $D = \mathbb{D}^d$, $A = A(D)$ and $\mu = \sigma$ the canonical probability measure on $\partial_{A(D)}$ as well as $\mathcal{H} = H^2(\sigma) = \overline{A(D)|_{\partial_{A(D)}}}^{\tau_{\|\cdot\|_{L^2(\sigma)}}}$. An operator $X \in B(H^2(\sigma))$ belongs to $\mathcal{T}_{\mathrm{a}}^{(\mathrm{a},0)}(T_z)$ if and only if $X = T_{f^*} + F$ for some $F \in \mathcal{S}_0(H^2(\sigma))$ and*

(i) if $n = 1$, f is the sum of a function in $H^\infty(\mathbb{D})$ and a rational function,

(ii) if $n \geq 2$, $f \in H^\infty(D)$.

Proof. The result follows from Theorem 7.14, Corollary 7.11, and Proposition 7.10 as well as Proposition 2.2 with the parapgraph before it in [45]. □

7.3. Schatten-class perturbations of analytic Toeplitz operators

As before, let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$.

The following proposition can be deduced from [50, Proposition 2.11].

7.16 Proposition (Hiai)**.** *The map*

$$\|\cdot\|_p : (B(\mathcal{H}), \tau_{\mathrm{WOT}}) \to [0, \infty],\ S \mapsto \|S\|_p$$

is lower semi-continuous.

For the proof of the upcoming proposition, we recall some basic facts about the Toeplitz projection established in [37]. Since $L^1(\mu)$ is separable and hence, I_μ is a separable metrizable space in the relative weak* topology, there exists a sequence $(\theta_k)_{k\in\mathbb{N}}$ in I_μ such that

$$W^*(\{\theta_k\ ;\ k \in \mathbb{N}\}) = W^*(I_\mu) = L^\infty(\mu).$$

Fix such a sequence $(\theta_k)_{k\in\mathbb{N}}$ and set

$$\Phi_{T,k} \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \frac{1}{k^k} \sum_{1\leq i_1,\ldots,i_k\leq k} T^*_{\theta_k^{i_k}\cdot\ldots\cdot\theta_1^{i_1}} X T_{\theta_1^{i_1}\cdot\ldots\cdot\theta_k^{i_k}}$$

for all $k \in \mathbb{N}$. A completely positive, unital projection

$$\Phi_T \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \tau_{w^*}\text{-}\lim_{i\in I} \Phi_{T,k_i}(X),$$

where $(\Phi_{T,k_i}(X))_{i\in I}$ is a τ_{w^*}-convergent subnet of $(\Phi_{T,k}(X))_{k\in\mathbb{N}}$ for all $X \in B(\mathcal{H})$, is called a *Toeplitz projection.* Furthermore, we have $\mathrm{Im}(\Phi_T) = \mathcal{T}^{(\mathrm{a})}(T)$.

7.17 Proposition. *For $1 \leq p < \infty$, the inclusion*

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T) \subset \mathcal{T}^{(\mathrm{a})}(T) + \mathcal{S}_p(\mathcal{H})$$

holds.

Proof. Let $1 \leq p < \infty$ and $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T)$. The map

$$C_X \colon H_A^\infty(\mu) \to \mathcal{S}_p(\mathcal{H}),\ g \mapsto [X, T_g],$$

is well-defined and linear. The continuity will be shown using the closed graph theorem. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $H_A^\infty(\mu)$ with $\tau_{\|\cdot\|_{L^\infty(\mu)}}$-limit g such that $(C_X(g_k))_{k\in\mathbb{N}}$ converges in $\tau_{\|\cdot\|_p}$ to some $S \in \mathcal{S}_p(\mathcal{H})$. With

$$\|C_X(g) - C_X(g_k)\| \leq 2\,\|X\|\,\|T_{g-g_k}\| \leq 2\,\|X\|\,\|g - g_k\|_{L^\infty(\mu)} \to 0$$

as $k \to \infty$ we obtain that

$$\begin{aligned} \|C_X(g) - S\| &\leq \|C_X(g) - C_X(g_k)\| + \|C_X(g_k) - S\| \\ &\leq \|C_X(g) - C_X(g_k)\| + \|C_X(g_k) - S\|_p \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Hence $S = C_X(g)$.

Since $\Phi_{T,k}(X) - X$ lies in the convex hull of $\{T_\theta^* C_X(\theta)\ ;\ \theta \in I_\mu\}$, denoted by $\operatorname{Conv}(\{T_\theta^* C_X(\theta)\ ;\ \theta \in I_\mu\})$, for all $k \geq 1$, we obtain that

$$\Phi_T(X) - X \in \overline{\operatorname{Conv}}^{\tau_{w^*}}(\{T_\theta^* C_X(\theta)\ ;\ \theta \in I_\mu\}),$$

i.e., there exists a net $(S_i)_{i\in I}$ in $\operatorname{Conv}(\{T_\theta^* C_X(\theta)\ ;\ \theta \in I_\mu\})$ such that

$$\Phi_T(X) - X = \tau_{w^*}\text{-}\lim_{i\in I} S_i = \tau_{\mathrm{WOT}}\text{-}\lim_{i\in I} S_i.$$

Furthermore, we have

$$\|S_i\|_p \leq \|C_X\|$$

for all $i \in I$ and thus

$$\|\Phi_T(X) - X\|_p = \left\|\tau_{\mathrm{WOT}}\text{-}\lim_{i\in I} S_i\right\|_p \leq \liminf_{i\in I} \|S_i\|_p \leq \|C_X\|,$$

where we have used Proposition 7.16. We conclude that

$$X = \Phi_T(X) + (X - \Phi_T(X)) \in \mathcal{T}^{(\mathrm{a})}(T) + \mathcal{S}_p(\mathcal{H}). \qquad \square$$

The upcoming result is our main result about Schatten-class perturbations of analytic Toeplitz operators.

