Toeplitz and Hankel operators on weighted Bergman spaces and the Fock space

Master's thesis

submitted by

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Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Saarbrücken, 20.08.2014

Dominik Schillo

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Contents

Introduction

The study of Toeplitz and Hankel operators was started by taking a closer look at so called Toeplitz and Hankel matrices. These infinite matrices are constant on each line parallel to the main diagonal or depend only on the sum of the coordinates, respectively. It turned out (cf. [6]) that these can also be seen as representations of the operators

$$T_f \colon H^2(\mathbb{T}) \to H^2(\mathbb{T}),\ g \mapsto P(fg)$$

and

$$H_f \colon H^2(\mathbb{T}) \to L^2(\mathbb{T}),\ g \mapsto (I-P)(fg),$$

where $L^2(\mathbb{T})$ denotes the Lebesgue space of square integrable functions on the unit circle and P denotes the orthogonal projection from $H^2(\mathbb{T})$ to the Hardy space on the unit circle, for some essentially bounded function f.
Since the Hardy space can be replaced by the Bergman space and the Fock space, one can ask the question which operator theoretical properties the corresponding operators have. For instance, one can ask under which conditions these operators are compact. If we also drop the restriction that the domains of the functions lie in the complex plane $\mathbb{C}$, i.e., if we allow domains to be in $\mathbb{C}^n$ for a positive integer n, the study of Toeplitz and Hankel operators intersects with the theory of complex analysis in several variables.
Towards the end of the last century K. Stroethoff and D. Zheng gave a characterization of compact Toeplitz and Hankel operators for the two spaces of analytic functions mentioned above in [12] and [10]. With the help of this characterization they were able to prove the following explicit formula for the essential spectrum of a Toeplitz operator with special symbols.

Theorem. *Let f be an essentially bounded function on the unit ball $\mathbb{B}_n$ such that H_f is compact. If we denote by $\widetilde{f}$ the Berezin transform of f, then we have*

$$\sigma_e(T_f) = \bigcap_{0<r<1} \operatorname{cl}(\widetilde{f}(\mathbb{B}_n \setminus r\mathbb{B}_n)),$$

where $\sigma_e(T_f)$ denotes the essential spectrum of T_f.

The result for the other cases looks similar. Since the Berezin tranform of such a function f is continuous, the connectedness of the essential spectrum for such Toeplitz operators follows at once.

In this thesis we present the above results in detail and prove a similar result for weighted Bergman spaces as conjectured in [9] and [11] .
The first chapter contains some basic facts of weighted Bergman spaces and the Fock space. Furthermore, we will introduce the Berezin transform of a function and recall some basics about Toeplitz, Hankel and Hilbert-Schmidt operators. The characterization of compact Toeplitz and Hankel operators and some of its corollaries are presented in Chapter 2, where the interplay between the automorphisms and the reproducing kernels of the spaces becomes important. The last two chapters deal with the question when Hankel operators with Berezin transformed symbols are compact as well as the formula for the essential spectrum mentioned above.

Chapter 1

Preliminaries

In this chapter we gather some basic results about Möbius transformations, Bergman spaces over the unit ball and the polydisc in $\mathbb{C}^n$ ($n \in \mathbb{N}^*$) and the Fock space. We also recall the definitions of Toeplitz and Hankel operators as well as the definition of the Berezin transform. In the last section we introduce Hilbert-Schmidt operators.
For the rest of this thesis, let $n \in \mathbb{N}^*$ be a fixed positive integer. We denote by $\mathbb{B}_n = \{z \in \mathbb{C}^n;\ \|z\| < 1\}$ the open unit ball and by $\mathbb{D}^n = \{z \in \mathbb{C}^n;\ |z_i| < 1 \text{ for } 1 \leq i \leq n\}$ the (open unit) polydisc in $\mathbb{C}^n$.
Furthermore, we write, for a measure space (X, Σ, ν), $L^p(X, \nu) = L^p(\nu)$ $(1 \leq p \leq \infty)$ for the L^p-space over X relative to the measure ν and we identify an element $f \in L^p(\nu)$ with a representative such that $\sup_{x \in X} |f(x)| = \|f\|_{L^\infty(\nu)}$.
For a subset D of a topological space X, we denote by $\mathrm{cl}(D)$ and ∂D the topological closure and the topological boundary of D, respectively.

1.1 Möbius transformations

We first recall the definitions of the Möbius transformations for the unit ball and the polydisc.

Definition 1.1. (i) Let $\lambda \in \mathbb{B}_n$, $s_\lambda = (1 - |\lambda|^2)^{\frac{1}{2}}$ and

$$P_\lambda \colon \mathbb{C}^n \to \bigvee \{\lambda\},\ z \mapsto \begin{cases} \frac{\langle z, \lambda \rangle}{\langle \lambda, \lambda \rangle} \lambda, & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$$

We call the function

$$\varphi_\lambda^{\mathbb{B}_n} \colon \mathbb{B}_n \to \mathbb{C}^n,\ z \mapsto \frac{\lambda - P_\lambda z - s_\lambda (I - P_\lambda) z}{1 - \langle z, \lambda \rangle}$$

a *Möbius transformation* on the unit ball.

(ii) Let $\lambda \in \mathbb{D}^n$. We call the function

$$\varphi_\lambda^{\mathbb{D}^n} : \mathbb{D}^n \to \mathbb{C}^n,\ z \mapsto \left(\frac{\lambda_1 - z_1}{1 - z_1 \overline{\lambda_1}}, \ldots, \frac{\lambda_n - z_n}{1 - z_n \overline{\lambda_n}} \right)$$

a *Möbius transformation* on the polydisc.

Remark 1.2. If $n = 1$ and $\lambda \in \mathbb{D}$, then $\varphi_\lambda^{\mathbb{B}_n}$ and $\varphi_\lambda^{\mathbb{D}^n}$ coincide since $P_\lambda = I$ if $\lambda \neq 0$ and $P_\lambda = 0$ if $\lambda = 0$. Furthermore, we obtain the classical Möbius transformations from complex analysis in one variable.

To shorten the notation, we will write Ω for $\mathbb{B}_n$ or $\mathbb{D}^n$ throughout this thesis. The next proposition summarizes important properties of the Möbius transformation and will be used frequently.

Proposition 1.3. *If $\lambda \in \Omega$, then*

(i) $\varphi_\lambda(0) = \lambda$ *and* $\varphi_\lambda(\lambda) = 0$,

(ii) *(a)*

$$1 - \left\langle \varphi_\lambda^{\mathbb{B}_n}(z), \varphi_\lambda^{\mathbb{B}_n}(w) \right\rangle = \frac{(1 - \langle \lambda, \lambda \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, \lambda \rangle)(1 - \langle \lambda, w \rangle)}$$

for all $z, w \in \mathbb{B}_n$ and $\lambda \in \mathbb{B}_n$,

(b)

$$1 - \varphi_\lambda^{\mathbb{D}^n}(z)_i \, \overline{\varphi_\lambda^{\mathbb{D}^n}(w)_i} = \frac{(1 - |\lambda_i|^2)(1 - z_i \overline{w_i})}{(1 - z_i \overline{\lambda_i})(1 - \lambda_i \overline{w_i})} \quad (i = 1, \ldots, n)$$

for all $z, w \in \mathbb{D}^n$ and $\lambda \in \mathbb{D}^n$,

(iii) *(a)*

$$1 - \left| \varphi_\lambda^{\mathbb{B}_n}(z) \right|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \langle z, \lambda \rangle|^2}$$

for all $z \in \mathbb{B}_n$ and $\lambda \in \mathbb{B}_n$,

(b)

$$1 - \left| \varphi_\lambda^{\mathbb{D}^n}(z)_i \right|^2 = \frac{(1 - |\lambda_i|^2)(1 - |z_i|^2)}{\left|1 - z_i \overline{\lambda_i}\right|^2} \quad (i = 1, \ldots, n)$$

for all $z \in \mathbb{D}^n$ and $\lambda \in \mathbb{D}^n$,

(iv) φ_λ *is an involution, i.e.,* $\varphi_\lambda \circ \varphi_\lambda = \mathrm{id}$,

(v) $\varphi_\lambda \in \mathrm{Aut}(\Omega)$ *(the automorphism group of Ω),*

(vi) φ_λ *can be extended to a homeomorphism of* $\mathrm{cl}(\Omega)$ *onto* $\mathrm{cl}(\Omega)$ *(which we will denote with the same symbol).*

Proof. A proof for the case $\Omega = \mathbb{B}_n$ can be found in [8, Theorem 2.2.2]. For the polydisc, use the first case with $n = 1$ for each component. $\square$

Proposition 1.4. *Let $z \in \Omega$ and let $(\lambda_\alpha)_\alpha$ be a net in Ω with limit in $\partial\Omega$. Then $\varphi_{\lambda_\alpha}(z)$ converges to a point in $\partial\Omega$.*

Proof. Suppose that $z \in \Omega$ and let $(\lambda_\alpha)_\alpha$ be a net in Ω with limit $\lambda \in \partial\Omega$. Then, for $\Omega = \mathbb{B}_n$, we have

$$\begin{aligned}\left|\varphi_{\lambda_\alpha}^{\mathbb{B}_n}(z) - \lambda\right| &= \left|\frac{\lambda_\alpha - P_{\lambda_\alpha} z - s_{\lambda_\alpha} Q_{\lambda_\alpha} z - \lambda + \langle z, \lambda_\alpha\rangle \lambda}{1 - \langle z, \lambda_\alpha\rangle}\right| \\ &= \left|\frac{\langle z, \lambda_\alpha\rangle \left(\lambda - \frac{\lambda_\alpha}{|\lambda_\alpha|^2}\right) - s_{\lambda_\alpha} Q_{\lambda_\alpha} z + (\lambda_\alpha - \lambda)}{1 - \langle z, \lambda_\alpha\rangle}\right| \\ &\stackrel{\alpha}{\to} 0.\end{aligned}$$

For $\Omega = \mathbb{D}^n$, we observe that there exists an $i \in \{1, \ldots, n\}$ such that $\lambda_i \in \partial\mathbb{D}$. By the previous case, we have

$$\left|\varphi_{\lambda_{\alpha,i}}^{\mathbb{D}}(z_i) - \lambda_i\right| \to 0,$$

hence the result follows. $\square$

Notation 1.5. We write $\mathcal{O}(\Omega, \mathbb{C}^n)$ for the space of holomorphic functions from Ω to $\mathbb{C}^n$ equipped with the topology of uniform convergence on all compact subsets. We use $\mathcal{O}(\Omega)$ as a shortcut for $\mathcal{O}(\Omega, \mathbb{C})$.

We identity $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the map

$$\mathbb{R}^{2n} \to \mathbb{C}^n, \ (x_1, y_1, \ldots, x_n, y_n) \mapsto (x_1 + iy_1, \ldots, x_n + iy_n).$$

For a function $f \in \mathcal{O}(\Omega)$, we define for $j = 1, \ldots, n$

$$\partial_j f = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right).$$

If $F = (f_1, \ldots, f_n) \in \mathcal{O}(\Omega, \mathbb{C}^n)$, we denote by

$$J_{\mathbb{C}} F = (\partial_i f_j)_{i,j=1}^n$$

the complex Jacobian matrix. With the above identification we can also form the real Jacobian matrix $J_{\mathbb{R}} F$. The next result shows the connection between the determinants of both Jacobian matrices.

Proposition 1.6. *Let $F = (f_1, \ldots, f_n) \in \mathcal{O}(\Omega, \mathbb{C}^n)$ and $z \in \Omega$. Then*

$$\det((J_{\mathbb{R}} F)(z)) = |\det((J_{\mathbb{C}} F)(z))|^2.$$

A proof of this result can be found in [8, Section 1.3.6]. As an application of this proposition we obtain the following result.

Proposition 1.7. *(i) For $\lambda \in \mathbb{B}_n$, we have*

$$\det((J_{\mathbb{R}}\varphi_\lambda)(z)) = \left(\frac{1-|\lambda|^2}{|1-\langle z,\lambda\rangle|^2}\right)^{n+1}$$

for all $z \in \mathbb{B}_n$.

(ii) For $\lambda \in \mathbb{D}^n$, we have

$$\det((J_{\mathbb{R}}\varphi_\lambda)(z)) = \prod_{i=1}^{n}\left(\frac{|1-\lambda_i|^2}{\left|1-z_i\overline{\lambda_i}\right|^2}\right)^2$$

for all $z \in \mathbb{D}^n$.

Proof. Let $\lambda \in \Omega$.

(i) One can find a proof of the statement in [8, Theorem 2.2.6].

(ii) We observe that

$$\partial_i\varphi_\lambda(z)_j = \delta_{i,j}\frac{|\lambda_i|^2-1}{(1-z_i\overline{\lambda_i})^2} \quad (z \in \mathbb{D}^n)$$

for all $i,j = 1,\dots,n$. Hence with Proposition 1.6 we conclude

$$\begin{aligned}\det((J_{\mathbb{R}}\varphi_\lambda)(z)) &= \left|\det\left(\left(\delta_{i,j}\frac{|\lambda_i|^2-1}{(1-z_i\overline{\lambda_i})^2}\right)_{i,j=1}^{n}\right)\right|^2\\ &= \prod_{i=1}^{n}\left(\frac{1-|\lambda_i|^2}{\left|1-z_i\overline{\lambda_i}\right|^2}\right)^2.\end{aligned}$$

□

1.2 Bergman spaces

Let m be the Lebesgue measure on $\mathbb{C}^n$ and let $\gamma > -1$ or $\gamma \in \mathbb{R}^n$ with $\gamma_i > -1$ $(i = 1,\dots n)$ for $\Omega = \mathbb{B}_n$ or $\Omega = \mathbb{D}^n$, respectively.
In the following we denote by $L^p(V_\gamma) = L^p(\Omega, V_\gamma)$ $(1 \leq p \leq \infty)$ the L^p-space over Ω relative to the measure V_γ given by

$$V_\gamma = \frac{\Gamma(n+1+\gamma)}{\pi^n\Gamma(1+\gamma)}(1-|z|^2)^\gamma m|_{\mathbb{B}_n}$$

on the unit ball and

$$V_\gamma = \frac{1}{\pi^n} \prod_{i=1}^{n} \frac{\Gamma(2+\gamma_i)}{\Gamma(1+\gamma_i)} (1-|z_i|^2)^{\gamma_i} m|_{\mathbb{D}^n}$$

on the polydisc. One can check that V_γ is a probability measure on Ω. We use V as a shortcut for V_0.

Definition 1.8 (Bergman space)**.** For $1 \le p < \infty$, we define the *Bergman space* $A^p(V_\gamma)$ by

$$A^p(V_\gamma) = A^p(\Omega, V_\gamma) = L^p(V_\gamma) \cap \mathcal{O}(\Omega),$$

i.e., $A^p(V_\gamma)$ is the set of all equivalence classes in $L^p(V_\gamma)$ with a holomorphic representative.

Since the holomorphic representative of an equivalence class in $A^p(V_\gamma)$ is uniquely determined, we identify the elements in $A^p(V_\gamma)$ with their holomorphic representative.
Every element in $A^p(V_\gamma)$ satisfies a mean value formula in 0.

Proposition 1.9. *Every $g \in A^1(V_\gamma)$ satisfies*

$$g(0) = \int_\Omega g(w) \mathrm{d}V_\gamma(w).$$

Proof. Suppose $g \in A^1(V_\gamma)$.
We first consider the case $\Omega = \mathbb{B}_n$.
For $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}_n)$ and $0 < r < 1$, we have

$$\int_{\partial \mathbb{B}_n} f(r\xi) \mathrm{d}\sigma(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha r^{|\alpha|} \int_{\partial \mathbb{B}_n} \xi^\alpha \mathrm{d}\sigma(\xi) = a_0 = f(0).$$

Here σ denotes the normalized surface measure on $\partial \mathbb{B}_n$, and we have used [8, Proposition 1.4.8] to see that the series collapses to the summand with index $\alpha = 0$. For $f \in A^1(V_\gamma)$, integration in polar coordinates ([8, Section 1.4.3]) yields that

$$\begin{aligned}
\int_{\mathbb{B}_n} f \mathrm{d}V_\gamma &= c_\gamma \int_{\mathbb{B}_n} f (1-|z|^2)^\gamma \mathrm{d}V \\
&= c_\gamma 2n \int_0^1 r^{2n-1} (1-r^2)^\gamma \left(\int_{\partial \mathbb{B}_n} f(r\xi) \mathrm{d}\sigma(\xi) \right) \mathrm{d}r \\
&= c_\gamma 2n \left(\int_0^1 r^{2n-1} (1-r^2)^\gamma \mathrm{d}r \right) f(0) \\
&= f(0).
\end{aligned}$$

Here c_γ is a suitable normalization constant. To verify the last equality, apply the same calculations with $f \equiv 1$.
For $\Omega = \mathbb{D}^n$, we use the result above in each coordinate and apply Fubini's theorem. □

For further studies, we introduce two important functions.

Definition 1.10. (i) We call the function

$$^{\mathbb{B}_n}K^{(\gamma)}\colon \mathbb{B}_n \times \mathbb{B}_n,\ (z,w) \mapsto \frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma}}$$

the *reproducing kernel* for $A^2(\mathbb{B}_n, V_\gamma)$.