7.18 Theorem. *Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_A)$. Furthermore, let $p \in [1,\infty)$ and $X \in B(\mathcal{H})$. The following statements are equivalent:*

(i) $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T)$,

(ii) $X = T_Y + S$ *for some* $S \in \mathcal{S}_p(\mathcal{H})$ *and a symbol* $Y \in (U)'$ *with* $H_Y \in \mathcal{S}_p(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$.

If in addition $W^*(U)$ is a maximal abelian von Neumann algebra, then the above are also equivalent to

(iii) $X = T_f + S$ *for some* $S \in \mathcal{S}_p(\mathcal{H})$ *and* $f \in L^\infty(\mu)$ *with* $H_f \in \mathcal{S}_p(\mathcal{H}, \hat{\mathcal{H}} \ominus \mathcal{H})$.

Proof. (i) $\implies$ (ii): Let $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T)$. By Propositions 6.23 and 7.17, there exist $Y \in (U)'$ and $S \in \mathcal{S}_p(\mathcal{H})$ such that

$$X = T_Y + S.$$

With Lemma 7.9 we conclude that

$$H_Y = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta \in I_\mu} P_\theta H_Y = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta \in I_\mu} H_{\overline{\theta}}(H_{\overline{\theta}}^* H_Y) = \tau_{\mathrm{SOT}}\text{-}\lim_{\theta \in I_\mu} H_{\overline{\theta}}[T_Y, T_\theta].$$

Furthermore, we have $[T_Y, Z] \in \mathcal{S}_p(\mathcal{H})$ for all $Z \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{c})}(T)$. With the notations from the proof of Proposition 7.17, we obtain that

$$\begin{aligned}
\|H_Y\|_p &= \left\| \tau_{\mathrm{WOT}}\text{-}\lim_{\theta \in I_\mu} H_{\overline{\theta}}[T_Y, T_\theta] \right\|_p \\
&\leq \liminf_{\theta \in I_\mu} \|H_{\overline{\theta}}[T_Y, T_\theta]\|_p \\
&\leq \liminf_{\theta \in I_\mu} \|H_{\overline{\theta}}\| \, \|[T_Y, T_\theta]\|_p \\
&\leq \|C_{T_Y}\|,
\end{aligned}$$

where we have used Proposition 7.16.

(ii) $\implies$ (i): This implication follows from the fact that

$$[T_Y, T_g] = H_{\overline{g}}^* H_Y$$

holds for all $Y \in (U)'$ and $g \in H_A^\infty(\mu)$.

The rest follows from Proposition 6.23 (ii). $\square$

7.19 Corollary. *Let $d \geq 2$. If either*

$$D = \mathbb{B}_d \text{ and } 1 \leq p \leq 2d \quad \text{or} \quad D = \mathbb{D}^d \text{ and } 1 \leq p < \infty,$$

then the identity

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T_z) = \mathcal{T}_{\mathrm{a}}(T_z) + \mathcal{S}_p(H^2(\sigma))$$

holds, where σ is the canonical probability measure on the Shilov boundary $\partial_{A(D)}$ of the function algebra $A(D)$ and $H^2(\sigma) = \overline{A(D)|_{\partial_{A(D)}}}^{\|\cdot\|_{L^2(\sigma)}}$.

Proof. The inclusion

$$\mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T_z) \supset \mathcal{T}_{\mathrm{a}}(T_z) + \mathcal{S}_p(H^2(\sigma))$$

is clear.

For the other inclusion, let $X \in \mathcal{T}_{\mathrm{a}}^{(\mathrm{a},p)}(T_z)$. By Theorem 7.18, there exist $S \in \mathcal{S}_p(H^2(\sigma))$ and $f \in L^\infty(\mu)$ with $H_f \in \mathcal{S}_p(H^2(\sigma), L^2(\sigma) \ominus H^2(\sigma))$ such that $X = T_f + S$. In the ball-case, [40, Theorem 1.5] yields that $H_f = 0$. In the polydisc-case, [23, Corollary 5] yields again $H_f = 0$. In both cases we obtain $f \in H^\infty(\sigma)$. □

8. Toeplitz operators

The Hardy space $H^2(\mathbb{D})$ on the unit disc can be identified with the closed subspace $H^2(m)$ of $L^2(m)$ consisting of all functions which have vanishing negative Fourier coefficients. By the Brown-Halmos condition [16, Theorem 6], an operator $X \in B(H^2(m))$ is a Toeplitz operator, i.e., there exists $f \in L^\infty(m)$ such that $X = T_f$, if and only if

$$T_\theta^* X T_\theta - X = 0$$

for all inner functions θ on $\mathbb{D}$. In [32, Exercise 7.38], Douglas asked if compact perturbations of Toeplitz operators are the only operators such that $T_\theta^* X T_\theta - X$ is compact for every inner function θ. More than 30 years later in [64], Xia validated this conjecture.

The goal of this chapter is to give an analogue of this result for the ideal of Schatten-class operators in a more general setting.

Suppose for this chapter that $\mathcal{H}$ is Hilbert space, $D \subset \mathbb{C}^d$ is a bounded domain, and that $T \in B(\mathcal{H})^d$ is a regular $A(D)$-isometry with minimal normal extension $U \in B(\hat{\mathcal{H}})^d$ and scalar spectral measure $\mu \in M_1^+(\partial_{A(D)})$.

To obtain our main result, Theorem 8.12, we have to restrict ourselves to a special class of domain algebras $A(D)$, which will be introduced in the upcoming section.

The results of this chapter are included in [30].

8.1. Henkin measures

We start this section with some basic properties of the algebra of bounded analytic functions on D. For bounded domains $D \subset \mathbb{C}$, the following two results can be found in [24, Lemmas 14.1.5 and 14.1.6]. In the multivariable case they can be proved in exactly the same way.

8.1 Proposition. *The following statements hold:*

(i) the space $H^\infty(D) \subset L^\infty(D) = L^1(D)'$ is τ_{w^*}-closed,

(ii) the space $L^1(D)/^{\perp}H^\infty(D)$ is separable and

$$H^\infty(D) \cong \big(L^1(D)/^{\perp}H^\infty(D)\big)',$$

(iii) the closed norm unit ball of $H^\infty(D)$, $\overline{B}_1^{H^\infty(D)}(0)$, equipped with the relative topology of the weak topology of $H^\infty(D)$ is a compact metrizable space.*

8.2 Proposition. *Let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence in $H^\infty(D)$ and $\theta \in H^\infty(D)$. Then the following conditions are equivalent:*

(i) $\tau_{w^*}\text{-}\lim_{k\to\infty}\theta_k = \theta$ in $L^\infty(D)$,

(ii) $\sup_{k\in\mathbb{N}}\|\theta_k\|_{H^\infty(D)} < \infty$ and $\tau_c\text{-}\lim_{k\to\infty}\theta_k = \theta$,

(iii) $\sup_{k\in\mathbb{N}}\|\theta_k\|_{H^\infty(D)} < \infty$ and $\tau_\pi\text{-}\lim_{k\to\infty}\theta_k = \theta$.

Here, τ_π denotes the topology of pointwise convergence on $H^\infty(D)$ and τ_c denotes the topology of compact convergence on $H^\infty(D)$.

On the unit disc it is well known that the map

$$r_m\colon H^\infty(\mathbb{D}) \to L^\infty(m),\ f \mapsto f^*,$$

where f^* denotes the non-tangential limit of $f \in H^\infty(\mathbb{D})$ and m is the canonical probability measure on the unit circle $\mathbb{T}$, is isometric, τ_{w^*}-continuous and satisfies $r_m(f|_{\mathbb{D}}) = [f|_{\mathbb{T}}]$ for all $f \in A(\mathbb{D})$. The next definition is a generalization of this fact.

8.3 Definition. We call μ a *(faithful) Henkin measure* if there exists a contractive (isometric) τ_{w^*}-continuous algebra homomorphism

$$r_\mu\colon H^\infty(D) \to L^\infty(\mu),\ f \mapsto r_\mu(f) =: f^*$$

with $r_\mu(f|_D) = [f|_{\partial_{A(D)}}]$ for all $f \in A(D)$.

8.4 Remark. Let X, Y be Banach spaces and let $r\colon X' \to Y'$ be an isometric, τ_{w^*}-continuous linear map. Since r is the adjoint of a continuous map $r_*\colon Y \to X$, the image $r(X') \subset Y'$ is τ_{w^*}-closed. Since the relative topology of the weak* topology of Y' on $r(X')$ is the weak* topology of $r(X') \cong (Y/^{\perp}r(X'))'$, it follows that

$$r\colon X' \to r(X')$$

is a dual algebra isomorphism.

8.5 Proposition. *If we suppose that* $\mu \in M_1^+(\partial_{A(D)})$ *is a faithful Henkin measure, then we have*

$$H^\infty_{A(D)|_{\partial_{A(D)}}}(\mu) \subset \operatorname{Im}(r_\mu).$$

Proof. Since

$$A(D)|_{\partial_{A(D)}} \subset \operatorname{Im}(r_\mu)$$

and $\operatorname{Im}(r_\mu)$ is τ_{w^*}-closed by Remark 8.4, we obtain that

$$H^\infty_{A(D)|_{\partial_{A(D)}}}(\mu) = \overline{A(D)|}^{\tau_{w^*}}_{\partial_{A(D)}} \subset \operatorname{Im}(r_\mu). \qquad \square$$

The next lemma will enable us to switch between different kinds of limits.

8.6 Lemma. *Let* $(\theta_k)_{k\in\mathbb{N}}$ *be a sequence in* $\overline{B}_1^{H^\infty(D)}(0)$. *Then the following statements are equivalent:*

(i) there exists $w \in D$ *such that* $\lim_{k\to\infty} \theta_k(w) = 1$,

(ii) τ_{w^*}-$\lim_{k\to\infty} \theta_k = 1$ *in* $H^\infty(D)$.

Furthermore, if $\mu \in M_1^+(\partial_{A(D)})$ *is a faithful Henkin measure, then the above conditions are also equivalent to*

(iii) τ_{w^*}-$\lim_{k\to\infty} \theta_k^* = 1$ *in* $L^\infty(\mu)$.