(ii) We call the function

$$^{\mathbb{D}^n}K^{(\gamma)}\colon \mathbb{D}^n \times \mathbb{D}^n,\ (z,w) \mapsto \prod_{i=1}^{n}\left(\frac{1}{1-z_i\overline{w_i}}\right)^{2+\gamma_i}$$

the *reproducing kernel* for $A^2(\mathbb{D}^n, V_\gamma)$.

For shorter notation we will use $K^{(\gamma)}$ for $^{\mathbb{B}_n}K^{(\gamma)}$ and $^{\mathbb{D}^n}K^{(\gamma)}$. Furthermore we write $K_z^{(\gamma)}$ for $K^{(\gamma)}(\cdot, z)$ $(z \in \Omega)$.

Remark 1.11. (i) From the definition we see $K_\lambda^{(\gamma)}(z) = \overline{K_z^{(\gamma)}(\lambda)}$, for all $z, \lambda \in \Omega$.

(ii) For $z, \lambda \in \Omega$, we have by Proposition 1.3

$$K^{(\gamma)}_{\varphi_\lambda(z)}(\varphi_\lambda(z)) = \left|K_\lambda^{(\gamma)}(z)\right|^{-2} K_\lambda^{(\gamma)}(\lambda) K_z^{(\gamma)}(z)$$

and

$$K_\lambda^{(\gamma)}(\varphi_\lambda(z)) = K^{(\gamma)}_{\varphi_\lambda(0)}(\varphi_\lambda(z)) = K_\lambda^{(\gamma)}(z)^{-1} K_\lambda^{(\gamma)}(\lambda).$$

Proposition 1.12. *There exist constants $C_{\Omega,\gamma} > 0$ such that, for all $\lambda, z \in \Omega$,*

$$\left|K_\lambda^{(\gamma)}(z)\right| \le C_{\Omega,\gamma} K_\lambda^{(\gamma)}(\lambda).$$

Proof. Suppose $\lambda, z \in \Omega$.
For $\Omega = \mathbb{B}_n$, the Cauchy-Schwarz inequality implies that

$$|\langle z, \lambda\rangle| \le |z|\,|\lambda|$$

so that

$$1 - |\langle z,\lambda\rangle| \ge 1 - |z|\,|\lambda| \ge 1 - |\lambda|\,.$$

The triangle inequality shows

$$\begin{aligned}
\left|^{\mathbb{B}_n}K_\lambda^{(\gamma)}(z)\right| &\le \frac{1}{(1-|\langle z,\lambda\rangle|)^{n+1+\gamma}} \\
&\le \frac{1}{(1-|\lambda|)^{n+1+\gamma}} \\
&= \left(\frac{1+|\lambda|}{1-|\lambda|^2}\right)^{n+1+\gamma} \\
&\le \left(\frac{2}{1-|\lambda|^2}\right)^{n+1+\gamma} \\
&= 2^{n+1+\gamma} K_\lambda^{\mathbb{B}_n}(\lambda).
\end{aligned}$$

Similarly, we obtain, for $\Omega = \mathbb{D}^n$, the estimates

$$\begin{aligned}\left|{}^{\mathbb{D}_n}K_\lambda^{(\gamma)}(z)\right| &\leq \prod_{i=1}^{n}\left(\frac{1}{1-|\lambda_i|}\right)^{2+\gamma_i}\\ &= \prod_{i=1}^{n}\left(\frac{1+|\lambda_i|}{1-|\lambda_i|^2}\right)^{2+\gamma_i}\\ &\leq \prod_{i=1}^{n}\left(\frac{2}{1-|\lambda_i|^2}\right)^{2+\gamma_i}\\ &\leq 2^{n(2+\max_i\{\gamma_i\})}\ {}^{\mathbb{D}_n}K_\lambda^{(\gamma)}(\lambda).\end{aligned}$$

□

We can now reformulate Proposition 1.7.

Proposition 1.13. *For $\lambda \in \Omega$, we have*

$$(J_{\mathbb{R}}\varphi_\lambda)(z) = \frac{1}{K_\lambda^{(0)}(\lambda)}\left|K_\lambda^{(0)}(z)\right|^2$$

for all $z \in \Omega$.

Corollary 1.14. *For $h \in L^1(V_\gamma)$, we have*

$$\int_\Omega h(\varphi_\lambda(w))\mathrm{d}V_\gamma(w) = \frac{1}{K_\lambda^{(\gamma)}(\lambda)}\int_\Omega h(z)\left|K_\lambda^{(\gamma)}(z)\right|^2\mathrm{d}V_\gamma(z) \quad (\lambda\in\Omega).$$

Furthermore, the function $h\circ\varphi_\lambda$ ($\lambda \in \Omega$) lies in $L^1(V_\gamma)$.

Proof. The first result follows directly by the change of variables formula together with Proposition 1.3, Proposition 1.7 and Definition 1.10.
For the second statement, we observe, for $\lambda \in \Omega$,

$$\begin{aligned}\int_\Omega |h(\varphi_\lambda(w))|\,\mathrm{d}V_\gamma(w) &= \frac{1}{K_\lambda^{(\gamma)}(\lambda)}\int_\Omega |h(z)|\left|K_\lambda^{(\gamma)}(z)\right|^2\mathrm{d}V_\gamma(z)\\ &\leq C_{\Omega,\gamma}^2 K_\lambda^{(\gamma)}(\lambda)\int_\Omega |h(z)|\,\mathrm{d}V_\gamma(z) < \infty,\end{aligned}$$

where we have used Proposition 1.12. □

With this corollary we can prove that every function in $A^1(V_\gamma)$ satisfies a mean value formula in each point of Ω.

Proposition 1.15. *If $g \in A^1(V_\gamma)$, then*

$$g(\lambda) = \int_\Omega g(w)\overline{K_\lambda^{(\gamma)}(w)}\mathrm{d}V_\gamma(w)$$

for all $\lambda \in \Omega$.

Proof. Suppose that $g \in A^1(V_\gamma)$ and $\lambda \in \Omega$. Since there exists a $\delta_\lambda > 0$ such that $\left|K_\lambda^{(\gamma)}(z)\right| \geq \delta_\lambda$ for all $z \in \Omega$, the function

$$h_\lambda \colon \Omega \to \mathbb{C},\ z \mapsto \frac{K_\lambda^{(\gamma)}(\lambda)}{K_\lambda^{(\gamma)}(z)} g(z)$$

belongs to $A^1(V_\gamma)$ again. It follows from Corollary 1.14 that $h_\lambda \circ \varphi_\lambda \in A^1(V_\gamma)$. Using Proposition 1.9 and the change of variables formula from Corollary 1.14, we obtain that

$$\begin{aligned} g(\lambda) &= h_\lambda(\lambda) = (h_\lambda \circ \varphi_\lambda)(0) \\ &= \int_\Omega h_\lambda(\varphi_\lambda(u)) \mathrm{d}V_\gamma(u) \\ &= \int_\Omega g(w) \overline{K_\lambda^{(\gamma)}(w)} \mathrm{d}V_\gamma(w). \end{aligned}$$

□

Corollary 1.16. *Let $g \in A^2(V_\gamma)$. Then*

$$g(\lambda) = \left\langle g, K_\lambda^{(\gamma)} \right\rangle$$

for all $\lambda \in \Omega$.

Proof. Since $A^2(V_\gamma) \subset A^1(V_\gamma)$, this follows immediately from Proposition 1.15. □

Remark 1.17. For $z \in \Omega$, we obtain

$$\left\| K_z^{(\gamma)} \right\|^2 = \left\langle K_z^{(\gamma)}, K_z^{(\gamma)} \right\rangle = K_z^{(\gamma)}(z),$$

where we have used the above corollary.

Proposition 1.18. *The Bergman spaces $A^p(V_\gamma)$ $(1 \leq p < \infty)$ is a closed subspace of the Banach space $L^p(V_\gamma)$.*
In particular, for $p = 2$, the space $A^2(V_\gamma)$ is a Hilbert space with the inner product induced by $L^2(V_\gamma)$.

A proof of this statement can be found in [13, Corollary 2.5].

Corollary 1.19. *The Bergman space $A^2(V_\gamma)$ is a functional Hilbert space with reproducing kernel $K^{(\gamma)}$.*

Proof. By Corollary 1.16 we have

$$|f(z)| = \left| \left\langle f, K_z^{(\gamma)} \right\rangle \right| \leq \|f\|_{2,\gamma} \left\| K_z^{(\gamma)} \right\|_{2,\gamma} = \|f\|_{2,\gamma} K_z^{(\gamma)}(z)^{\frac{1}{2}}$$

for all $f \in A^2(V_\gamma)$ and $z \in \Omega$, where we have used Remark 1.17. Hence the point evaluations are continuous and the reproducing kernel of the functional Hilbert space $A^2(V_\gamma)$ is given by $K^{(\gamma)}$. □

Remark 1.20. An orthonormal basis for $A^2(V_\gamma)$ is given by $(z^\alpha / \|z^\alpha\|_2)_{\alpha\in\mathbb{N}^n}$. In particular, the polynomials are dense in $A^2(V_\gamma)$.

Proof. We first show that $(z^\alpha / \|z^\alpha\|_2)_{\alpha\in\mathbb{N}^n}$ is an orthonormal system.
This follows, for $\Omega = \mathbb{B}_n$, immediately by [13, Proposition 1.11] and the remark before it.
For $\Omega = \mathbb{D}^n$, the result follows by the first part with $n = 1$ and Fubini's theorem.
To see that $(z^\alpha / \|z^\alpha\|_2)_{\alpha\in\mathbb{N}^n}$ is complete, let $w \in \Omega$. The function $K_w \in A^2(V_\gamma)$ can be seen as a holonorphic function on an open subset of $\mathrm{cl}(\Omega)$. Hence K_w is uniformly continuous on $\mathrm{cl}(\Omega)$ and can be uniformly approximated by polynomials. Since the set $\{K_w;\ w \in \Omega\}$ is total, the polynomials are dense in $A^2(V_\gamma)$. □

To end this section, we take a look at the orthogonal projection from $L^2(V_\gamma)$ onto $A^2(V_\gamma)$ which is an integral operator.

Proposition 1.21. *The operator*

$$P_\gamma \colon L^2(V_\gamma) \to L^2(V_\gamma), g \mapsto P_\gamma g$$

with

$$P_\gamma g \colon \Omega \to \mathbb{C},\ z \mapsto \int_\Omega g(w)\overline{K_z^{(\gamma)}(w)}\mathrm{d}V_\gamma(w) \quad (g \in L^2(V_\gamma))$$

is the orthogonal projection from $L^2(V_\gamma)$ to $A^2(V_\gamma)$.

Proof. Let P'_γ be the orthogonal projection from $L^2(V_\gamma)$ to $A^2(V_\gamma)$ and let $f \in L^2(V_\gamma)$ as well as $z \in \Omega$. We obtain

$$P'_\gamma f(z) = \left\langle P'_\gamma f, K_z^{(\gamma)} \right\rangle = \left\langle f, P'_\gamma K_z^{(\gamma)} \right\rangle.$$

Since $K_z^{(\gamma)} \in A^2(V_\gamma)$, we have

$$P'_\gamma f(z) = \left\langle f, K_z^{(\gamma)} \right\rangle = \int_\Omega f(w)\overline{K_z^{(\gamma)}(w)}\mathrm{d}V_\gamma(w) = P_\gamma f(z).$$

Thus the result follows. □

1.3 Fock space

This section gives a brief overview about the defintion and the relevant properties of the Fock space.

Definition 1.22 (Fock space)**.** Let μ be the measure on $\mathbb{C}^n$ given by

$$\mu = \frac{\exp(-\frac{|z|^2}{2})}{(2\pi)^n} m.$$

For $1 \leq p < \infty$, we define the *Fock space* (or *Segal-Bargmann space*) $L_a^p(\mu)$ by

$$L_a^p(\mu) = L^p(\mu) \cap \mathcal{O}(\mathbb{C}^n),$$

i.e., $L_a^p(\mu)$ is the set of all equivalence classes in $L^p(\mu)$ with a holomorphic representative.

The next proposition can be proven similar to Proposition 1.9.

Proposition 1.23. *Every $g \in L_a^1(\mu)$ satisfies*

$$g(0) = \int_{\mathbb{C}^n} g(w) \mathrm{d}\mu(w).$$

A proof of the following two results can be found in [4, Section 7] (with $\alpha = \frac{1}{2}$).

Proposition 1.24. *The Fock space $L_a^2(\mu)$ is a functional Hilbert space with reproducing kernel*

$$K^{L_a^2(\mu)} \colon \mathbb{C}^n \times \mathbb{C}^n, \; (z, w) \mapsto \exp\left(\frac{\langle z, w\rangle}{2}\right).$$

To shorten the notation we will write K for $K^{L_a^2(\mu)}$ when it is clear that we consider the Fock space. As before we use K_z for $K(\cdot, z)$ $(z \in \mathbb{C}^n)$.

Remark 1.25. An orthonormal basis for $L_a^2(\mu)$ is given by $(z^\alpha / \|z^\alpha\|_2)_{\alpha \in \mathbb{N}^n}$. In particular, the polynomials are dense in $L_a^2(\mu)$.

The translations

$$\tau_z \colon \mathbb{C}^n \to \mathbb{C}^n, \; w \mapsto w + z, \quad (z \in \mathbb{C}^n)$$

will play the same role for the Fock space as the Möbius transformations for the Bergman space.

Proposition 1.26. *Let $h \colon \mathbb{C}^n \to \mathbb{C}$ be a measurable function and let $\lambda \in \mathbb{C}^n$ be such that $h \circ \tau_\lambda \in L^1(\mu)$. Then the identity*

$$\int_{\mathbb{C}^n} h(\tau_\lambda(w)) \mathrm{d}\mu(w) = \frac{1}{K_\lambda(\lambda)} \int_{\mathbb{C}^n} h(z) \left|K_z(\lambda)\right|^2 \mathrm{d}\mu(z)$$

holds.

Proof. Suppose that $h \in L^1(\mu)$ and $\lambda \in \mathbb{C}^n$. Then

$$\begin{aligned}
&\int_{\mathbb{C}^n} h \circ \tau_\lambda(w) \mathrm{d}\mu(w) \\
&= \int_{\mathbb{C}^n} h \circ \tau_\lambda(w) \exp\left(-\frac{|w|^2}{2}\right) \frac{1}{(2\pi)^n} \mathrm{d}m(w) \\
&= \int_{\mathbb{C}^n} h(z) \exp\left(-\frac{|z-\lambda|^2}{2}\right) \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\
&= \int_{\mathbb{C}^n} h(z) \exp\left(-\frac{|z|^2}{2}\right) \exp\left(-\frac{|\lambda|^2}{2}\right) \exp\left(\mathrm{Re}(\langle \lambda, z\rangle)\right) \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\
&= \int_{\mathbb{C}^n} h(z) \left|K_z(\lambda)\right|^2 K_\lambda(\lambda)^{-1} \mathrm{d}\mu(z),
\end{aligned}$$

where we have used the transformation $z = \tau_\lambda(w)$. □

Furthermore, we denote by P the orthogonal projection from $L^2(\mu)$ onto $L_a^2(\mu)$.

1.4 Toeplitz and Hankel operators

Now we introduce the class of operators which we are interested in. We use the notation $L_a^p(\rho) \subset L^p(\rho)$ simultaneously for the weighted Bergman spaces on $\mathbb{B}_n, \mathbb{D}^n$ or the Fock spaces on $\mathbb{C}^n$ and write P for the orthogonal projection from $L^2(\rho)$ onto $L_a^2(\rho)$. The underlying domain $\mathbb{B}_n, \mathbb{D}^n$ or $\mathbb{C}^n$ will be written as G and for the automorphisms $\varphi_\lambda, \tau_\lambda$ we write κ_λ $(\lambda \in G)$. In all three cases, the space

$$H^\infty(G) = L^\infty(\rho) \cap \mathcal{O}(G)$$

consists of the bounded analytic functions on G. Of course, $H^\infty(G) = \mathbb{C}$ in the Fock space case.

Definition 1.27 (Toeplitz and Hankel operators)**.** For $f \in L^\infty(\rho)$, we define the *Toeplitz operator* $T_f \colon L_a^2(\rho) \to L_a^2(\rho)$ and the *Hankel Operator* $H_f \colon L_a^2(\rho) \to L_a^2(\rho)^\perp$ via

$$T_f = PM_f|_{L_a^2(\rho)} \quad \text{and} \quad H_f = (I-P)M_f|_{L_a^2(\rho)},$$

where

$$M_f \colon L^2(\rho) \to L^2(\rho),\ g \mapsto fg$$

is the multiplication operator with symbol f. We call f the *symbol* of the Toeplitz or Hankel operator, respectively.

Remark 1.28. Let $f \in L^\infty(\rho), g \in L^2_a(\rho)$ and $z \in G$. Since

$$\langle P(fg), K_z \rangle = \langle fg, K_z \rangle ,$$

we have

$$(T_f g)(z) = \langle T_f g, K_z \rangle = \int_G f(w)g(w)\overline{K_z(w)}\mathrm{d}\rho(w).$$

The next propositions describe some basic properties of Toeplitz and Hankel operators.