Proof. Let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence in $\overline{B}_1^{H^\infty(D)}(0)$.

(i) $\implies$ (ii): Let $(\theta_{k_l})_{l\in\mathbb{N}}$ be a subsequence of $(\theta_k)_{k\in\mathbb{N}}$. By Montel's theorem, there exists a τ_{w^*}convergent subsequence $(\theta_{k_{l_m}})_{m\in\mathbb{N}}$ of $(\theta_{k_l})_{l\in\mathbb{N}}$. Let $\theta \in H^\infty(D)$ be its τ_{w^*}-limit. Then $\|\theta\|_{H^\infty(D)} \le 1$, and, by Proposition 8.2, we have

$$\theta(w) = \lim_{m\to\infty} \theta_{k_{l_m}}(w) = 1.$$

By the maximum modulus principle, we obtain $\theta \equiv 1$. Hence, every subsequence of $(\theta_k)_{k\in\mathbb{N}}$ has a τ_{w^*}-convergent subsequence with limit $\theta \equiv 1$. But then

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \theta_k = 1 \text{ in } H^\infty(D).$$

(ii) $\implies$ (i): This follows immediately from Proposition 8.2.

(iii) $\iff$ (ii): Since

$$r_\mu \colon H^\infty(D) \to \operatorname{Im}(r_\mu)$$

is a τ_{w^*}-homeomorphism by Remark 8.4, the result follows. $\square$

8.2. Products of inner functions

Let $D \subset \mathbb{C}^d$ be a bounded domain and let $A = A(D)$. Let $T \in B(\mathcal{H})^d$ be a regular $A(D)$-isometry such that its scalar spectral measure μ is a faithful Henkin measure.

Since infinite products of inner functions are essential in the construction of our central proposition (Proposition 8.10), we state some properties of such products in this section. We start with an observation about finite products.

8.7 Remark. Let $N \in \mathbb{N}$, $x_0, \dots, x_N \in [-1, \infty)$, and $y_0, \dots, y_N \in \mathbb{C}$. If we set $q_N = \prod_{i=0}^N (1 + x_i)$, $p_N = \prod_{i=0}^N (1 + y_i)$ as well as $\tilde{p}_N = \prod_{i=0}^N (1 + |y_i|)$, then

$$|p_N - 1| \leq \tilde{p}_N - 1 \quad \text{and} \quad q_N \leq \exp\left(\sum_{i=0}^N x_i\right).$$

8.8 Lemma. *Let $w \in D$ and let $(\eta_N)_{N \in \mathbb{N}}$ be a sequence in $\mathcal{O}(D)$ with $|\eta_{N+1}| \leq |\eta_N|$ for all $N \in \mathbb{N}$ and such that the limit $\lim_{N \to \infty} \eta_N(w)$ exists in $\mathbb{C} \setminus \{0\}$. Then:*

(i) Every subsequence of $(\eta_N)_{N \in \mathbb{N}}$ has a τ_c-convergent subsequence.

(ii) If η and $\tilde{\eta}$ are τ_c-limits of two subsequences of $(\eta_N)_{N \in \mathbb{N}}$, then $\eta = \tilde{\eta}$.

(iii) The sequence $(\eta_N)_{N \in \mathbb{N}}$ is τ_c-convergent to some function $\eta \in \mathcal{O}(D)$ with $\eta(w) \neq 0$.

Proof. (i) Let $(\eta_{N_k})_{k \in \mathbb{N}}$ be a subsequence of $(\eta_N)_{N \in \mathbb{N}}$ and let $K \subset D$ be compact. Then

$$|\eta_{N_k}(z)| \leq |\eta_0(z)| \leq \|\eta_0\|_K$$

for all $k \in \mathbb{N}$ and $z \in K$. By Montel's theorem, the sequence $(\eta_{N_k})_{k \in \mathbb{N}}$ has a τ_c-convergent subsequence.

(ii) Let $(\eta_{N_k})_{k \in \mathbb{N}}$ and $(\eta_{\tilde{N}_k})_{k \in \mathbb{N}}$ be subsequences of $(\eta_N)_{N \in \mathbb{N}}$ with τ_c-limits η and $\tilde{\eta}$, respectively. By assumption, we have

$$\eta(w) = \tilde{\eta}(w) = \lim_{N \to \infty} \eta_N(w) \neq 0$$

and, since $|\eta_{N+1}| \leq |\eta_N|$ for all $N \in \mathbb{N}$,

$$|\eta(z)| = \lim_{N \to \infty} |\eta_N(z)| = |\tilde{\eta}(z)|$$

for all $z \in D$. Hence,

$$\left|\frac{\eta}{\tilde{\eta}}\right| \equiv 1$$

on the complement of the zero set $Z(\tilde{\eta})$ of $\tilde{\eta}$ in D. By Riemann's extension theorem and the open mapping principle (cf. [36]), it follows that

$$\frac{\eta}{\tilde{\eta}} = \tau$$

on $D \setminus Z(\tilde{\eta})$ for some $\tau \in \mathbb{T}$. But since $\eta(w) = \tilde{\eta}(w) \neq 0$, we obtain that $\eta = \tilde{\eta}$.

(iii) The assertion in (iii) follows immediately from the preceding parts and the condition that $\lim_{N\to\infty} \eta_N(w)$ exists in $\mathbb{C} \setminus \{0\}$. □

Denote by

$$I_D = \{\theta \in H^\infty(D) \;;\; \theta^* \in I_\mu\}$$

the set of *inner functions on* D with respect to μ.

8.9 Proposition. *Let $(\theta_j)_{j\in\mathbb{N}}$ be a sequence in $\overline{B}_1^{H^\infty(D)}(0)$ such that, for some point $w \in D$, $\theta_j(w) \neq 0$ for all $j \in \mathbb{N}$ and*

$$\sum_{j=0}^{\infty} |1 - \theta_j(w)| < \infty.$$

Then:

(i) The sequence $(\eta_N)_{N\in\mathbb{N}} = \left(\prod_{j=0}^{N} \theta_j\right)_{N\in\mathbb{N}}$ is τ_{w^*}-convergent in $H^\infty(D)$ to some function $\eta \in \overline{B}_1^{H^\infty(D)}(0)$ with $\eta(w) \neq 0$.

(ii) For each $N \in \mathbb{N}$, the infinite product

$$\rho_N = \prod_{j=N+1}^{\infty} \theta_j \in \overline{B}_1^{H^\infty(D)}(0)$$

converges uniformly on all compact subsets of D and

$$\tau_{w^*}\text{-}\lim_{N\to\infty} \rho_N^* = 1 \quad \text{in } L^\infty(\mu).$$

(iii) We have

$$\tau_{\|\cdot\|_{L^2(\mu)}}\text{-}\lim_{N\to\infty} \eta_N^* = \eta^*.$$

(iv) If $(\theta_j)_{j\in\mathbb{N}}$ is a sequence in I_D, then $\eta \in I_D$.

Proof. (i) It is clear that $\lim_{N\to\infty}\eta_N(w) \in \mathbb{C}\setminus\{0\}$ exists (cf. [60, Theorem 15.4]). Furthermore, we have

$$|\eta_{N+1}| = |\eta_N|\,|\theta_{N+1}| \leq |\eta_N| \leq \ldots \leq |\eta_0| = |\theta_0| \leq 1$$

for all $N \in \mathbb{N}$. With Lemma 8.8 and Proposition 8.2 it follows that $(\eta_N)_{N\in\mathbb{N}}$ converges in τ_{w^*} to some function $\eta \in \overline{B}_1^{H^\infty(D)}(0)$ with $\eta(w) \neq 0$.

(ii) Part (i) applied to the sequences $(\theta_j)_{j\geq N+1}$ yields that the products

$$\rho_N = \prod_{j=N+1}^{\infty} \theta_j \in \overline{B}_1^{H^\infty(D)}(0)$$

converge uniformly on all compact subsets, or equivalently, with respect to the weak* topology of $H^\infty(D)$ (cf. Proposition 8.2). By Remark 8.7, we have

$$\begin{aligned}
|\rho_N(w) - 1| &= \lim_{k\to\infty}\left|\prod_{j=N+1}^{k} 1 + (\theta_j(w) - 1) - 1\right| \\
&\leq \lim_{k\to\infty}\left(\prod_{j=N+1}^{k} (1 + |\theta_j(w) - 1|) - 1\right) \\
&\leq \lim_{k\to\infty} \exp\left(\sum_{j=N+1}^{k} |\theta_j(w) - 1|\right) - 1 \\
&\to 0
\end{aligned}$$

as $N \to \infty$, i.e.,

$$\lim_{N\to\infty} \rho_N(w) = 1.$$

Hence,

$$\tau_{w^*}\text{-}\lim_{N\to\infty} \rho_N^* = 1 \text{ in } L^\infty(\mu)$$

by Lemma 8.6.

(iii) Since

$$\eta^* = \eta_N^* \rho_N^*$$

for all $N \in \mathbb{N}$, we obtain that

$$\begin{aligned}
\|\eta_N^* - \eta\|_{L^2(\mu)} &= \|\eta_N^*(1 - \rho_N^*)\|_{L^2(\mu)} \\
&\leq \|\eta_N^*\|_{L^\infty(\mu)}\,\|1 - \rho_N^*\|_{L^2(\mu)} \\
&\leq \|1 - \rho_N^*\|_{L^2(\mu)} \\
&\to 0
\end{aligned}$$

as $N \to \infty$ by the previous parts and Lemma 6.26.

(iv) Suppose that $\theta_j \in I_D$ for all $j \in \mathbb{N}$. Since $\eta_N^* = \prod_{j=0}^N \theta_j^* \in I_\mu$ for each N and since by (iii) the sequence $(\eta_N^*)_{N\in\mathbb{N}}$ has a subsequence that converges pointwise μ-almost everywhere to η^*, it follows that $\eta^* = \tau_{w^*}\text{-}\lim_{N\to\infty} \eta_N^* \in I_\mu$. Thus $\eta \in I_D$. □

8.3. Schatten-class perturbations of Toeplitz operators

Let $T \in B(\mathcal{H})^d$ be a regular A-isometry with respect to $A = A(D)$, where $D \subset \mathbb{C}^d$ is a bounded domain such that the associated scalar spectral measure $\mu \in M_1^+(\partial_{A(D)})$ is a faithful Henkin probability measure on D.

The following proposition is an adaption of [64, Lemma 5].