Proposition 1.29. *If $f, g \in L^\infty(\rho)$, then*

$$T_{gf} - T_g T_f = H^*_{\overline{g}} H_f.$$

Proof. Let $f, g \in L^\infty(\rho)$ and $h, k \in L^2_a(\rho)$. Then

$$\begin{aligned}
&\left\langle H^*_{\overline{g}} H_f h, k \right\rangle \\
&= \langle H_f h, H_{\overline{g}} k \rangle \\
&= \langle M_f h, M_{\overline{g}} k \rangle - \langle PM_f h, M_{\overline{g}} k \rangle - \langle M_f h, PM_{\overline{g}} k \rangle + \langle PM_f h, PM_{\overline{g}} k \rangle \\
&= \langle M_g M_f h, k \rangle - \langle M_g PM_f h, k \rangle - \langle PM_f h, PM_{\overline{g}} k \rangle + \langle PM_f h, PM_{\overline{g}} k \rangle \\
&= \langle PM_g M_f h, k \rangle - \langle M_g PM_f h, k \rangle \\
&= \langle (T_{gf} - T_g T_f) h, k \rangle .
\end{aligned}$$

□

Proposition 1.30. *For $f \in L^\infty(\rho)$ and $g, h \in H^\infty(G)$, the identities*

(i) $T_{fh} = T_f T_h$,

(ii) $T_{\overline{g}f} = T^*_g T_f$

hold.

Proof. Let $f \in L^\infty(\rho), g, h \in H^\infty(G)$ and $k \in L^2_a(\rho)$. Then

(i) $T_{fh}k = P(fhk) = P(fPhk) = PM_f PM_h k = T_f T_h k$.

(ii) $T_{\overline{g}f} = T^*_{\overline{f}g} = (T_{\overline{f}} T_g)^* = T^*_g T_f$.

□

Proposition 1.31. *Let $f \in L^\infty(\rho)$. For each $\lambda \in G$, the following identities hold:*

$$T_f(K_\lambda) = (P(f \circ \kappa_\lambda) \circ \kappa_\lambda^{-1}) K_\lambda \tag{1.1}$$

and

$$H_f(K_\lambda) = (f - P(f \circ \kappa_\lambda) \circ \kappa_\lambda^{-1}) K_\lambda. \tag{1.2}$$

Proof. Let $f \in L^\infty(\rho)$ and $\lambda \in G$.

(i) For the Bergman space let $g \in A^2(V_\gamma)$ and $h \in L^2(\rho)$. Then by Corollary 1.14 we have

$$\begin{aligned}&\left\langle (h\circ\varphi_\lambda)K_\lambda^{(\gamma)}, (g\circ\varphi_\lambda)K_\lambda^{(\gamma)}\right\rangle\\ =&\int_\Omega h(\varphi_\lambda(w))K_\lambda^{(\gamma)}(w)\overline{g(\varphi_\lambda(w))K_\lambda^{(\gamma)}(w)}\mathrm{d}V_\gamma(w)\\ =&\int_\Omega ((h\overline{g})\circ\varphi_\lambda)(w)\left|K_\lambda^{(\gamma)}(w)\right|^2\mathrm{d}V_\gamma(w)\\ =&K_\lambda^{(\gamma)}(\lambda)\int_\Omega (h\overline{g})(u)\mathrm{d}V_\gamma(u)\\ =&K_\lambda^{(\gamma)}(\lambda)\left\langle h, g\right\rangle\\ =&K_\lambda^{(\gamma)}(\lambda)\left\langle P_\gamma h, g\right\rangle.\end{aligned}$$

Replacing h once by f and once by Pf, we obtain

$$\begin{aligned}\left\langle (f\circ\varphi_\lambda)K_\lambda^{(\gamma)}, (g\circ\varphi_\lambda)K_\lambda^{(\gamma)}\right\rangle &= K_\lambda^{(\gamma)}(\lambda)\left\langle P_\gamma f, g\right\rangle\\ &= \left\langle ((P_\gamma f)\circ\varphi_\lambda)K_\lambda^{(\gamma)}, (g\circ\varphi_\lambda)K_\lambda^{(\gamma)}\right\rangle.\end{aligned}$$

Corollary 1.14 shows that $(g/K_\lambda^{(\gamma)})\circ\varphi_\lambda \in A^2(V_\gamma)$. If we replace f by $f\circ\varphi_\lambda$ ($\in L^\infty(V_\gamma)$) and g by $(g/K_\lambda^{(\gamma)})\circ\varphi_\lambda$, we obtain that

$$\begin{aligned}\left\langle (P_\gamma(f\circ\varphi_\lambda)\circ\varphi_\lambda)K_\lambda^{(\gamma)}, g\right\rangle &= \left\langle fK_\lambda^{(\gamma)}, g\right\rangle\\ &= \left\langle P_\gamma(fK_\lambda^{(\gamma)}), g\right\rangle\end{aligned}$$

holds for all $g \in A^2(V_\gamma)$. Finally we have

$$T_f(K_\lambda^{(\gamma)}) = P_\gamma(fK_\lambda^{(\gamma)}) = (P_\gamma(f\circ\varphi_\lambda)\circ\varphi_\lambda)K_\lambda^{(\gamma)}.$$

(ii) For the Fock space, let $z \in \mathbb{C}^n$. We see that

$$\begin{aligned}K_w(z) &= K_\lambda(z)K_{w-\lambda}(z)\\ &= \overline{K_z(\lambda)}K_{w-\lambda}(z-\lambda)K_w(\lambda)K_\lambda(\lambda)^{-1}\end{aligned}$$

for all $w \in \mathbb{C}^n$, and therefore by Remark 1.28 that

$$\begin{aligned}T_f(K_\lambda)(z) &= \frac{K_\lambda(z)}{K_\lambda(\lambda)}\int_{\mathbb{C}^n} f(w)\overline{K_{z-\lambda}(w-\lambda)}\left|K_w(\lambda)\right|^2\mathrm{d}\mu(w)\\ &= K_\lambda(z)\int_{\mathbb{C}^n} f\circ\tau_\lambda(u)\overline{K_{z-\lambda}(u)}\mathrm{d}\mu(u)\\ &= K_\lambda(z)\int_{\mathbb{C}^n} P(f\circ\tau_\lambda)(u)\overline{K_{z-\lambda}(u)}\mathrm{d}\mu(u)\\ &= K_\lambda(z)P(f\circ\tau_\lambda)(z-\lambda)\\ &= P(f\circ\tau_\lambda)\circ\tau_{-\lambda}(z)K_\lambda(z),\end{aligned}$$

where we have used Proposition 1.26.

□

1.5 Berezin transform

We first recall the definition of the Berezin transform.

Definition 1.32. Let $k_\lambda = \frac{K_\lambda}{\|K_\lambda\|_2}$ $(\lambda \in G)$. Then $k_\lambda \in H^\infty(\Omega) \subset A^2(V_\gamma)$ for $\Omega = \mathbb{B}_n$ or $\Omega = \mathbb{D}^n$ and $k_\lambda \in L_a^2(\mu)$ for $G = \mathbb{C}^n$. Hence in the Bergman space case, for each $g \in L^2(V_\gamma)$, we obtain a well-defined function

$$\widetilde{g}\colon \Omega \to \mathbb{C},\ \lambda \mapsto \langle gk_\lambda, k_\lambda\rangle = \frac{1}{K_\lambda(\lambda)} \int_\Omega g(z) \left|K_\lambda^{(\gamma)}(z)\right|^2 \mathrm{d}V_\gamma.$$

In the Fock space case, the same formula makes sense and defines a function $\widetilde{g}\colon \mathbb{C}^n \to \mathbb{C}$ at least for each function $g \in L^\infty(\mu)$. The function $\widetilde{g}$ is called the *Berezin transform* of g.

Remark 1.33. Let $g \in L^\infty(\rho)$ and $\lambda \in G$.

(i) We obtain

$$\begin{aligned}\widetilde{g}(\lambda) = \langle gk_\lambda, k_\lambda\rangle &= \frac{1}{K_\lambda(\lambda)} \int_G g(z) \left|K_\lambda(z)\right|^2 \mathrm{d}\rho(z)\\ &= \int_G g \circ \kappa_\lambda \mathrm{d}\rho = \langle g \circ \kappa_\lambda, K_0\rangle = P(g\circ\kappa_\lambda)(0),\end{aligned}$$

where we have used Corollaries 1.14 and Proposition 1.26. In the case $g \in H^\infty(G)$, it follows that

$$\widetilde{g}(\lambda) = (g\circ\kappa_\lambda)(0) = g(\lambda).$$

For the Bergman space, these results hold also for $g \in A^2(\Omega)$.

(ii) We have

$$\begin{aligned}\widetilde{\overline{g}}(\lambda) &= \langle \overline{g}k_\lambda, k_\lambda\rangle = \langle k_\lambda, gk_\lambda\rangle\\ &= \overline{\langle gk_\lambda, k_\lambda\rangle} = \overline{\widetilde{g}}(\lambda).\end{aligned}$$

(iii) The Cauchy-Schwarz inequality yields that

$$|\widetilde{g}(\lambda)| \le \|gk_\lambda\|_2 \|k_\lambda\|_2 \le \|g\|_\infty,$$

and therefore

$$\|\widetilde{g}\|_\infty \le \|g\|_\infty.$$

The next propostions are basic but helpful results about the Berezin transform of a function and the interplay with automorphisms and orthogonal projections.

Proposition 1.34. *For $f \in L^\infty(\rho)$, we have*

$$(f \circ \kappa_\lambda)\tilde{}\,(w) = \widetilde{f}(\kappa_\lambda(w))$$

for all $\lambda, w \in G$.

Proof. Let $\lambda, w \in G$.

(i) Bergman space: We first claim that the function

$$U = \varphi_{\varphi_\lambda(w)} \circ (\varphi_\lambda \circ \varphi_w)$$

is in $\mathrm{Aut}(\Omega)$, unitary and linear. For $\Omega = \mathbb{B}_n$, this follows by [8, Theorem 2.2.5]) and Proposition 1.3 (i). For $\Omega = \mathbb{D}^n$, the statement is also true, since the Möbius transformations act in each variable separately. By Proposition 1.3 we have

$$\varphi_\lambda \circ \varphi_w = \varphi_{\varphi_\lambda(w)} \circ U$$

and therefore (cf. Remark 1.33 (i))

$$\begin{aligned}(f \circ \varphi_\lambda)\tilde{}\,(w) &= \int_\Omega (f \circ \varphi_\lambda) \circ \varphi_w \mathrm{d}V_\gamma \\ &= \int_\Omega f \circ \varphi_{\varphi_\lambda(w)} \circ U \mathrm{d}V_\gamma \\ &= \int_\Omega f \circ \varphi_{\varphi_\lambda(w)} \mathrm{d}V_\gamma \\ &= \widetilde{f}(\varphi_\lambda(w)),\end{aligned}$$

where we have used the change of variable formula in the third step.

(ii) Fock space: We have

$$\begin{aligned}(f \circ \tau_\lambda)\tilde{}\,(w) &= \int_{\mathbb{C}^n} (f \circ \tau_\lambda) \circ \tau_w \mathrm{d}\mu \\ &= \int_{\mathbb{C}^n} f \circ (\tau_{\lambda+w}) \mathrm{d}\mu \\ &= \widetilde{f}(\lambda + w) \\ &= \widetilde{f}(\tau_\lambda(w)).\end{aligned}$$

□

Proposition 1.35. *For $g \in L^2_a(\rho)$, we have $P(\overline{g}) = \overline{g(0)}$.*

Proof. Let $g \in L_a^2(\rho)$ and $\beta \in \mathbb{N}^n$. Since $gz^\beta \in L_a^1(\rho)$, we obtain as an application of the mean value formula (cf. Propositions 1.9 and 1.23) that

$$\begin{aligned}\left\langle P(\overline{g}), z^\beta \right\rangle &= \left\langle \overline{g}, z^\beta \right\rangle \\ &= \int_G \overline{g}\ \overline{z^\beta} \mathrm{d}\rho \\ &= \overline{\int_G g z^\beta \mathrm{d}\rho} \\ &= \overline{(gz^\beta)(0)} \\ &= \delta_{\beta,0}\overline{g(0)}.\end{aligned}$$

Thus, Remarks 1.20 and 1.25 yield that

$$P(\overline{g}) = \sum_{\alpha \in \mathbb{N}^n} \left\langle P(\overline{g}), \frac{z^\alpha}{\|z^\alpha\|_2} \right\rangle \frac{z^\alpha}{\|z^\alpha\|_2} = \overline{g(0)}.$$

□

Proposition 1.36. *The identity*

$$P(\overline{P(\overline{f} \circ \kappa_\lambda)}) = \widetilde{f}(\lambda)$$

holds for all $f \in L^\infty(\rho)$ and $\lambda \in G$.

Proof. Let $f \in L^\infty(\rho)$ and $\lambda \in G$. Then $P(\overline{f} \circ \kappa_\lambda) \in L_a^2(\rho)$ and therefore

$$\begin{aligned}P(\overline{P(\overline{f} \circ \kappa_\lambda)}) &= \overline{P(\overline{f} \circ \kappa_\lambda)(0)} = \overline{\langle P(\overline{f} \circ \kappa_\lambda), 1\rangle} \\ &= \overline{\langle \overline{f} \circ \kappa_\lambda, 1\rangle} = \langle f \circ \kappa_\lambda, 1\rangle \\ &= \widetilde{f}(\lambda)\end{aligned}$$

where we have used Proposition 1.35 in the first step and Remark 1.33 (i) in the last step. □

1.6 Hilbert-Schmidt operators

In this section we gather the definition and some properties of Hilbert-Schmidt operators.

Definition 1.37 (Hilbert-Schmidt operator)**.** Let H be a Hilbert space, A a bounded linear operator on H and $(e_i)_i$ an orthonormal basis for H. The operator A is called a *Hilbert-Schmidt operator*, if

$$\|A\|_2 = \left(\sum_i \|Ae_i\|^2 \right)^{\frac{1}{2}}$$

is finite.

Remark 1.38. Let A be a Hilbert-Schmidt operator on a Hilbert space H. If $(e_i)_i$ and $(f_j)_j$ are orthonormal bases of H, then

$$\sum_i \|Ae_i\|^2 = \sum_i \sum_j |\langle Ae_i, f_j\rangle|^2 = \sum_j \|A^* f_j\|^2 .$$

In particular, the Hilbert-Schmidt norm $\|A\|_2$ is independent of the orthormal basis chosen to define it.

Lemma 1.39. *For a σ-finite and separable measure space (X, Σ, ν), consider $a \in L^2(X \times X, \nu \otimes \nu)$. Then*

$$A\colon L^2(X,\nu) \to L^2(X,\nu),\ g \mapsto Ag$$

with

$$Ag\colon X \to \mathbb{C},\ x \mapsto \int_X a(x,y)g(y)\mathrm{d}\nu(y) \quad (g \in L^2(X,\nu))$$

defines a Hilbert-Schmidt operator with $\|A\|_2 = \|a\|_2$.

Proof. This follows by [3, Theorem 4.5] (and its proof). □

Proposition 1.40. *Every Hilbert-Schmidt operator on a separable Hilbert space is compact.*

Proof. Let A be a Hilbert-Schmidt operator on a Hilbert space H and let $(e_k)_{k\in\mathbb{N}}$ be a orthonormal basis of H. We will show that A can be approximated by operators with finite rank. Define

$$A_k\colon H \to H,\ h \mapsto \sum_{i=0}^{k} \langle h, e_i\rangle Ae_i,$$

which is an operator with finite rank. For $h \in H$ with $\|h\| = 1$, we have

$$\begin{aligned}
\|A - A_k\| &\le \|(A - A_k)h\| \\
&= \left\| \sum_{i=k+1}^{\infty} \langle h, e_i\rangle Ae_i \right\| \le \sum_{i=k+1}^{\infty} |\langle h, e_i\rangle| \, \|Ae_i\| \\
&\le \left(\sum_{i=k+1}^{\infty} |\langle h, e_i\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i=k+1}^{\infty} \|Ae_i\|^2 \right)^{\frac{1}{2}} \\
&\le \|h\| \left(\sum_{i=k+1}^{\infty} \|Ae_i\|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=k+1}^{\infty} \|Ae_i\|^2 \right)^{\frac{1}{2}} \\
&\to 0
\end{aligned}$$

as $k \to \infty$, where we have used the Cauchy-Schwarz inequality. □

Chapter 2

Compactness of Toeplitz and Hankel operators

The goal of this chapter is to give a characterization of compact Toeplitz and Hankel operators. Since the proofs differ in the Bergman space and Fock space case, we split the proofs. The last section contains some corollaries, which will be useful in the next chapters. Throughout the chapter, we consider f to be in $L^\infty(\rho)$.

2.1 The main result

We use

$$\lambda \to \Delta$$

as a shortcut for the statements

(i) For all sequences $(\lambda_k)_k$ in Ω with limit point in $\partial\Omega$ (Bergman space),

(ii) For all sequences $(\lambda_k)_k$ in $\mathbb{C}^n$ with $|\lambda_k| \to \infty$ as $k \to \infty$ (Fock space).

Proposition 2.1. *We have*

$$k_\lambda \to 0$$

weakly in $L_a^2(\rho)$ as $\lambda \to \Delta$.