8.10 Proposition. *Let $p \in [1, \infty]$ and $X \in \mathcal{T}^{(\mathrm{a},p)}(T)$. Then, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|T_{\theta^*}^* X T_{\theta^*} - X\|_p \leq \varepsilon$$

for all $\theta \in I_D$ with $\left|\int_{\partial_{A(D)}} 1 - \theta^* \,\mathrm{d}\mu\right| \leq \delta$.

Proof. We first notice that if $(\theta_k)_{k\in\mathbb{N}}$ is a sequence in $\overline{B}_1^{H^\infty(D)}(0)$ such that

$$\lim_{k\to\infty} \int_{\partial_{A(D)}} \theta_k^* \,\mathrm{d}\mu = 1,$$

then, by Lemma 6.26, we obtain that

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \theta_k^* = 1 \quad \text{in } L^\infty(\mu).$$

By Lemma 8.6, there exists $w \in D$ such that

$$\lim_{k\to\infty} \theta_k(w) = 1.$$

We use this observation to prove the claim of the proposition by contradiction.

Assume that there is an $\varepsilon > 0$ such that there exist $w \in D$ and, for all $k \in \mathbb{N}$, a function $\theta_k \in I_D$ with

$$|1 - \theta_k(w)| \leq \frac{1}{2^k} \quad \text{and} \quad \left\|T_{\theta_k^*}^* X T_{\theta_k^*} - X\right\|_p > \varepsilon.$$

We construct a strictly increasing sequence $(k(j))_{j\geq 1}$ of natural numbers and a sequence $(F_j)_{j\geq 1}$ of orthogonal projections with finite rank which are pairwise orthogonal and satisfy the inequalities

$$\begin{aligned} \|F_{j+1}W_jF_{j+1}\|_p &\geq \frac{\varepsilon}{2}, \\ \|(\mathrm{id}_{\mathcal{H}} - F_{j+1})W_j\|_p &\leq 2^{-j}, \\ \|W_j(\mathrm{id}_{\mathcal{H}} - F_{j+1})\|_p &\leq 2^{-j} \end{aligned}$$

for all $j \geq 1$, where

$$W_j = T^*_{\eta_j^*}\left(T^*_{\theta^*_{k(j+1)}} X T_{\theta^*_{k(j+1)}} - X\right) T_{\eta_j^*} \quad \text{and} \quad \eta_j = \prod_{m=1}^{j} \theta_{k(m)}$$

for all $j \geq 1$.

Set $k(1) = 1$ and $F_1 = 0$.

Suppose $k(1) < \ldots < k(j)$ and $F_1, \ldots, F_j$ with the desired properties are already defined. We set

$$G_j = \sum_{m=1}^{j} F_m.$$

Then, by Lemmas 6.27 and 8.6, we obtain

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T^*_{\theta^*_k} X T_{\theta^*_k} - X = 0 \quad \text{and} \quad \tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} T^*_{\theta^*_k} X^* T_{\theta^*_k} - X^* = 0,$$

and hence, by Lemma 6.5,

$$\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} W_{j,k} = 0 \quad \text{and} \quad \tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} W^*_{j,k} = 0$$

with

$$W_{j,k} = T^*_{\eta_j^*}\left(T^*_{\theta^*_k} X T_{\theta^*_k} - X\right) T_{\eta_j^*} \in \mathcal{S}_p(\mathcal{H})$$

for all $k \geq 0$. Since G_j has finite rank and hence lies in $\mathcal{S}_p(\mathcal{H})$, it follows from Proposition 6.4 that

$$\lim_{k\to\infty}\left(\|G_jW_{j,k}\|_p + \|W_{j,k}G_j\|_p\right) = \lim_{k\to\infty}\left(\left\|W^*_{j,k}G_j\right\|_p + \|W_{j,k}G_j\|_p\right) = 0.$$

By Lemma 6.27 (iii), we have

$$\begin{aligned} 0 &\leq \left\|T^*_{\theta^*_k} X T_{\theta^*_k} - X\right\|_p - \|W_{j,k}\|_p \\ &\leq \left\|T^*_{\theta^*_k} X T_{\theta^*_k} - X - W_{j,k}\right\|_p \\ &= \left\|T^*_{\eta_j^*}\left(T^*_{\theta^*_k} X T_{\theta^*_k} - X\right) T_{\eta_j^*} - \left(T^*_{\theta^*_k} X T_{\theta^*_k} - X\right)\right\|_p \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Thus, we can choose $k(j+1) > k(j)$ such that

$$\begin{aligned} \|W_j\|_p &\geq \frac{5}{6}\varepsilon, \\ \|G_j W_j\|_p &\leq \min\left(2^{-j-1}, \frac{1}{12}\varepsilon\right), \\ \|W_j G_j\|_p &\leq \min\left(2^{-j-1}, \frac{1}{12}\varepsilon\right). \end{aligned}$$

Since by Proposition 6.4

$$\|P_k W_j P_k - W_j\|_p \leq \|(P_k - \mathrm{id}_{\mathcal{H}})W_j\|_p + \|W_j(P_k - \mathrm{id}_{\mathcal{H}})\|_p \to 0$$

as $k \to \infty$ for each sequence $(P_k)_{k\in\mathbb{N}}$ of orthogonal projections on $\mathcal{H}$ with $\tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} P_k = \mathrm{id}_{\mathcal{H}}$, there exists a finite-rank orthogonal projection $G \geq G_j$ with

$$\begin{aligned} \|G W_j G\|_p &\geq \frac{9}{12}\varepsilon, \\ \|(\mathrm{id}_{\mathcal{H}} - G)W_j\|_p &\leq 2^{-j-1}, \\ \|W_j(\mathrm{id}_{\mathcal{H}} - G)\|_p &\leq 2^{-j-1}. \end{aligned}$$