Proof. Let $g \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial. With Remark 1.17 we obtain that

$$\langle g, k_\lambda \rangle = \left\langle g, \frac{K_\lambda}{\|K_\lambda\|_2} \right\rangle = K_\lambda(\lambda)^{-\frac{1}{2}} g(\lambda).$$

For the Bergman space, we obtain

$$\langle g, k_\lambda \rangle = K_\lambda(\lambda)^{-\frac{1}{2}} g(\lambda) \to 0$$

as $\lambda \to \Delta$, since $K_\lambda(\lambda) \to \infty$ as $\lambda \to \Delta$ and g is bounded.
For the Fock space, we see

$$\langle g, k_\lambda \rangle = \exp\left(-\frac{1}{4}|\lambda|^2\right) g(\lambda) \to 0$$

as $\lambda \to \Delta$.
By Remarks 1.20 and 1.25 the polynomials are dense in $L^2_a(\rho)$. Hence k_λ converges weakly to 0 as $\lambda \to \Delta$. $\square$

Remark 2.2. An operator T on a Hilbert space H is compact if and only if

$$\|Th_k\| \to 0$$

as $k \to \infty$ for all sequences $(h_k)_{k \in \mathbb{N}}$ in H which converge weakly to 0.

Theorem 2.3. *Let $f \in L^\infty(\rho)$ and Q be either P or $I - P$. The statements*

(i) $QM_f|_{L^2_a(\rho)}$ is compact;

(ii) $\|QM_f k_\lambda\|_2 \to 0$ as $\lambda \to \Delta$;

(iii) $\|Q(f \circ \kappa_\lambda)\|_2 \to 0$ as $\lambda \to \Delta$

are equivalent.

Proof. Suppose that$f \in L^\infty(\rho)$.
(i) implies (ii): Let $QM_f|_{L^2_a(\rho)}$ be compact. By Proposition 2.1 and the above remark we have

$$\|QM_f k_\lambda\|_2 \to 0$$

as $\lambda \to \Delta$.
(ii) implies (iii): Let $\lambda \in G$. By Proposition 1.31 we obtain that

$$QM_f(K_\lambda) = (Q(f \circ \kappa_\lambda) \circ \kappa_\lambda^{-1})K_\lambda.$$

Hence

$$\begin{aligned}
\|QM_f k_\lambda\|_2^2 &= \left\| QM_f \left(\frac{K_\lambda}{\|K_\lambda\|_2} \right) \right\|_2^2 \\
&= \frac{1}{K_\lambda(\lambda)} \int_G \left| (Q(f \circ \kappa_\lambda) \circ \kappa_\lambda^{-1})(z) \right|^2 |K_\lambda(z)|^2 \, \mathrm{d}\rho(z) \\
&= \int_G |Q(f \circ \kappa_\lambda)(z)|^2 \, \mathrm{d}\rho(z) \\
&= \|Q(f \circ \kappa_\lambda)\|_2^2,
\end{aligned}$$

where we have used Corollary 1.14 and Proposition 1.26. This leads to

$$\|Q(f \circ \kappa_\lambda)\|_2 \to 0 \ (\lambda \to \Delta).$$

The remaining implication will be shown in the next sections.

2.2 Bergman space

To verify the last implication in Theorem 2.3 we need some results from function theory on the unit ball as well as some integral estimates.

Lemma 2.4. *For the function*

$$M_\Omega\colon [1,\infty)\times(0,\infty)\to\mathbb{R}\cup\{\infty\},$$
$$(q,\varepsilon)\mapsto \sup_{\lambda\in\Omega}\int_\Omega \left|K_\lambda^{(\gamma)}(w)\right|^{(1-2\varepsilon)q} K_w^{(\gamma)}(w)^{\varepsilon q}\mathrm{d}V_\gamma(w),$$

there exist $q_\Omega > 1$ and $\varepsilon_\Omega > 0$ such that $M_\Omega(q_\Omega,\varepsilon_\Omega) < \infty$ and $M_\Omega(1,\varepsilon_\Omega) < \infty$.

Before proving this statement, we recall the following result.

Proposition 2.5 (Theorem 1.12 in [13])**.** *Let $c \in \mathbb{R}$ and $t > -1$ and define the function*

$$J_{c,t}\colon \mathbb{B}_n\to\mathbb{C},\ z\mapsto \int_{\mathbb{B}_n}\frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\mathrm{d}V(w).$$

(i) If $c < 0$, then $J_{c,t}$ is bounded in $\mathbb{B}_n$.

(ii) If $c = 0$, then

$$\frac{J_{c,t}(z)}{\log\left(\frac{1}{1-|z|^2}\right)}$$

has a positive finte limit as $|z|\uparrow 1$.

Proof of Lemma 2.4. First we consider the case $\Omega = \mathbb{B}_n$.
Note that

$$M_{\mathbb{B}_n}(q,\varepsilon) = \frac{\Gamma(n+1+\gamma)}{n!\Gamma(1+\gamma)}\sup_{\lambda\in\mathbb{B}_n} J_{c,t}(\lambda)$$

with

$$t = \gamma-(n+1+\gamma)\varepsilon q$$

and

$$c = (n+1+\gamma)(q-\varepsilon q-1).$$

The conditions $t > -1$ and $c < 0$ are equivalent to

$$q-1 < \varepsilon q < \frac{1+\gamma}{(n+1+\gamma)}.$$

Thus, if we set

$$\varepsilon_{\mathbb{B}_n} = \frac{1}{2\left(\left(\frac{n}{1+\gamma}\right)+1\right)}, \quad q_{\mathbb{B}_n} = \frac{2\left(\frac{n}{1+\gamma}\right)+\frac{3}{2}}{2\left(\frac{n}{1+\gamma}\right)+1},$$

then the above chain of inequalities is satisfied with $(q,\varepsilon)=(1,\varepsilon_{\mathbb{B}_n})$ and $(q,\varepsilon)=(q_{\mathbb{B}_n},\varepsilon_{\mathbb{B}_n})$.
For the polydisc, choose

$$\varepsilon_{\mathbb{D}^n} = \frac{1}{2\left(\left(\frac{1}{1+\gamma_0}\right)+1\right)}, \quad q_{\mathbb{D}^n} = \frac{2\left(\frac{1}{1+\gamma_0}\right)+\frac{3}{2}}{2\left(\frac{1}{1+\gamma_0}\right)+1},$$

where $\gamma_0 = \min_{i=1,\dots,n}\gamma_i$. Then

$$q-1 < \varepsilon q < \frac{1+\gamma_i}{(n+1+\gamma_i)}$$

and the same inequalities with $q=1$ hold for all $i=1,\dots,n$. Since

$$M_{\mathbb{D}^n}(q,\varepsilon) = \prod_{i=1}^{n} M_{\mathbb{D}}(q,\varepsilon)$$

for all $(q,\varepsilon)\in[1,\infty)\times(0,\infty)$, the result follows. □

Lemma 2.6. *For a nonnegative measurable function H on $\Omega\times\Omega$, $1<q<\infty$, $p=\frac{q}{q-1}$ and $\varepsilon>0$, we have, for all $w\in\Omega$,*

$$\begin{aligned}&\int_\Omega H(w,\varphi_w(z))\left|K_w^{(\gamma)}(z)\right| K_z^{(\gamma)}(z)^\varepsilon \mathrm{d}V_\gamma(z)\\ \le& K_w^{(\gamma)}(w)^\varepsilon M_\Omega(q,\varepsilon)^{\frac{1}{q}}\left(\int_\Omega H(w,z)^p \mathrm{d}V_\gamma(z)\right)^{\frac{1}{p}}.\end{aligned}$$

Proof. Let H be a nonnegative measurable function on $\Omega\times\Omega$, $1<q<\infty$, $p=\frac{q}{q-1}$, $\varepsilon>0$ and $w\in\Omega$. With Corollary 1.14 and Remark 1.11 we obtain

$$\begin{aligned}&\int_\Omega H(w,\varphi_w(z))\left|K_w^{(\gamma)}(z)\right| K_z^{(\gamma)}(z)^\varepsilon \mathrm{d}V_\gamma(z)\\ =&\int_\Omega H(w,u)\left|K_w^{(\gamma)}(\varphi_w(u))\right| K_{\varphi_w(u)}^{(\gamma)}(\varphi_w(u))^\varepsilon \frac{\left|K_w^{(\gamma)}(u)\right|^2}{K_w^{(\gamma)}(w)}\mathrm{d}V_\gamma(u)\\ =&K_w^{(\gamma)}(w)^\varepsilon\int_\Omega H(w,u)\left|K_w^{(\gamma)}(u)\right|^{1-2\varepsilon} K_u^{(\gamma)}(u)^\varepsilon \mathrm{d}V_\gamma(u)\\ \le&K_w^{(\gamma)}(w)^\varepsilon M_\Omega(q,\varepsilon)^{\frac{1}{q}}\left(\int_\Omega H(w,z)^p\mathrm{d}V_\gamma(z)\right)^{\frac{1}{p}},\end{aligned}$$

where we have used the substitution $z=\varphi_w(u)$ and the Hölder inequality. □

Lemma 2.7. *Let $\varepsilon > 0$. Then, for all $z \in \Omega$,*

$$\int_\Omega |P_\gamma(f \circ \varphi_w)(\varphi_w(z))| \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)$$
$$\leq M_\Omega(1,\varepsilon)^2 \|f\|_{\infty,\gamma} K_z^{(\gamma)}(z)^\varepsilon$$

and

$$\int_\Omega |f(z) - P_\gamma(f \circ \varphi_w)(\varphi_w(z))| \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)$$
$$\leq 2M_\Omega(1,\varepsilon)^2 \|f\|_{\infty,\gamma} K_z^{(\gamma)}(z)^\varepsilon.$$

Proof. Let $\varepsilon > 0$ and $z \in \Omega$.

(i) With Proposition 1.31 and Remark 1.28 we see, for $w \in \Omega$,

$$\begin{aligned} |P_\gamma(f \circ \varphi_w)(\varphi_w(z))| \left|K_w^{(\gamma)}(z)\right| &= \left|T_f(K_w^{(\gamma)})(z)\right| \\ &= \left|\int_\Omega f(u) K_w^{(\gamma)}(u) \overline{K_z^{(\gamma)}(u)} \mathrm{d}V_\gamma(u)\right| \\ &\leq \|f\|_{\infty,\gamma} \int_\Omega \left|K_w^{(\gamma)}(u)\right| \left|K_z^{(\gamma)}(u)\right| \mathrm{d}V_\gamma(u) \end{aligned}$$

and therefore, using Fubini's theorem, we find that

$$\int_\Omega |P_\gamma(f \circ \varphi_w)(\varphi_w(z))| \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)$$
$$\leq \|f\|_{\infty,\gamma} \int_\Omega \left|K_z^{(\gamma)}(u)\right| \left(\int_\Omega \left|K_w^{(\gamma)}(u)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)\right) \mathrm{d}V_\gamma(u).$$

By the proof of Lemma 2.6 we see that, if we choose $H \equiv 1$, the result there holds also for $q = 1$. Thus

$$\|f\|_{\infty,\gamma} \int_\Omega \left|K_z^{(\gamma)}(u)\right| \left(\int_\Omega \left|K_w^{(\gamma)}(u)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)\right) \mathrm{d}V_\gamma(u)$$
$$\leq \|f\|_{\infty,\gamma} \int_\Omega \left|K_z^{(\gamma)}(u)\right| M_\Omega(1,\varepsilon) K_u^{(\gamma)}(u)^\varepsilon \mathrm{d}V_\gamma(u)$$
$$\leq M_\Omega(1,\varepsilon)^2 \|f\|_{\infty,\gamma} K_z^{(\gamma)}(z)^\varepsilon.$$

(ii) If we use part (i) (with $f \equiv 1$) and the triangle inequality, we obtain

$$\int_\Omega |f(z) - P_\gamma(f \circ \varphi_w)(\varphi_w(z))| \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)$$
$$\leq \int_\Omega \left(\|f\|_{\infty,\gamma} + |P_\gamma(f \circ \varphi_w)(\varphi_w(z))|\right) \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)$$
$$\leq 2M_\Omega(1,\varepsilon)^2 \|f\|_{\infty,\gamma} K_z^{(\gamma)}(z)^\varepsilon.$$

□

Proposition 2.8. *For $q \in (1, \infty)$, we have*

$$\|P_\gamma f\|_{q,\gamma} \le C_{q,\gamma} \|f\|_{\infty,\gamma}$$

with a suitable constant C_q.

Proof. Let $q \in (1, \infty)$ and let $f \in L^\infty(V_\gamma)$. Since

$$\|P_\gamma f\|_{q,\gamma}^q \le \|f\|_{\infty,\gamma}^q \int_\Omega \left(\int_\Omega \left| K_z^{(\gamma)}(w) \right| \mathrm{d}V_\gamma(w) \right)^q \mathrm{d}V_\gamma(z),$$

it is enough to show that the integral on the right side is bounded.
For $\Omega = \mathbb{B}_n$, we see by Proposition 2.5 that

$$C = \lim_{|z|\uparrow 1} \frac{\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{n+1+\gamma}} \mathrm{d}m(w)}{\log\left(\frac{1}{1-|z|^2}\right)} \in (0, \infty)$$

exists. Choose $R > 1$ such that

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{n+1+\gamma}} \mathrm{d}m(w) \le 2C \log\left(\frac{1}{1-|z|^2}\right)$$

for $R < |z| < 1$. For $|z| \le R$, the same integral can be estimated by

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{n+1+\gamma}} \mathrm{d}m(w) \le \frac{1}{(1-R)^{n+1+\gamma}} \int_{\mathbb{B}_n} (1-|w|^2)^\gamma \mathrm{d}m(w).$$

Since $\log\left(\frac{e}{1-|z|^2}\right)(1-R)^{n+1+\gamma}$ is continuous and never zero, for $|z| \le R$, there exists a constant $C' > 0$ such that

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{n+1+\gamma}} \mathrm{d}m(w) \le C' \log\left(\frac{e}{1-|z|^2}\right)$$

for all $z \in \mathbb{B}_n$.
Choose $s > 0$ such that $\gamma - sq > -1$. Since

$$\lim_{t\to\infty} \log(t) t^{-s} = 0,$$

the function

$$\left(\frac{\log\left(\frac{e}{1-|z|^2}\right)}{\left(\frac{e}{1-|z|^2}\right)^s} \right)^q$$

is bounded for $z \in \mathbb{B}_n$. Hence there is a constant $C'' > 0$ such that

$$\left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{n+1+\gamma}} \mathrm{d}m(w) \right)^q \left(1-|z|^2\right)^\gamma \le C'' \left(1-|z|^2\right)^{\gamma-sq}$$

for $z \in \mathbb{B}_n$. It follows that

$$\int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{n+1+\gamma}} \mathrm{d}m(w) \right)^q \left(1-|z|^2\right)^\gamma \mathrm{d}m(z) < \infty.$$

For $\Omega = \mathbb{D}^n$, we observe that

$$\begin{aligned} &\int_{\mathbb{D}^n} \left(\int_{\mathbb{D}^n} \left| K_z^{(\gamma)}(w) \right| \mathrm{d}V_\gamma(w) \right)^q \mathrm{d}V_\gamma(z) \\ =& c_\gamma \prod_{i=1}^n \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{\left(1-|w_i|^2\right)^{\gamma_i}}{(1-z_i w_i)^{2+\gamma_i}} \mathrm{d}m(w_i) \right)^q \left(1-|z_i|^2\right)^{\gamma_i} \mathrm{d}m(z_i) \end{aligned}$$

with a suitable constant c_γ. Hence the result follows by the previous proof for the unit ball with $n = 1$. □

Proposition 2.9 (Schur Test). *Let (X, Σ, ν) be a measure space and let $F\colon X \times X \to \mathbb{C}$ be a measurable function. If there are a strictly positive measurable function h on X and positive numbers α and β such that*

$$\int_X |F(x,y)|\, h(y) \mathrm{d}\nu(y) \le \alpha h(x) \quad (\textit{for almost every } x \in X)$$

and

$$\int_X |F(x,y)|\, h(x) \mathrm{d}\nu(x) \le \beta h(y) \quad (\textit{for almost every } y \in X),$$

then

$$A\colon L^2(X,\nu) \to L^2(X,\nu),\ g \mapsto Ag$$

with

$$Ag\colon X \to \mathbb{C},\ x \mapsto \int_X F(x,y) g(y) \mathrm{d}\nu(y) \quad (g \in L^2(X,\nu))$$

defines a bounded linear operator with $\|A\|^2 \le \alpha\beta$.

Proof. Let $g \in L^2(X,\nu)$. Then by Hölder's inequality and Fubini's theorem we have

$$\begin{aligned}
\|Ag\|_2^2 &= \int_X \left| \int_X F(x,y)g(y)\mathrm{d}\nu(y) \right|^2 \mathrm{d}\nu(x) \\
&\leq \int_X \left(\int_X |F(x,y)|\,|g(y)|\,\mathrm{d}\nu(y) \right)^2 \mathrm{d}\nu(x) \\
&= \int_X \left(\int_X \sqrt{|F(x,y)|}\sqrt{h(y)} \left(\sqrt{\frac{|F(x,y)|}{h(y)}}\,|g(y)| \right) \mathrm{d}\nu(y) \right)^2 \mathrm{d}\nu(x) \\
&\leq \int_X \left(\int_X |F(x,y)|\,h(y)\mathrm{d}\nu(y) \right) \left(\int_X \frac{|F(x,y)|}{h(y)}\,|g(y)|^2\,\mathrm{d}\nu(y) \right) \mathrm{d}\nu(x) \\
&\leq \int_X \alpha h(x) \left(\int_X \frac{|F(x,y)|}{h(y)}\,|g(y)|^2\,\mathrm{d}\nu(y) \right) \mathrm{d}\nu(x) \\
&= \alpha \int_X \frac{|g(y)|^2}{h(y)} \left(\int_X |F(x,y)|\,h(x)\mathrm{d}\nu(x) \right) \mathrm{d}\nu(y) \\
&\leq \alpha \int_X \frac{|g(y)|^2}{h(y)} \beta h(y)\mathrm{d}\nu(y) \\
&= \alpha\beta \int_X |g(y)|^2\,\mathrm{d}\nu(y).
\end{aligned}$$

□

We now present the main result of this section.

Theorem 2.10. *Let Q_γ be either P_γ or $I - P_\gamma$. The statements*

(i) $Q_\gamma M_f|_{A^2(V_\gamma)}$ *is compact;*

(ii) $\left\| Q_\gamma M_f k_\lambda^{(\gamma)} \right\|_{2,\gamma} \to 0$ *as* $\lambda \to \partial\Omega$*;*

(iii) $\|Q_\gamma(f \circ \varphi_\lambda)\|_{2,\gamma} \to 0$ *as* $\lambda \to \partial\Omega$*;*

(iv) $\|Q_\gamma(f \circ \varphi_\lambda)\|_{p,\gamma} \to 0$ *as* $\lambda \to \partial\Omega$ *for all* $p \in [1,\infty)$

are equivalent.

Proof. In Section 2.1 we proved already that (i) implies (ii) and that (ii) implies (iii).
(iii) implies (iv): Suppose that $\|Q_\gamma(f \circ \varphi_\lambda)\|_{2,\gamma} \to 0$ as $\lambda \to \partial\Omega$ and let $p \in [1,\infty)$ be arbitrary. If $p \leq 2$, then

$$\|Q_\gamma(f \circ \varphi_\lambda)\|_{p,\gamma} \leq \|Q_\gamma(f \circ \varphi_\lambda)\|_{2,\gamma} \to 0$$

as $\lambda \to \partial\Omega$, since $L^p(V_\gamma) \subset L^2(V_\gamma)$ and the inclusion mapping is contractive. Otherwise, if $p > 2$, then for $F = |Q_\gamma(f \circ \varphi_\lambda)|^p$

$$\begin{aligned}\int_\Omega F \mathrm{d}V_\gamma &= \int_\Omega F^{\frac{1}{2p}} F^{1-\frac{1}{2p}} \mathrm{d}V_\gamma \\ &\leq \left(\int_\Omega F^{\frac{2}{p}} \mathrm{d}V_\gamma\right)^{\frac{1}{4}} \left(\int_\Omega F^{\frac{4}{3}\left(1-\frac{1}{2p}\right)} \mathrm{d}V_\gamma\right)^{\frac{3}{4}}\end{aligned}$$

by Hölder's inequality. This can be rewritten as

$$\|Q_\gamma(f \circ \varphi_\lambda)\|_{p,\gamma}^p \leq \|Q_\gamma(f \circ \varphi_\lambda)\|_{2,\gamma}^{\frac{1}{2}} \|Q_\gamma(f \circ \varphi_\lambda)\|_{q,\gamma}^{p-\frac{1}{2}}$$

with $q = \frac{2}{3}(2p-1) > 1$. Thus by Proposition 2.8 we obtain

$$\|Q_\gamma(f \circ \varphi_\lambda)\|_{p,\gamma}^p \leq \|Q_\gamma(f \circ \varphi_\lambda)\|_{2,\gamma}^{\frac{1}{2}} \Big((C_{q,\gamma} + 1) \|f\|_{\infty,\gamma}\Big)^{p-\frac{1}{2}} \to 0$$

as $\lambda \to \partial\Omega$.

(iv) implies (i): Suppose that (iv) holds. By Schauder's theorem it suffices to show that $(Q_\gamma M_f|_{A^2(V_\gamma)})^*$ is compact. Furthermore, we only need to prove that one can approximate $(Q_\gamma M_f|_{A^2(V_\gamma)})^*$ by compact operators in the operator norm. For $g \in L^2(V_\gamma)$, we have $(Q_\gamma M_f|_{A^2(V_\gamma)})^* g \in A^2(V_\gamma)$ and

$$\begin{aligned}((Q_\gamma M_f|_{A^2(V_\gamma)})^* g)(w) &= \left\langle (Q_\gamma M_f|_{A^2(V_\gamma)})^* g, K_w^{(\gamma)} \right\rangle = \left\langle g, Q_\gamma M_f K_w^{(\gamma)} \right\rangle \\ &= \left\langle g, (Q_\gamma(f \circ \varphi_w) \circ \varphi_w) K_w^{(\gamma)} \right\rangle \\ &= \int_\Omega g(z) \overline{Q_\gamma(f \circ \varphi_w)(\varphi_w(z)) K_w^{(\gamma)}(z)} \mathrm{d}V_\gamma(z),\end{aligned}$$

for $w \in \Omega$, where we have used Proposition 1.31. We will show that, for $0 < r < 1$, the operator

$$S_r \colon L^2(V_\gamma) \to L^2(V_\gamma),\ g \mapsto S_r g$$

with

$$S_r g \colon \Omega \to \mathbb{C},\ w \mapsto \chi_{r\Omega}(w) \int_\Omega g(z) \overline{Q_\gamma(f \circ \varphi_w)(\varphi_w(z)) K_w^{(\gamma)}(z)} \mathrm{d}V_\gamma(z)$$

is compact. With Corollary 1.14 we obtain

$$\begin{aligned}&\int_\Omega \left(\int_\Omega \chi_{r\Omega}(w) \left|Q_\gamma(f \circ \varphi_w)(\varphi_w(z)) K_w^{(\gamma)}(z)\right|^2 \mathrm{d}V_\gamma(z)\right) \mathrm{d}V_\gamma(w) \\ =&\int_\Omega \chi_{r\Omega}(w) K_w^{(\gamma)}(w) \|Q_\gamma(f \circ \varphi_w)\|_{2,\gamma}^2 \mathrm{d}V_\gamma(w) \\ =&\int_{r\Omega} K_w^{(\gamma)}(w) \|Q_\gamma(f \circ \varphi_w)\|_{2,\gamma}^2 \mathrm{d}V_\gamma(w) \\ \leq& K_r^{(\gamma)}(r) \|f\|_{\infty,\gamma}^2 < \infty,\end{aligned}$$

i.e., $\chi_{r\Omega}(w)\overline{Q_\gamma(f\circ\varphi_w)(\varphi_w(z))K_w^{(\gamma)}(z)} \in L^2(\Omega\times\Omega, V_\gamma\otimes V_\gamma)$ (Tonelli) and therefore S_r is a Hilbert-Schmidt operator by Lemma 1.39. By Proposition 1.40 S_r is compact. If we set

$$F\colon \Omega\times\Omega\to\mathbb{C},\ (w,z)\mapsto \chi_{\Omega\setminus r\Omega}(w)\overline{Q_\gamma(f\circ\varphi_w)(\varphi_w(z))K_w^{(\gamma)}(z)},$$

we obtain

$$(((Q_\gamma M_f|_{A^2(V_\gamma)})^* - S_r)g)(w) = \int_\Omega F(w,z)g(z)\mathrm{d}V_\gamma(z),$$

for all $g\in L^2(V_\gamma)$ and $w\in\Omega$. To complete the proof, we verify

$$\lim_{r\to 1}\left\|S_r - (Q_\gamma M_f|_{A^2(V_\gamma)})^*\right\| = 0.$$

Set $\varepsilon=\varepsilon_\Omega, q=q_\Omega$ with $\varepsilon_\Omega, q_\Omega$ as in Lemma 2.4 and let $p=\frac{q}{q-1}$. For $w,z\in\Omega$, we define

(i) $h(w) = K_w^{(\gamma)}(w)^\varepsilon$,

(ii) $H(w,z) = \chi_{\Omega\setminus r\Omega}(w)\left|Q_\gamma(f\circ\varphi_w)(z)\right|$,

(iii) $\alpha = M_\Omega(q,\varepsilon)^{\frac{1}{q}}\sup\Big\{\left\|Q_\gamma(f\circ\varphi_\lambda)\right\|_{p,\gamma};\ \lambda\in\Omega\setminus r\Omega\Big\}$,

(iv) $\beta = 2M_\Omega(1,\varepsilon)^2\left\|f\right\|_{\infty,\gamma}$.

By Lemma 2.6 we have, for $w\in\Omega$,

$$\begin{aligned}
&\int_\Omega \left|F(w,z)\right| h(z)\mathrm{d}V_\gamma(z)\\
=&\int_\Omega \chi_{\Omega\setminus r\Omega}(w)\left|Q_\gamma(f\circ\varphi_w)(\varphi_w(z))\right|\left|K_w^{(\gamma)}(z)\right| K_z^{(\gamma)}(z)^\varepsilon \mathrm{d}V_\gamma(z)\\
=&\int_\Omega H(w,\varphi_w(z))\left|K_w^{(\gamma)}(z)\right| K_z^{(\gamma)}(z)^\varepsilon \mathrm{d}V_\gamma(z)\\
\le& K_w^{(\gamma)}(w)^\varepsilon M_\Omega(q,\varepsilon)^{\frac{1}{q}}\left(\int_\Omega H(w,z)^p\mathrm{d}V_\gamma(z)\right)^{\frac{1}{p}}\\
=& K_w^{(\gamma)}(w)^\varepsilon M_\Omega(q,\varepsilon)^{\frac{1}{q}}\left(\int_\Omega \chi_{\Omega\setminus r\Omega}(w)\left|Q_\gamma(f\circ\varphi_w)(z)\right|^p\mathrm{d}V_\gamma(z)\right)^{\frac{1}{p}}\\
\le& K_w^{(\gamma)}(w)^\varepsilon M_\Omega(q,\varepsilon)^{\frac{1}{q}}\sup\Big\{\left\|Q_\gamma(f\circ\varphi_\lambda)\right\|_{p,\gamma};\ \lambda\in\Omega\setminus r\Omega\Big\}\\
=&\alpha h(w)
\end{aligned}$$

as well as, for $z \in \Omega$,

$$\begin{aligned}&\int_\Omega |F(w,z)|\, h(w) \mathrm{d}V_\gamma(w)\\ =&\int_\Omega \chi_{\Omega\setminus r\Omega}(w)\, |Q_\gamma(f\circ\varphi_w)(\varphi_w(z))| \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w)\\ \le& 2M_\Omega(1,\varepsilon)^2 \|f\|_{\infty,\gamma} K_z^{(\gamma)}(z)^\varepsilon\\ =&\beta h(z)\end{aligned}$$

by Lemma 2.7. Thus the conditions of the Schur Test (Proposition 2.9) are satisfied and by hypothesis

$$\alpha \to 0$$

as $r \to 1$. Thus

$$\left\|(Q_\gamma M_f|_{A^2(V_\gamma)})^* - S_r\right\|^2 \le \alpha\beta \to 0$$

as $r \to 1$. □

Corollary 2.11. *Let $g \in C(\mathrm{cl}(\mathbb{B}_n))$. Then H_g is compact.*

Proof. Suppose $g \in C(\mathrm{cl}(\mathbb{B}_n))$ and $z \in \mathbb{B}_n$. Let $(\lambda_k)_k$ be a sequence in $\mathbb{B}_n$ with limit $\lambda \in \partial\mathbb{B}_n$. Then by the proof of Proposition 1.4 and the dominated convergence theorem, we have

$$\lim_{k\to\infty} \|g\circ\varphi_{\lambda_k} - g(\lambda_k)\|_{2,\gamma} = 0,$$

since $g \in C(\mathrm{cl}(\mathbb{B}_n))$. From this, we obtain with $g(\lambda_k) \in A^2(V_\gamma)$ $(k \in \mathbb{N})$

$$\begin{aligned}\|(I-P_\gamma)\,(g\circ\varphi_{\lambda_k})\|_{2,\gamma} &= \|(I-P_\gamma)(g\circ\varphi_{\lambda_k} - g(\lambda_k))\|_{2,\gamma}\\ &\le \|g\circ\varphi_{\lambda_k} - g(\lambda_k)\|_{2,\gamma} \to 0\end{aligned}$$

as $k \to \infty$. By Theorem 2.3 H_g is compact. □

The following Lemma will be useful for corollaries of the preceding theorem.

Lemma 2.12. *The map*

$$\widetilde{\cdot}: L^2(V_\gamma) \to L^2(V_\gamma),\ g \mapsto \widetilde{g}$$

is a well-defined bounded linear operator.

Proof. Let $g \in L^2(V_\gamma), \varepsilon = \varepsilon_\Omega$ as in Lemma 2.4 and let $C_{\Omega,\gamma}$ be the constant from Proposition 1.12. For $w, z \in \Omega$, define

(i) $F(w,z) = \frac{\left|K_w^{(\gamma)}(z)\right|^2}{K_w^{(\gamma)}(w)}$,

(ii) $h(w) = K_w^{(\gamma)}(w)^\varepsilon$,

(iii) $\alpha = \beta = C_{\Omega,\gamma} M_\Omega(1,\varepsilon)^2$.

With Proposition 1.12 we see, for $w, z \in \Omega$,

$$F(w,z) = \frac{\left|K_w^{(\gamma)}(z)\right|^2}{K_w^{(\gamma)}(w)} \leq C_{\Omega,\gamma} \left|K_w^{(\gamma)}(z)\right|$$

and hence by Lemma 2.7 that, for $z \in \Omega$,

$$\begin{aligned}\int_\Omega |F(w,z)|\, h(w) \mathrm{d}V_\gamma(w) &\leq C_{\Omega,\gamma} \int_\Omega \left|K_w^{(\gamma)}(z)\right| K_w^{(\gamma)}(w)^\varepsilon \mathrm{d}V_\gamma(w) \\ &\leq C_{\Omega,\gamma} M_\Omega(1,\varepsilon)^2 K_z^{(\gamma)}(z)^\varepsilon\end{aligned}$$

as well as, for $w \in \Omega$,

$$\begin{aligned}\int_\Omega |F(w,z)|\, h(z) \mathrm{d}V_\gamma(z) &\leq C_{\Omega,\gamma} \int_\Omega \left|K_w^{(\gamma)}(z)\right| K_z^{(\gamma)}(z)^\varepsilon \mathrm{d}V_\gamma(z) \\ &\leq C_{\Omega,\gamma} M_\Omega(1,\varepsilon)^2 K_w^{(\gamma)}(w)^\varepsilon.\end{aligned}$$

Thus the conditions of the Schur test (Proposition 2.9) are satisfied, i.e., the map $\widetilde{\cdot}$ is a bounded linear operator on $L^2(V_\gamma)$ with

$$\|\widetilde{g}\|_{2,\gamma} \leq C_{\Omega,\gamma} M_\Omega(1,\varepsilon_\Omega)^2 \|g\|_{2,\gamma},$$

for all $g \in L^2(V_\gamma)$. □

2.3 Fock space

For the Fock space, we proceed similarly to the previous chapter, i.e., we begin by stating some integral estimates.

Lemma 2.13. *We have*

$$\int_{\mathbb{C}^n} |K_\lambda(z)|^p \, e^{-\frac{a}{2}\langle z,z\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(z) = \left(\frac{1}{a}\right)^n e^{\frac{p^2}{8a}\langle \lambda,\lambda\rangle},$$

for all $\lambda \in \mathbb{C}^n$, $a > 0$ and $p \geq 0$.
Furthermore, we have

$$|(Pf)(\lambda)| \leq \|f\|_\infty \, e^{\frac{1}{8}\langle \lambda,\lambda\rangle}$$

for all $\lambda \in \mathbb{C}^n$.