Furthermore, we set

$$F_{j+1} = G - G_j.$$

Then F_{j+1} is orthogonal to $F_1, \ldots, F_j$ and

$$\begin{aligned} \|F_{j+1} W_j F_{j+1}\|_p &= \|(G - G_j)W_j(G - G_j)\|_p \\ &\geq \|G W_j G\|_p - \|G_j W_j G\|_p - \|G W_j G_j\|_p - \|G_j W_j G_j\|_p \\ &\geq \|G W_j G\|_p - \frac{1}{4}\varepsilon \\ &\geq \frac{9}{12}\varepsilon - \frac{1}{4}\varepsilon \\ &= \frac{1}{2}\varepsilon. \end{aligned}$$

We also have

$$\begin{aligned} \|(\mathrm{id}_{\mathcal{H}} - F_{j+1})W_j\|_p &= \|(\mathrm{id}_{\mathcal{H}} - G + G_j)W_j\|_p \\ &\leq \|(\mathrm{id}_{\mathcal{H}} - G)W_j\|_p + \|G_j W_j\|_p \\ &\leq 2^{-j-1} + 2^{-j-1} \\ &= 2^{-j} \end{aligned}$$

as well as

$$\begin{aligned}\|W_j(\mathrm{id}_{\mathcal{H}} - F_{j+1})\|_p &= \|W_j(\mathrm{id}_{\mathcal{H}} - G + G_j)\|_p \\ &\leq \|W_j(\mathrm{id}_{\mathcal{H}} - G)\|_p + \|W_j G_j\|_p \\ &\leq 2^{-j-1} + 2^{-j-1} \\ &= 2^{-j}.\end{aligned}$$

Thus, F_{j+1} fullfills the desired properties.

For $j \geq 1$, define

$$S_j = W_j - F_{j+1} W_j F_{j+1} \in \mathcal{S}_p(\mathcal{H}).$$

Then

$$\begin{aligned}\|S_j\|_p &= \|W_j - F_{j+1} W_j F_{j+1}\|_p \\ &\leq \|(\mathrm{id}_{\mathcal{H}} - F_{j+1}) W_j\|_p + \|F_{j+1} W_j (\mathrm{id}_{\mathcal{H}} - F_{j+1})\|_p \\ &\leq 2^{-j} + 2^{-j} \\ &= 2^{-j+1}\end{aligned}$$

for all $j \geq 1$, and hence,

$$S = \tau_{\|\cdot\|_p}\text{-}\sum_{j=1}^{\infty} S_j \in \mathcal{S}_p(\mathcal{H})$$

is well defined. As an orthogonal direct sum of a bounded sequence of operators, the series

$$B = \tau_{\mathrm{SOT}}\text{-}\sum_{j=1}^{\infty} F_{j+1} W_j F_{j+1} \in B(\mathcal{H})$$

exists. The operator B does not lie in $\mathcal{S}_p(\mathcal{H})$.

To see this, assume $B \in \mathcal{S}_p(\mathcal{H})$ and observe that

$$\frac{1}{2}\varepsilon \leq \|F_{j+1} W_j F_{j+1}\|_p = \|F_{j+1} B F_{j+1}\|_p \leq \|F_{j+1} B\|_p$$

for all $j \geq 1$. Since

$$\tau_{\mathrm{SOT}}\text{-}\lim_{j \to \infty} F_{j+1} = 0,$$

we obtain by Proposition 6.4 the contradiction

$$\frac{1}{2}\varepsilon \leq \|F_{j+1} B\|_p \to 0$$

as $j \to \infty$. Hence, $B \notin \mathcal{S}_p(\mathcal{H})$.

With
$$\sum_{j=1}^{\infty} \left|1-\theta_{k(j)}(w)\right| \le \sum_{j=1}^{\infty} 2^{-j} = 1,$$
it follows from Proposition 8.9 that the sequence $(\eta_n^*)_{n\in\mathbb{N}}$ converges in $\tau_{\|\cdot\|_{L^2(\mu)}}$ to
$$\eta^* = \left(\prod_{j=1}^{\infty} \theta_{k(j)}\right)^* \in I_\mu,$$
and with Lemma 6.24 we obtain that
$$\tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} T_{\eta_n^*} = T_{\eta^*} \quad \text{and} \quad \tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} T_{\eta_n^*}^* = T_{\eta^*}^*.$$

By construction, we have
$$\begin{aligned}
S+B &= \tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} W_j \\
&= \tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} T_{\eta_j^*}^*(T_{\theta_{k(j+1)}^*}^* X T_{\theta_{k(j+1)}^*} - X)T_{\eta_j^*} \\
&= \tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} T_{\eta_{j+1}^*}^* X T_{\eta_{j+1}^*} - T_{\eta_j^*}^* X T_{\eta_j^*} \\
&= \tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} \left(T_{\eta_{n+1}^*}^* X T_{\eta_{n+1}^*} - T_{\eta_1^*}^* X T_{\eta_1^*}\right) \\
&= T_{\eta^*}^* X T_{\eta^*} - T_{\eta_1^*}^* X T_{\eta_1^*} \\
&= \left(T_{\eta^*}^* X T_{\eta^*} - X\right) - \left(T_{\eta_1^*}^* X T_{\eta_1^*} - X\right).
\end{aligned}$$

Thus, we obtain the contradiction that
$$\mathcal{S}_p(\mathcal{H}) \not\ni B = (T_{\eta^*}^* X T_{\eta^*} - X) - (T_{\eta_1^*}^* X T_{\eta_1^*} - X) - S \in \mathcal{S}_p(\mathcal{H}). \qquad \square$$

In our setting, the measure μ has an additional nice property.