Proof. Suppose $\lambda \in \mathbb{C}^n, a > 0$ and $p \geq 0$. We obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} |K_\lambda(z)|^p \, e^{-\frac{a}{2}\langle z,z\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\
=&\int_{\mathbb{C}^n} \left| K_\lambda\left(\frac{1}{\sqrt{a}}u\right)\right|^p e^{-\frac{1}{2}\langle u,u\rangle} \frac{1}{(2\pi)^n} \left(\frac{1}{\sqrt{a}}\right)^{2n} \mathrm{d}m(u) \\
=&\left(\frac{1}{a}\right)^n \int_{\mathbb{C}^n} \left| K_\lambda\left(\frac{1}{\sqrt{a}}u\right)\right|^p \mathrm{d}\mu(u) \\
=&\left(\frac{1}{a}\right)^n \int_{\mathbb{C}^n} e^{\frac{p}{2}\operatorname{Re}\left(\left\langle \frac{1}{\sqrt{a}}u,\lambda\right\rangle\right)} \mathrm{d}\mu(u) \\
=&\left(\frac{1}{a}\right)^n \int_{\mathbb{C}^n} e^{\operatorname{Re}\left(\left\langle u,\frac{p}{2\sqrt{a}}\lambda\right\rangle\right)} \mathrm{d}\mu(u) \\
=&\left(\frac{1}{a}\right)^n \int_{\mathbb{C}^n} \left| K\left(u, \frac{p}{2\sqrt{a}}\lambda\right)\right|^2 \mathrm{d}\mu(u) \\
=&\left(\frac{1}{a}\right)^n K\left(\frac{p}{2\sqrt{a}}\lambda, \frac{p}{2\sqrt{a}}\lambda\right) \\
=&\left(\frac{1}{a}\right)^n e^{\frac{p^2}{8a}\langle\lambda,\lambda\rangle},
\end{aligned}$$

where we have used the transformation $u = \sqrt{a}z$ and Proposition 1.26 with $h \equiv 1$.
Furthermore, we have

$$Pf(\lambda) = \langle Pf, K_\lambda\rangle = \langle f, K_\lambda\rangle = \int_{\mathbb{C}^n} f(z)\overline{K_\lambda(z)}\mathrm{d}\mu(z)$$

and therefore

$$|Pf(\lambda)| \leq \|f\|_\infty \int_{\mathbb{C}^n} |K_\lambda(z)| \, e^{-\frac{1}{2}\langle z,z\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(z) = \|f\|_\infty \, e^{\frac{1}{8}\langle\lambda,\lambda\rangle},$$

where we have used the first identity with $p = a = 1$. □

Lemma 2.14. *For a nonnegative measurable function H on $\mathbb{C}^n \times \mathbb{C}^n$ with $H(w,z) \leq Be^{\frac{\langle z,z\rangle}{8}}$ for a constant B and all $z, w \in \mathbb{C}^n$, there exists a constant $C_{n,B}$ such that*

$$\begin{aligned}
&\int_{\mathbb{C}^n} H(w, \tau_{-w}(z)) \, |K_w(z)| \, K_z(z)^{\frac{1}{2}} \mathrm{d}\mu(z) \\
\leq &C_{n,B} K_w(w)^{\frac{1}{2}} \left(\int_{\mathbb{C}^n} H(w,z)^2 \mathrm{d}\mu(z)\right)^{\frac{1}{4}}
\end{aligned}$$

for all $w \in \mathbb{C}^n$.

Proof. Let H be as described and $w \in \mathbb{C}^n$. With the change of variables formula, we obtain

$$
\begin{aligned}
&\int_{\mathbb{C}^n} H(w, \tau_{-w}(z)) \left|K_w(z)\right| K_z(z)^{\frac{1}{2}} \mathrm{d}\mu(z) \\
&= \int_{\mathbb{C}^n} H(w, \tau_{-w}(z)) \exp\left(\frac{1}{2}\operatorname{Re}(\langle z, w\rangle) + \frac{1}{4}\langle z, z\rangle - \frac{1}{2}\langle z, z\rangle\right) \frac{\mathrm{d}m(z)}{(2\pi)^n} \\
&= \int_{\mathbb{C}^n} H(w, u) \exp\left(\frac{1}{2}\operatorname{Re}(\langle u + w, w\rangle) - \frac{1}{4}\langle u + w, u + w\rangle\right) \frac{\mathrm{d}m(u)}{(2\pi)^n} \\
&= \int_{\mathbb{C}^n} H(w, u) \exp\left(\frac{1}{4}|w|^2 - \frac{1}{4}|u|^2\right) \frac{\mathrm{d}m(u)}{(2\pi)^n} \\
&= K_w(w)^{\frac{1}{2}} \int_{\mathbb{C}^n} H(w, u) e^{-\frac{\langle u,u\rangle}{4}} \frac{1}{(2\pi)^n} \mathrm{d}m(u),
\end{aligned}
$$

where we have used the substitution $z = \tau_w(u)$. The Hölder inequality and Lemma 2.13 (with $a = \frac{1}{6}$ and $p = 0$) yields that

$$
\begin{aligned}
&\int_{\mathbb{C}^n} H(w, u) e^{-\frac{\langle u,u\rangle}{4}} \frac{1}{(2\pi)^n} \mathrm{d}m(u) \\
&= \int_{\mathbb{C}^n} H(w, u) e^{-\frac{3}{16}\langle u,u\rangle} e^{-\frac{1}{16}\langle u,u\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(u) \\
&\leq \left(\int_{\mathbb{C}^n} H(w, u)^4 e^{-\frac{3}{4}\langle u,u\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(u)\right)^{\frac{1}{4}} \left(\int_{\mathbb{C}^n} e^{-\frac{\langle u,u\rangle}{12}} \frac{1}{(2\pi)^n} \mathrm{d}m(u)\right)^{\frac{3}{4}} \\
&\leq \left(B^2 \int_{\mathbb{C}^n} H(w, u)^2 e^{-\frac{\langle u,u\rangle}{2}} \frac{1}{(2\pi)^n} \mathrm{d}m(u)\right)^{\frac{1}{4}} 6^{\frac{3}{4}n} \\
&= C_{n,B} \left(\int_{\mathbb{C}^n} H(w, z)^2 \mathrm{d}\mu(z)\right)^{\frac{1}{4}}
\end{aligned}
$$

with $C_{n,B} = 6^{\frac{3}{4}n} B^{\frac{1}{2}}$. □

Lemma 2.15. *For $z \in \mathbb{C}^n$, we have*

$$
\int_{\mathbb{C}^n} \left|P(f \circ \tau_w)(\tau_{-w}(z))\right| \left|K_w(z)\right| K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w) \leq 2^{2n} \|f\|_\infty K_z(z)^{\frac{1}{2}}
$$

and

$$
\int_{\mathbb{C}^n} \left|(I - P)(f \circ \tau_w)(\tau_{-w}(z))\right| \left|K_w(z)\right| K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w) \leq 2^{2n+1} \|f\|_\infty K_z(z)^{\frac{1}{2}}.
$$

Proof. Let $z \in \mathbb{C}^n$. As in Lemma 2.7 it suffices to show the first inequality and the same calculation (with $\varepsilon = \frac{1}{2}$) shows that

$$
\begin{aligned}
&\int_{\mathbb{C}^n} \left|P(f \circ \tau_w)(\tau_w(z))\right| \left|K_w(z)\right| K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w) \\
&\leq \|f\|_\infty \int_{\mathbb{C}^n} \left|K_z(u)\right| \left(\int_{\mathbb{C}^n} \left|K_w(u)\right| K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w)\right) \mathrm{d}\mu(u).
\end{aligned}
$$

Hence Lemma 2.13 (with $a = \frac{1}{2}$ and $p = 1$) yields that

$$\begin{aligned}&\int_{\mathbb{C}^n} |P(f \circ \tau_w)(\tau_w(z))|\,|K_w(z)|\,K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w)\\ \leq &\|f\|_\infty \int_{\mathbb{C}^n} |K_z(u)|\,2^n K_u(u)^{\frac{1}{2}} \mathrm{d}\mu(u)\\ =&2^{2n} \|f\|_\infty K_z(z)^{\frac{1}{2}}.\end{aligned}$$

□

With these estimates the proof of the following thoerom is similiar to the proof of Theorem 2.10.

Theorem 2.16. *Let Q be either P or $I - P$. The following statements are equivalent:*

(i) $QM_f|_{L_a^2(\mu)}$ *is compact.*

(ii) $\|QM_f k_\lambda\|_2 \to 0$ *as* $|\lambda| \to \infty$.

(iii) $\|Q(f \circ \tau_\lambda)\|_2 \to 0$ *as* $|\lambda| \to \infty$.

Proof. In Section 2.1 we proved already that (i) implies (ii) and that (ii) implies (iii).
Suppose that (iii) holds. By Schauder's theorem it suffices to show that $(QM_f|_{L_a^2(\mu)})^*$ is compact. Furthermore, we only need to prove that we can approximate $(QM_f|_{L_a^2(\mu)})^*$ by compact operators in the operator norm. For $g \in L^2(\mu)$, we have $(QM_f|_{L_a^2(\mu)})^* g \in L_a^2(\mu)$ and

$$\begin{aligned}((QM_f|_{L_a^2(\mu)})^* g)(w) &= \left\langle (QM_f|_{L_a^2(\mu)})^* g, K_w \right\rangle = \langle g, QM_f K_w\rangle\\ &= \langle g, (Q(f \circ \tau_w) \circ \tau_{-w}) K_w\rangle\\ &= \int_\Omega g(z) \overline{Q(f \circ \tau_w)(\tau_{-w}(z)) K_w(z)} \mathrm{d}\mu(z),\end{aligned}$$

where we have used Proposition 1.31. We will show that, for $R \in (0, \infty)$, the operator

$$S_R \colon L^2(\mu) \to L^2(\mu),\ g \mapsto S_R g,$$

with

$$(S_R g)(w) = \chi_{R\mathbb{B}_n}(w) \int_{\mathbb{C}^n} g(z) \overline{Q(f \circ \tau_w)(\tau_{-w}(z)) K_w(z)} \mathrm{d}\mu(z),$$

is compact. With Proposition 1.26 we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\chi_{R\mathbb{B}_n}(w)\left|Q(f\circ\tau_w)(\tau_{-w}(z))K_w(z)\right|^2\mathrm{d}\mu(z)\right)\mathrm{d}\mu(w)\\
=&\int_{\mathbb{C}^n}\chi_{R\mathbb{B}_n}(w)K_w(w)\left\|Q(f\circ\tau_w)\right\|_2^2\mathrm{d}\mu(w)\\
=&\int_{R\mathbb{B}_n}K_w(w)\left\|Q(f\circ\tau_w)\right\|_2^2\mathrm{d}\mu(w)\\
\leq&K_R(R)\left\|f\right\|_\infty^2<\infty,
\end{aligned}$$

i.e., $\chi_{R\mathbb{B}_n}(w)\overline{Q(f\circ\tau_w)(\tau_{-w}(z))K_w(z)}\in L^2(\mathbb{C}^n\times\mathbb{C}^n,\mu\otimes\mu)$ (Tonelli) and therefore S_R is a Hilbert-Schmidt operator by Lemma 1.39. By Proposition 1.40 S_R is compact. If we set

$$F\colon\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C},\ (w,z)\mapsto\chi_{\mathbb{C}^n\setminus R\mathbb{B}_n}(w)\overline{Q(f\circ\tau_w)(\tau_{-w}(z))K_w(z)},$$

we obtain

$$(((QM_f|_{L_a^2(\mu)})^*-S_R)g)(w)=\int_{\mathbb{C}^n}F(w,z)g(z)\mathrm{d}\mu(z)$$

for all $g\in L^2(\mu)$ and $w\in\mathbb{C}^n$. To complete the proof, we verify that

$$\lim_{R\to\infty}\left\|S_R-(QM_f|_{L_a^2(\mu)})^*\right\|=0.$$

Therefore, for $w,z\in\mathbb{C}^n$, we define

(i) $h(w)=K_w(w)^{\frac{1}{2}}$,

(ii) $B=2\left\|f\right\|_\infty$,

(iii) $H(w,z)=\chi_{\mathbb{C}^n\setminus R\mathbb{B}_n}(w)\left|Q(f\circ\tau_w)(z)\right|$,

(iv) $\alpha=C_{n,B}\sup\left\{\left\|Q(f\circ\tau_\lambda)\right\|_2^{\frac{1}{2}};\ \lambda\in\mathbb{C}^n\setminus R\mathbb{B}_n\right\}$,

(v) $\beta=2^{2n+1}\left\|f\right\|_\infty$.

By Lemma 2.14 and 2.13 we have, for $w \in \mathbb{C}^n$,

$$\begin{aligned}
&\int_{\mathbb{C}^n} |F(w,z)|\, h(z)\mathrm{d}\mu(z) \\
=&\int_{\mathbb{C}^n} \chi_{\mathbb{C}^n\setminus R\mathbb{B}_n}(z)\, |Q(f\circ\tau_w)(\tau_{-w}(z))|\, |K_w(z)|\, K_z(z)^{\frac{1}{2}}\mathrm{d}\mu(z) \\
=&\int_{\mathbb{C}^n} H(w,\tau_{-w}(z))\, |K_w(z)|\, K_z(z)^{\frac{1}{2}}\mathrm{d}\mu(z) \\
\leq& C_{n,B} K_w(w)^{\frac{1}{2}} \left(\int_{\mathbb{C}^n} H(w,z)^2 \mathrm{d}\mu(z)\right)^{\frac{1}{4}} \\
=& C_{n,B} K_w(w)^{\frac{1}{2}} \left(\int_{\mathbb{C}^n} \chi_{\mathbb{C}^n\setminus R\mathbb{B}_n}(w)\, |Q(f\circ\tau_w)(z)|^2\, \mathrm{d}\mu(z)\right)^{\frac{1}{4}} \\
\leq& C_{n,B} K_w(w)^{\frac{1}{2}} \sup\left\{ \|Q(f\circ\tau_\lambda)\|_2^{\frac{1}{2}}\, ;\ \lambda\in\mathbb{C}^n\setminus R\mathbb{B}_n \right\} \\
=&\alpha h(w)
\end{aligned}$$

as well as, for $z \in \mathbb{C}^n$,

$$\begin{aligned}
&\int_{\mathbb{C}^n} |F(w,z)|\, h(w)\mathrm{d}\mu(w) \\
=&\int_{\mathbb{C}^n} \chi_{\mathbb{C}^n\setminus R\mathbb{B}_n}(w)\, |Q(f\circ\tau_w)(\tau_{-w}(z))|\, |K_w(z)|\, K_w(w)^{\frac{1}{2}}\mathrm{d}\mu(w) \\
\leq& 2^{2n+1}\, \|f\|_\infty\, K_z(z)^{\frac{1}{2}} \\
=&\beta h(z)
\end{aligned}$$

by Lemma 2.15. Thus the conditions of the Schur test (Proposition 2.9) are satisfied and by hypothesis

$$\alpha \to 0$$

as $R \to \infty$. Thus

$$\left\|(Q_\gamma M_f|_{L^2_a(\mu)})^* - S_R\right\|^2 \leq \alpha\beta \to 0$$

as $R \to \infty$. □

The following lemma together with Lemma 2.12 will be used to proof Proposition 2.22.

Lemma 2.17. *For $g \in L^\infty(\mu)$, there is a constant $C_{g,n}$ such that*

$$\|\widetilde{g}\circ\tau_\lambda - P(g\circ\tau_\lambda)\|_2 \leq C_{g,n}\, \|(I-P)\,(g\circ\tau_\lambda)\|_2^{\frac{1}{4}}$$

for all $\lambda \in \mathbb{C}^n$.

Proof. Suppose $g \in L^\infty(\mu)$ and $\lambda \in \mathbb{C}^n$. With Lemma 2.13 and Remark 1.33, we obtain

$$\begin{aligned}|\widetilde{g}(w) - (Pg)(w)| &\leq |\widetilde{g}(w)| + |(Pg)(w)| \\ &\leq \|g\|_\infty + \|g\|_\infty e^{\frac{1}{8}\langle w,w\rangle} \\ &\leq 2\|g\|_\infty e^{\frac{1}{8}\langle w,w\rangle}.\end{aligned}$$

On the other hand, the function $(Pg)K_w$ lies in $L^2_a(\mu)$ for all $w \in \mathbb{C}^n$, since each factor is holomorphic and

$$\begin{aligned}\int_{\mathbb{C}^n} |(Pg)(z)|^2 |K_w(z)|^2 \,\mathrm{d}\mu(z) &\leq \|g\|_\infty^2 \int_{\mathbb{C}^n} e^{\frac{1}{4}\langle z,z\rangle} |K_w(z)|^2 \,\mathrm{d}\mu(z) \\ &= \|g\|_\infty^2 \int_{\mathbb{C}^n} |K_w(z)|^2 e^{-\frac{1}{4}\langle z,z\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\ &= \|g\|_\infty^2 2^n e^{\langle w,w\rangle} < \infty\end{aligned}$$

for all $w \in \mathbb{C}^n$, where we have used Lemma 2.13 twice. Hence

$$\begin{aligned}(Pg)(w)e^{\frac{1}{2}\langle w,w\rangle} &= (Pg)(w)K_w(w) \\ &= \langle (Pg)K_w, K_w\rangle \\ &= \int_{\mathbb{C}^n} (Pg)(z)\,|K_w(z)|^2 \,\mathrm{d}\mu(z).\end{aligned}$$

The definition of the Berezin transform yields that

$$|\widetilde{g}(w) - (Pg)(w)| \leq e^{-\frac{1}{2}\langle w,w\rangle} \int_{\mathbb{C}^n} |g(z) - (Pg)(z)|\,|K_w(z)|^2 \,\mathrm{d}\mu(z).$$

If we combine the two inequalities, we obtain, using Fubini's theorem and Lemma 2.13, that

$$\begin{aligned}&\|\widetilde{g} - Pg\|_2^2 \\ =&\int_{\mathbb{C}^n} |\widetilde{g}(w) - (Pg)(w)|^2 e^{-\frac{1}{2}\langle w,w\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(w) \\ \leq&2\|g\|_\infty \int_{\mathbb{C}^n} |g(z) - (Pg)(z)| \left(\int_{\mathbb{C}^n} |K_w(z)|^2 e^{-\frac{7}{8}\langle w,w\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(w) \right) \mathrm{d}\mu(z) \\ =&2\|g\|_\infty \int_{\mathbb{C}^n} |g(z) - (Pg)(z)| \left(\frac{4}{7}\right)^n e^{\frac{2}{7}\langle z,z\rangle} \mathrm{d}\mu(z) \\ =&2\left(\frac{4}{7}\right)^n \|g\|_\infty \int_{\mathbb{C}^n} |g(z) - (Pg)(z)|\, e^{-\frac{3}{14}\langle z,z\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(z).\end{aligned}$$

The Hölder inequality yields that

$$\begin{aligned}
&\int_{\mathbb{C}^n} |g(z)-(Pg)(z)|\, e^{-\frac{3}{14}\langle z,z\rangle} \frac{1}{(2\pi)^n}\mathrm{d}m(z)\\
=&\int_{\mathbb{C}^n} e^{-\frac{3}{112}\langle z,z\rangle}\, |g(z)-(Pg)(z)|\, e^{-\frac{3}{16}\langle z,z\rangle} \frac{1}{(2\pi)^n}\mathrm{d}m(z)\\
\leq&\left(\int_{\mathbb{C}^n} e^{-\frac{1}{28}\langle z,z\rangle}\frac{\mathrm{d}m(z)}{(2\pi)^n}\right)^{\frac{3}{4}}\left(\int_{\mathbb{C}^n}|g(z)-(Pg)(z)|^4\, e^{-\frac{3}{4}\langle z,z\rangle}\frac{\mathrm{d}m(z)}{(2\pi)^n}\right)^{\frac{1}{4}}\\
=&c_n\left(\int_{\mathbb{C}^n}|g(z)-(Pg)(z)|^4\, e^{-\frac{3}{4}\langle z,z\rangle}\frac{1}{(2\pi)^n}\mathrm{d}m(z)\right)^{\frac{1}{4}}\\
\leq&c_n\left(\int_{\mathbb{C}^n}4\,\|g\|_\infty^2\,|g(z)-(Pg)(z)|^2\, e^{-\frac{1}{2}\langle z,z\rangle}\frac{1}{(2\pi)^n}\mathrm{d}m(z)\right)^{\frac{1}{4}}\\
=&2^{\frac{1}{2}}c_n\,\|g\|_\infty^{\frac{1}{2}}\,\|g-Pg\|_2^{\frac{1}{2}}
\end{aligned}$$

with $c_n = \left(\int_{\mathbb{C}^n} e^{-\frac{1}{28}\langle z,z\rangle}\frac{1}{(2\pi)^n}\mathrm{d}m(z)\right)^{\frac{3}{4}}$. Hence

$$\begin{aligned}
\|\widetilde{g}-Pg\|_2^2 &\leq 2\left(\frac{4}{7}\right)^n \|g\|_\infty\, 2^{\frac{1}{2}}c_n\,\|g\|_\infty^{\frac{1}{2}}\,\|g-Pg\|_2^{\frac{1}{2}}\\
&= 2^{\frac{3}{2}}\left(\frac{4}{7}\right)^n c_n\,\|g\|_\infty^{\frac{3}{2}}\,\|g-Pg\|_2^{\frac{1}{2}}.
\end{aligned}$$

The result follows now by Proposition 1.34 with $C_{g,n} = 2^{\frac{3}{4}}\left(\frac{4}{7}\right)^{\frac{n}{2}} c_n^{\frac{1}{2}}\,\|g\|_\infty^{\frac{3}{4}}$. $\square$

2.4 Corollaries

We now present some useful corollaries of the main theorem. Mainly the connection between the compactness of Toeplitz and Hankel operators and the Berezin transform of its symbol will be established.

Corollary 2.18. *The operator $M_f|_{L_a^2(\rho)}$ is compact if and only if*

$$\widetilde{|f|^2}(\lambda)\to 0$$

as $\lambda\to\Delta$, or equivalently, if T_f and H_f are both compact.

Proof. For $\lambda\in G$, we have with $M_f|_{L_a^2(\rho)} = T_f + H_f$

$$\begin{aligned}
\widetilde{|f|^2}(\lambda) &= \left\langle |f|^2 k_\lambda, k_\lambda\right\rangle = \langle f k_\lambda, f k_\lambda\rangle\\
&= \|M_f k_\lambda\|_2^2 = \|T_f k_\lambda\|_2^2 + \|H_f k_\lambda\|_2^2.
\end{aligned}$$

For the if-part, we observe

$$\|T_f k_\lambda\|_2\to 0,\ \|H_f k_\lambda\|_2\to 0 \quad (\lambda\to\Delta)$$

since $\widetilde{|f|^2}(\lambda) \to 0$ as $\lambda \to \Delta$. By Theorem 2.3 the operators T_f and H_f are compact so that $M_f|_{L^2_a(\rho)} = T_f + H_f$ is also compact.
For the only if-part, we see

$$\widetilde{|f|^2}(\lambda) = \|M_f k_\lambda\|_2^2 \to 0$$

as $\lambda \to \Delta$, where we have used Proposition 2.1 and Remark 2.2. □

Corollary 2.19. *If f vanishes outside a compact set, then T_f and H_f are both compact.*

Proof. By Corollary 2.18 it suffices to show that $\widetilde{|f|^2}(\lambda) \to 0$ as $\lambda \to \Delta$. To this end, we observe

$$\begin{aligned}\widetilde{|f|^2}(\lambda) &= \frac{1}{K_\lambda(\lambda)} \int_G |f(z)|^2 \, |K_\lambda(z)|^2 \, \mathrm{d}\rho(z) \\ &\leq \frac{1}{K_\lambda(\lambda)} \int_D |f(z)|^2 \, |K_\lambda(z)|^2 \, \mathrm{d}\rho(z) \\ &\leq \frac{1}{K_\lambda(\lambda)} \, \|f\|_\infty^2 \int_D |K_\lambda(z)|^2 \, \mathrm{d}\rho(z)\end{aligned}$$

with $D = \operatorname{supp}(f)$.

(i) Bergman space: By Proposition 1.12 we have

$$\int_D \left| K_\lambda^{(\gamma)}(z) \right|^2 \mathrm{d}V_\gamma(z) \leq C_{\Omega,\gamma}^2 \int_D K_z^{(\gamma)}(z)^2 \mathrm{d}V_\gamma(z).$$

Since D is compact, the right side is bounded by a constant $C > 0$. Therefore we have

$$\widetilde{|f|^2}(\lambda) \leq \frac{C}{K_\lambda(\lambda)} \, \|f\|_\infty^2 \to 0$$

as $\lambda \to \partial\Omega$.

(ii) Fock space: By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}&\frac{1}{K_\lambda(\lambda)} \int_D |K_\lambda(z)|^2 \, \mathrm{d}\mu(z) \\ &= \int_D \exp\left(\operatorname{Re}\left(\langle z, \lambda \rangle\right) - \frac{|z|^2}{2} - \frac{|\lambda|^2}{2} \right) \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\ &\leq \int_D \exp\left(|z| \, |\lambda| - \frac{|z|^2}{2} - \frac{|\lambda|^2}{2} \right) \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\ &= \int_D \exp\left(-\frac{1}{2} \left(|z| + |\lambda| \right)^2 \right) \frac{1}{(2\pi)^n} \mathrm{d}m(z) \\ &\leq \frac{1}{(2\pi)^n} m(D) \exp\left(-\frac{1}{2} |\lambda|^2 \right) \to 0\end{aligned}$$

as $|\lambda| \to \infty$.

□

Notation 2.20. Let H_1, H_2 be Hilbert spaces and $\mathfrak{L}(H_1, H_2)$ be the Banach space of all linear bounded operators from H_1 to H_2. Furthermore, we denote by $\mathfrak{K}(H_1, H_2) \subset \mathfrak{L}(H_1, H_2)$ the closed subspace of linear compact operators from H_1 to H_2. If $H = H_1 = H_2$, we use $\mathfrak{L}(H)$ for the Banach algebra $\mathfrak{L}(H, H)$ and $\mathfrak{K}(H)$ for the closed ideal $\mathfrak{K}(H, H)$.
We write

$$AQ(\rho) = \left\{ f \in L^\infty(\rho);\ H_f \in \mathfrak{K}(L_a^2(\rho), L_a^2(\rho)^\perp) \right\}.$$

Proposition 2.21. *The set $AQ(\rho)$ is a closed subalgebra of $L^\infty(\rho)$.*

Proof. Consider the map

$$H\colon L^\infty(\rho) \to \mathfrak{L}(L_a^2(\rho), L_a^2(\rho)^\perp),\ f \mapsto H_f,$$

which is linear and a contraction since

$$\|H(f)\| = \|H_f\| \le \|(1-P)M_f\| \le \|M_f\| \le \|f\|_\infty \quad (f \in L^\infty(\rho)).$$

Therefore

$$AQ(\rho) = H^{-1}(\mathfrak{K}(L_a^2(\rho), L_a^2(\rho)^\perp))$$

is a closed subspace of $L^\infty(\rho)$. The identity

$$\begin{aligned} H_{fg}(h) &= (1-P)M_{fg}(h) \\ &= (1-P)M_f(1-P)M_g(h) + (1-P)M_f P M_g(h) \\ &= (1-P)M_f H_g(h) + H_f T_g(h) \end{aligned}$$

for all $f, g \in L^\infty(\rho)$ and $h \in L_a^2(\rho)$, shows that $AQ(\rho)$ is an algebra. □

Proposition 2.22. *If H_f is compact, i.e., $f \in AQ(\rho)$, then*

(i) $T_{f-\widetilde{f}}$ is compact,

(ii) $\widetilde{f} \in AQ(\rho)$.

Proof. Let $f \in AQ(\rho)$.

(i) We want to show that

$$\left\| P(f \circ \kappa_\lambda - \widetilde{f} \circ \kappa_\lambda) \right\|_2 \to 0$$

as $\lambda \to \Delta$. Then Theorem 2.3 shows that $T_{f-\widetilde{f}}$ is compact. Since P is continuous, it is enough to prove that

$$\left\| f \circ \kappa_\lambda - \widetilde{f} \circ \kappa_\lambda \right\|_2 \to 0$$

as $\lambda \to \Delta$. With the triangle inequality, we have

$$\left\| f \circ \kappa_\lambda - \widetilde{f} \circ \kappa_\lambda \right\|_2 \le \left\| \widetilde{f} \circ \kappa_\lambda - P(f \circ \kappa_\lambda) \right\|_2 + \| P(f \circ \kappa_\lambda) - f \circ \kappa_\lambda \|_2 ,$$

where the last term tends to 0 as $\lambda \to \Delta$ by Theorem 2.3.
For the Bergman space, we observe that by Lemma 2.12 it suffices to show that $\widetilde{g} = \widetilde{f} \circ \varphi_\lambda - P_\gamma(f \circ \varphi_\lambda)$ with $g = f \circ \varphi_\lambda - P_\gamma(f \circ \varphi_\lambda)$. But this follows immediately by Remark 1.33 and Proposition 1.34.
For the Fock space, Lemma 2.17 and Theorem 2.3 again gives us the result.

(ii) By Theorem 2.3 and by part (i) and its proof, we obtain

$$\begin{aligned}\left\| (I-P)(\widetilde{f} \circ \kappa_\lambda) \right\|_2 &\le \left\| \widetilde{f} \circ \kappa_\lambda - P(f \circ \kappa_\lambda) \right\|_2 + \left\| P(f \circ \kappa_\lambda - \widetilde{f} \circ \kappa_\lambda) \right\|_2 \\ &\to 0\end{aligned}$$

as $\lambda \to \Delta$.

□

For later use we state the following result which was established in the proof of Proposition 2.22.

Corollary 2.23. *We have*

$$\| g \circ \kappa_\lambda - \widetilde{g} \circ \kappa_\lambda \|_2 \to 0$$

as $\lambda \to \Delta$, *for all* $g \in AQ(\rho)$.

The next two results help us to strengthen the statement in Corollary 2.27 in the Fock space case.

Proposition 2.24. *For a bounded Lipschitz continuous function g with Lipschitz constant L, there is a constant $c_n > 0$ such that $\|H_g\| \le c_n \cdot L$.*

Proof. Let g be a bounded Lipschitz continuous function with Lipschitz constant L and $z \in \mathbb{C}^n$. For $w \in \mathbb{C}^n$, we define

(i) $F(w, z) = (g(z) - g(w))\, \overline{K_z(w)}$,

(ii) $h(w) = K_w(w)^{\frac{1}{2}}$,

(iii) $c_n = \int_{\mathbb{C}^n} |u|\, e^{-\frac{1}{4}\langle u,u \rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(u)$,

(iv) $\alpha = \beta = c_n L$.

Then with the transformation $u = z - w$ we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} |g(z) - g(w)| \, |K_z(w)| \, K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w) \\
\leq& L \int_{\mathbb{C}^n} |z - w| \, |K_z(w)| \, K_w(w)^{\frac{1}{2}} \mathrm{d}\mu(w) \\
=& L \int_{\mathbb{C}^n} |z - w| \, e^{-\frac{1}{4}(\langle w,w\rangle - 2\,\mathrm{Re}(\langle w,z\rangle))} \frac{1}{(2\pi)^n} \mathrm{d}m(w) \\
=& L \int_{\mathbb{C}^n} |u| \, e^{-\frac{1}{4}(\langle z-u,z-u\rangle - 2\,\mathrm{Re}(\langle z-u,z\rangle))} \frac{1}{(2\pi)^n} \mathrm{d}m(u) \\
=& L e^{\frac{1}{4}\langle z,z\rangle} \int_{\mathbb{C}^n} |u| \, e^{-\frac{1}{4}\langle u,u\rangle} \frac{1}{(2\pi)^n} \mathrm{d}m(u) \\
=& c_n L K_z(z)^{\frac{1}{2}}.
\end{aligned}$$

Thus the conditions of the Schur test (Proposition 2.9) are satisfied, i.e., together with Remark 1.28 the result follows. □

Notation 2.25. We define inductively

$$\widetilde{f}^{(0)} = f, \quad \widetilde{f}^{(l+1)} = (\widetilde{f}^{(l)})^{\sim} \quad (l \geq 0).$$

Corollary 2.26. *We have* $\left\| H_{\widetilde{f}^{(m)}} \right\| \to 0$ *as* $m \to \infty$.

Proof. Let $z, w \in \mathbb{C}^n$ and $k \in \mathbb{N}^*$. By [1, Lemma 2] we have

$$\left| \widetilde{f}^{(k)}(z) - \widetilde{f}^{(k)}(w) \right| \leq 2(2\pi)^{-\frac{1}{2}} \|f\|_\infty k^{-\frac{1}{2}} |z - w|,$$

i.e., $\widetilde{f}^{(k)}$ is Lipschitz continuous with Lipschitz constant $2(2\pi)^{-\frac{1}{2}} \|f\|_\infty k^{-\frac{1}{2}}$. Proposition 2.24 yields that

$$\left\| H_{\widetilde{f}^{(m)}} \right\| \leq 2c_n(2\pi)^{-\frac{1}{2}} \|f\|_\infty m^{-\frac{1}{2}} \to 0$$

as $m \to \infty$. □

Corollary 2.27. *The following statements are equivalent:*

(i) H_f *and* $H_{\overline{f}}$ *are compact;*

(ii) $\left\| f \circ \kappa_\lambda - \widetilde{f}(\lambda) \right\|_2 \to 0$ *as* $\lambda \to \Delta$;

(iii) $\widetilde{|f|^2}(\lambda) - \left| \widetilde{f}(\lambda) \right|^2 \to 0$ *as* $\lambda \to \Delta$.