8.11 Lemma. *The measure μ is continuous.*

Proof. Assume that there exists $z \in \partial_{A(D)}$ such that $\mu(\{z\}) > 0$. Since the map $r_\mu \colon H^\infty(D) \to L^\infty(\mu)$ is injective and since by Propositions 6.17 and 8.5
$$L^\infty(\mu) = W^*(I_\mu) = W^*(r_\mu(I_D)),$$

there exists a non-constant function $\theta \in I_D$. But then

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \theta^k = 0,$$

and therefore,

$$\tau_{w^*}\text{-}\lim_{k\to\infty} r_\mu(\theta^k) = 0.$$

Since $|r_\mu(\theta)(z)| = 1$, we conclude that

$$\begin{aligned}
\mu(\{z\}) &= \mu(\{z\})\, |r_\mu(\theta)(z)|^k \\
&= \left| r_\mu(\theta^k)(z)\mu(\{z\}) \right| \\
&= \left| \left\langle \chi_{\{z\}}, r_\mu(\theta^k) \right\rangle_{\langle L^1(\mu), L^\infty(\mu)\rangle} \right| \\
&\to 0
\end{aligned}$$

as $k \to \infty$. Hence $\mu(\{z\}) = 0$, which is a contradiction. $\square$

We are now able to state our main theorem.

8.12 Theorem. *Let $1 \le p < \infty$ and $X \in B(\mathcal{H})$. The following statements are equivalent:*

(i) $X \in \mathcal{T}^{(\mathrm{a},p)}(T)$,

(ii) $X = T_Y + S$ for some $S \in \mathcal{S}_p(\mathcal{H})$ and a symbol $Y \in (U)'$.

If in addition $W^*(U)$ is a maximal abelian von Neumann algebra, then the above are also equivalent to

(iii) $X = T_f + S$ for some $S \in \mathcal{S}_p(\mathcal{H})$ and $f \in L^\infty(\mu)$.

Proof. (i) $\implies$ (ii): Let $X \in \mathcal{T}^{(\mathrm{a},p)}(T)$. By Proposition 8.10, there exists $1 > \delta > 0$ such that

$$\|T^*_{\theta^*} X T_{\theta^*} - X\|_p \le 1$$

for all $\theta \in I_D$ with $\left|\int_{\partial_{A(D)}} 1 - \theta^* \,\mathrm{d}\mu\right| \le \delta$. Set $\alpha = 1 - \delta/2$. Since the triple $(A(D)|_{\partial_{A(D)}}, \partial_{A(D)}, \mu)$ is regular in the sense of Aleksandrov and μ is continuous, by Proposition 8.5, there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in I_D with $\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha$ in $L^\infty(\mu)$ (cf. [4, Corollary 29]). By passing to a subsequence, we can achieve that $\left|\int_{\partial_{A(D)}} 1 - \alpha_k^* \,\mathrm{d}\mu\right| \le \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$\tilde{X} = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T^*_{\alpha_k^*} X T_{\alpha_k^*} \in B(\mathcal{H})$$

exists. By Proposition 7.16, we conclude that

$$\left\|X - \tilde{X}\right\|_p = \left\|\tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} X - T^*_{\alpha^*_k} X T_{\alpha^*_k}\right\|_p \leq \liminf_{k\to\infty} \left\|X - T^*_{\alpha^*_k} X T_{\alpha^*_k}\right\|_p \leq 1.$$

Hence, $X - \tilde{X} \in \mathcal{S}_p(\mathcal{H})$ and the result follows with Proposition 6.30 (ii).

(ii) $\implies$ (i): This is clear.

The rest follows from Proposition 6.23 (ii). □

It follows from the results in Section 5 of [27] that the canonical probability measure σ on the Shilov boundary of the domain algebra $A(D)$ over a smooth strictly pseudoconvex or bounded symmetric and circled domain $D \subset \mathbb{C}^d$ is a faithful Henkin measure. Since σ is a scalar spectral measure of the minimal normal extension $M_z \in B(L^2(\sigma))^d$ of the regular $A(D)$-isometry $T_z \in B(H^2(\sigma))^d$ and since $W^*(M_z) \subset B(L^2(\sigma))$ is a maximal abelian von Neumann algebra, Theorem 8.12 applies to this setting and yields the following corollary.

8.13 Corollary. *Let $D \subset \mathbb{C}^d$ be a smooth strictly pseudoconvex domain or a bounded symmetric and circled domain, and let $1 \leq p < \infty$. Then a given operator $X \in B(H^2(\sigma))$ is of the form*

$$X = T_f + S,$$

where $T_f \in B(H^2(\sigma))$ is a Toeplitz operator with symbol $f \in L^\infty(\sigma)$ and $S \in \mathcal{S}_p(H^2(\sigma))$, if and only if $T_{\theta^*} X T_{\theta^*} - X \in \mathcal{S}_p(H^2(\sigma))$ for every inner function θ on D.

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