For the Fock space, these conditions are also equivalent to

(iv) H_f *or* $H_{\overline{f}}$ *is compact.*

Proof. (i) implies (ii): With Propositions 1.36 we obtain

$$\begin{aligned}
\left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2 &\le \|f\circ\kappa_\lambda-P(f\circ\kappa_\lambda)\|_2+\left\|P(f\circ\kappa_\lambda)-\widetilde{f}(\lambda)\right\|_2\\
&=\|f\circ\kappa_\lambda-P(f\circ\kappa_\lambda)\|_2+\left\|P(f\circ\kappa_\lambda-\overline{P(\overline{f}\circ\kappa_\lambda)})\right\|_2\\
&\le \|f\circ\kappa_\lambda-P(f\circ\kappa_\lambda)\|_2+\|P\|\left\|f\circ\kappa_\lambda-\overline{P(\overline{f}\circ\kappa_\lambda)}\right\|_2\\
&\le \|f\circ\kappa_\lambda-P(f\circ\kappa_\lambda)\|_2+\left\|\overline{f}\circ\kappa_\lambda-P(\overline{f}\circ\kappa_\lambda)\right\|_2\\
&\to 0
\end{aligned}$$

as $\lambda\to\Delta$, since H_f and $H_{\overline{f}}$ are compact and Theorem 2.3.
(ii) implies (i): Suppose $\left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2\to 0$ as $\lambda\to\Delta$. By Remark 1.33 we first obtain

$$\begin{aligned}
\|f\circ\kappa_\lambda-P(f\circ\kappa_\lambda)\|_2 &\le \left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2+\left\|P(f\circ\kappa_\lambda)-\widetilde{f}(\lambda)\right\|_2\\
&=\left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2+\left\|P(f\circ\kappa_\lambda-\widetilde{f}(\lambda))\right\|_2\\
&\le \left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2+\|P\|\left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2\\
&\to 0
\end{aligned}$$

as $\lambda\to\Delta$ and secondly

$$\begin{aligned}
\left\|\overline{f}\circ\kappa_\lambda-\widetilde{\overline{f}}(\lambda)\right\|_2 &= \left\|\overline{f}\circ\kappa_\lambda-\overline{\widetilde{f}}(\lambda)\right\|_2\\
&=\left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2\\
&\to 0
\end{aligned}$$

as $\lambda\to\Delta$. Thus by Theorem 2.3 H_f and $H_{\overline{f}}$ are both compact.
(ii) and (iii) are equivalent: Since

$$\begin{aligned}
&\left\|f\circ\kappa_\lambda-\widetilde{f}(\lambda)\right\|_2^2\\
=&\int_{\mathbb{C}^n}\left|f\circ\kappa_\lambda(w)-\widetilde{f}(\lambda)\right|^2\mathrm{d}\mu(w)\\
=&\int_{\mathbb{C}^n}|f\circ\kappa_\lambda(w)|^2-2\operatorname{Re}\left(f\circ\kappa_\lambda(w)\overline{\widetilde{f}(\lambda)}\right)+\left|\widetilde{f}(\lambda)\right|^2\mathrm{d}\mu(w)\\
=&\widetilde{|f|^2}(\lambda)-2\operatorname{Re}\left(\overline{\widetilde{f}(\lambda)}\int_{\mathbb{C}^n}f\circ\kappa_\lambda(w)\mathrm{d}\mu(w)\right)+\left|\widetilde{f}(\lambda)\right|^2\\
=&\widetilde{|f|^2}(\lambda)-2\operatorname{Re}\left(\overline{\widetilde{f}(\lambda)}\widetilde{f}(\lambda)\right)+\left|\widetilde{f}(\lambda)\right|^2\\
=&\widetilde{|f|^2}(\lambda)-\left|\widetilde{f}(\lambda)\right|^2,
\end{aligned}$$

the equivalence of (ii) and (iii) is obvious.
That (i) implies (iv) is clear.
(iv) implies (i): Without loss of generality suppose $H_{\overline{f}}$ is compact. We see with the same calculation as in Proposition 2.22 and Remark 1.33 (ii) that

$$\left\|(I-P)\overline{f}\circ\tau_\lambda - \widetilde{\overline{f}}\circ\tau_\lambda\right\|_2 \leq \left\|\overline{f}\circ\tau_\lambda - \widetilde{\overline{f}}\circ\tau_\lambda\right\|_2 = \left\|f\circ\tau_\lambda - \widetilde{f}\circ\tau_\lambda\right\|_2 \to 0$$

as $|\lambda| \to \infty$ and hence by Theorem 2.3 $H_{f-\widetilde{f}}$ is compact. Furthermore

$$H_{f-\widetilde{f}^{(m)}} = \sum_{i=0}^{m-1} H_{\widetilde{f}^{(i)}-\widetilde{f}^{(i+1)}} = H_{f-\widetilde{f}} + \sum_{i=1}^{m-1} H_{(\widetilde{f}^{(i-1)}-\widetilde{f}^{(i)})^{\sim}}$$

for all $m \in \mathbb{N}$. Thus $H_{f-\widetilde{f}^{(m)}}$ is compact, since every summand is compact by using Proposition 2.22 iterative. Thus

$$\left\|H_f - H_{f-\widetilde{f}^{(m)}}\right\| = \left\|H_{\widetilde{f}^{(m)}}\right\| \to 0$$

as $m \to \infty$ by Corollary 2.26 and therefore H_f can be approximated in the operator norm by a sequence of compact operators. Hence H_f is compact.
$\square$

Chapter 3

Hankel operators with Berezin symbol

In this chapter we deal with the question when the Hankel operator $H_{\widetilde{f}}$ of a symbol $f \in L^\infty(\rho)$ is compact. This will help us in the next chapter to take a closer look at the essential spectrum of Toeplitz operators with symbols in $AQ(\rho)$.
The following lemma is a standard result in measure theory.

Lemma 3.1. *Let T be a metric space, (X, Σ, ν) be a measure space and $g\colon T \times X \to \mathbb{C}$ with*

(i) for all $t \in T$, we have $g(t, \cdot) \in \mathcal{L}^1(X)$,

(ii) for ν-almost all $x \in X$, the function $g(\cdot, x)\colon T \to \mathbb{C}$ is continuous in $t_0 \in T$,

(iii) there exist a neighborhood U of t_0 and a nonnegative integrable function h such that, for all $t \in U$,

$$|g(t, \cdot)| \leq h$$

holds ν-a.e.

Then, the function

$$G\colon T \to \mathbb{C},\ t \mapsto \int_X g(t, x) \mathrm{d}\nu(x)$$

is continuous in $t_0 \in T$.

Proof. Let $(t_k)_{k \in \mathbb{N}}$ be a sequence in U with $\lim_{k \to \infty} t_k = t_0$. Define $g_k = g(t_k, \cdot)$ for all $k \in \mathbb{N}$. By (iii) we can use the dominated convergence theorem which yields that

$$\begin{aligned}\lim_{k\to\infty} G(t_k) &= \lim_{k\to\infty} \int_X g(t_k, x)\mathrm{d}\nu(x) \\ &= \int_X \lim_{k\to\infty} g(t_k, x)\mathrm{d}\nu(x) \\ &= \int_X g(t_0, x)\mathrm{d}\nu(x) \\ &= G(t_0),\end{aligned}$$

since $g(\cdot, x)$ is continuous in $t_0 \in T$ for ν-almost all $x \in X$ by (ii). $\square$

To formulate and prove our main result in this chapter, a closer look at the properties of the Berezin transform is necessary.

Notation 3.2. We denote by $C_b(G)$ the algebra of all complex valued bounded continuous functions on G.

Proposition 3.3. *If $f \in L^\infty(\rho)$, then $\widetilde{f} \in C_b(G)$.*

Proof. Suppose $f \in L^\infty(\rho)$. Remark 1.33 shows $\widetilde{f} \in L^\infty(G)$. It is left to show that $\widetilde{f}$ is continuous.

(i) Bergman space: Since

$$\widetilde{f}(\lambda) = \int_\Omega f(w) \frac{\left|K_\lambda^{(\gamma)}(w)\right|^2}{K_\lambda^{(\gamma)}(\lambda)} \mathrm{d}V_\gamma(w) \quad (\lambda \in \Omega),$$

we define

$$g\colon \Omega \times \Omega \to \mathbb{C},\ (\lambda, w) \mapsto f(w) \frac{\left|K_\lambda^{(\gamma)}(w)\right|^2}{K_\lambda^{(\gamma)}(\lambda)}$$

and want to apply the last lemma.
Obviously, $g(\lambda, \cdot) \in L^\infty(V_\gamma) \subset L^1(V_\gamma)$ is integrable for all $\lambda \in \Omega$). For all $\lambda_0 \in \Omega$ and $w \in \Omega$, the function

$$\Omega \to \mathbb{C},\ \lambda \mapsto \frac{\left|K_\lambda^{(\gamma)}(w)\right|^2}{K_\lambda^{(\gamma)}(\lambda)}$$

is continuous in λ_0 and therefore $g(\cdot, w)$ is continuous in λ_0.
Let $\lambda \in \Omega$. Then there exists a neighborhood U of λ_0 such that $\mathrm{cl}(U) \subset \Omega$ is compact. Since the function

$$\Omega \to \mathbb{C},\ \lambda \mapsto K_\lambda^{(\gamma)}(\lambda)$$

is continuous, we obtain

$$M = \max_{\lambda \in \mathrm{cl}(U)} \left(K_\lambda^{(\gamma)}(\lambda) \right) < \infty.$$

Therefore we define the nonnegative integrable function

$$h \colon \Omega \to \Omega,\ w \mapsto \|f\|_\infty C_{\Omega,\gamma}^2 M.$$

It follows

$$|g(\lambda, w)| \leq \|f\|_\infty C_{\Omega,\gamma}^2 M,$$

for all $\lambda \in U$ and $w \in \Omega$, where we have used Proposition 1.12. Now we can apply the last lemma, which ends the proof.

(ii) Fock space: Since the function

$$\mathbb{C}^n \to L_a^2(\mu),\ \lambda \mapsto K_\lambda$$

is conjugate holomorphic and has no zeros (cf. [4, Proposition 1.4]), the functions

$$\mathbb{C}^n \to L_a^2(\mu),\ \lambda \mapsto gk_\lambda$$

and

$$\mathbb{C}^n \to \mathbb{C},\ \lambda \mapsto \langle gk_\lambda, k_\lambda \rangle$$

are C^∞.

□

Notation 3.4. By $\beta(G)$ we denote the Stone-Čech compactification of G. Via the identification

$$\beta(G)^G = \prod_G \beta(G)$$

we endow $\beta(G)^G$ with the product topology.

Remark 3.5. Every $f \in C_b(G)$ has a unique continuous extension f^β to $\beta(G)$.

Notation 3.6. Let $(\lambda_\alpha)_\alpha$ be a net in G. We use

$$\lambda_\alpha \to \Delta$$

as a shortcut for the statements

(i) The net $(\lambda_\alpha)_\alpha$ converges to a point in $\partial\Omega$ (Bergman space),

(ii) The net $(|\lambda_\alpha|)_\alpha$ converges to ∞ (Fock space).

Definition 3.7. (i) We define

$$\Phi = \{\varphi \in \beta(\Omega)^{\Omega};\ \text{there is a net } (\lambda_\alpha)_\alpha \text{ such that } \lambda_\alpha \to \partial\Omega \text{ and } \varphi_{\lambda_\alpha} \to \varphi \text{ in } \beta(\Omega)^{\Omega}\}.$$

and

$$T = \{\tau \in \beta(\mathbb{C}^n)^{\mathbb{C}^n};\ \text{there is a net } (\lambda_\alpha)_\alpha \text{ such that } |\lambda_\alpha| \to \infty \text{ and } \tau_{\lambda_\alpha} \to \tau \text{ in } \beta(\mathbb{C}^n)^{\mathbb{C}^n}\}.$$

To shorten the notation, we write Ψ for Φ or T.
Furthermore, we call the images of the elements in Ψ the *Ψ-parts* of $\beta(G)$.

(ii) Moreover, we write

$$AO\Psi = \{f \in C_b(G);\ f^\beta \circ \kappa \in H^\infty(G) \text{ for all } \kappa \in \Psi\}$$

(*analytic on Ψ-parts*) and

$$CO\Psi = \{f \in C_b(G);\ f^\beta \circ \kappa \text{ is constant for all } \kappa \in \Psi\}$$

(*constant on Ψ-parts*).

Remark 3.8. Since $\mathcal{O}(D) \cap \overline{\mathcal{O}(D)} = \mathbb{C}$ for each domain $D \subset \mathbb{C}^n$, it follows that

$$CO\Psi = AO\Psi \cap \overline{AO\Psi}.$$

By Liouville's theorem the sets AOT and COT coincide.

Lemma 3.9. *(i) The space $(\beta(G))^G$ is compact. Thus, for every net $(\lambda_\alpha)_\alpha$ with $\lambda_\alpha \to \Delta$, the net $(\kappa_{\lambda_\alpha})_\alpha$ in $(\beta(G))^G$ has a convergent subnet which converges in $(\beta(G))^G$ to a $\kappa \in \Psi$.*

(ii) The sets $AO\Psi$ and $CO\Psi$ (together with the usual addition and multiplication) are algebras.

Proof. (i) This follows immediately by Tychonoff's theorem.

(ii) Let $f, g \in AO\Psi$. Since $(f+g)^\beta = f^\beta + g^\beta$ and $(fg)^\beta = f^\beta g^\beta$ (the extensions are unique), the result follows by the observation that $H^\infty(G)$ and $\mathbb{C}$ are algebras.

□

Notation 3.10 (Bergman metric). By b we denote the Bergman metric on Ω.

The following properties of the Bergman metric can be found in [2, Section 1 & 2].

Remark 3.11. (i) The Bergman metric is Möbius invariant and induces the natural topology on Ω.

(ii) Every function $f \in L^\infty(V_\gamma)$ has bounded mean oscillation, i.e., $f \in BMO(\Omega)$.

Lemma 3.12. *If $f \in L^\infty(V_\gamma)$, then*

$$\left|\widetilde{f}(z) - \widetilde{f}(w)\right| \leq C_f b(z,w),$$

for all $z, w \in \Omega$.

Lemma 3.13. *Let $f \in L^\infty(\rho)$ and $(\omega_\alpha)_\alpha$ be a net in G with $\omega_\alpha \to \Delta$ such that $\kappa_{\omega_\alpha} \to \kappa$ in $\beta(G)^G$. Then $(\widetilde{f} \circ \kappa_{\omega_\alpha})_\alpha$ converges uniformly on all compact subsets of G to $\widetilde{f}^\beta \circ \kappa$.*
In particular, we have $\widetilde{f}^\beta \circ \kappa \in C_b(G)$.

Proof. Suppose that $f \in L^\infty(\rho)$ and that $(\omega_\alpha)_\alpha$ is a net in G with $\omega_\alpha \to \Delta$ such that $\kappa_{\omega_\alpha} \to \kappa$ in $\beta(G)^G$. It is clear that $(\widetilde{f} \circ \kappa_{\omega_\alpha})_\alpha$ converges pointwise to $\widetilde{f}^\beta \circ \kappa$, so that by the Arzelà–Ascoli theorem it suffices to show that $(\widetilde{f} \circ \kappa_{\omega_\alpha})_\alpha$ is equicontinuous on G.

(i) Bergman space: By Lemma 3.12, we have

$$\begin{aligned}\left|(\widetilde{f} \circ \varphi_{\omega_\alpha})(z) - (\widetilde{f} \circ \varphi_{\omega_\alpha})(w)\right| &\leq C_f b(\varphi_{\omega_\alpha}(z), \varphi_{\omega_\alpha}(w)) \\ &= C_f b(z,w),\end{aligned}$$

for all $z, w \in \Omega$, since Remark 3.11 holds.

(ii) Fock space: By [1, Lemma 2], we have

$$\begin{aligned}\left|(\widetilde{f} \circ \tau_{\omega_\alpha})(z) - (\widetilde{f} \circ \tau_{\omega_\alpha})(w)\right| &\leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} |\tau_{\omega_\alpha}(z) - \tau_{\omega_\alpha}(w)| \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} |z - w| \,.\end{aligned}$$

Therefore $(\widetilde{f} \circ \kappa_{\omega_\alpha})_\alpha$ is equicontinuous. □

Corollary 3.14. *Let $f \in L^\infty(\rho)$ and $(\omega_\alpha)_\alpha$ be a net in G with $\omega_\alpha \to \Delta$ such that $\kappa_{\omega_\alpha} \to \kappa$ in $\beta(G)^G$. Then $(\widetilde{f} \circ \kappa_{\omega_\alpha})_\alpha$ converges in $L^2(\rho)$ to $\widetilde{f}^\beta \circ \kappa$.*

Proof. For every compact set $C \subset G$, we obtain

$$\begin{aligned}&\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_2^2 \\ &\leq \int_C \left|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right|^2 \mathrm{d}\rho + \rho(G \setminus C) \left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_{G\setminus C}^2 \\ &\leq \int_C \left|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right|^2 \mathrm{d}\rho + \rho(G \setminus C) 4 \left\|f\right\|_\infty^2 .\end{aligned}$$

Let $\varepsilon > 0$. By Lemma 3.13 there exists a α_0 such that

$$\int_C \left|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right|^2 \mathrm{d}\rho \leq \varepsilon^2 \int_C \mathrm{d}\rho \leq \varepsilon^2$$

for all $\alpha \geq \alpha_0$. Since ρ is regular, we can choose the compact set C in such way that $\rho(G \setminus C_0) < \varepsilon$. Thus for all $\alpha \geq \alpha_0$

$$\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_2^2 \leq \varepsilon^2 + 4\varepsilon \left\|f\right\|_\infty^2 .$$

□

The question formulated at the beginning of this chapter is answered by the following theorem.

Theorem 3.15. *Let $f \in L^\infty(\rho)$. Then we have*

(i) $H_{\widetilde{f}}$ is compact if and only if $\widetilde{f} \in AO\Psi$,

(ii) $H_{\widetilde{f}}$ and $H_{\overline{\widetilde{f}}}$ are both compact if and only if $\widetilde{f} \in CO\Psi$.

Proof. Let $\kappa \in \Psi$ and let $(\omega_\alpha)_\alpha$ be a net in G with $\omega_\alpha \to \Delta$ and $\kappa_{\omega_\alpha} \to \kappa$ in $\beta(G)^G$. The chain of inequalities

$$\begin{aligned}
&\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - P(\widetilde{f} \circ \kappa_{\omega_\alpha})\right\|_2 \\
\leq &\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_2 + \left\|\widetilde{f}^\beta \circ \kappa - P(\widetilde{f}^\beta \circ \kappa)\right\|_2 + \left\|P(\widetilde{f}^\beta \circ \kappa - \widetilde{f} \circ \kappa_{\omega_\alpha})\right\|_2 \\
\leq &2\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_2 + \left\|\widetilde{f}^\beta \circ \kappa - P(\widetilde{f}^\beta \circ \kappa)\right\|_2 \\
\leq &2\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_2 + \left\|\widetilde{f}^\beta \circ \kappa - \widetilde{f} \circ \kappa_{\omega_\alpha}\right\|_2 + \left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - P(\widetilde{f} \circ \kappa_{\omega_\alpha})\right\|_2 \\
&+ \left\|P(\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa)\right\|_2 \\
\leq &4\left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - \widetilde{f}^\beta \circ \kappa\right\|_2 + \left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - P(\widetilde{f} \circ \kappa_{\omega_\alpha})\right\|_2
\end{aligned}$$

shows in view of Corollary 3.14 that

$$\lim_\alpha \left\|\widetilde{f} \circ \kappa_{\omega_\alpha} - P(\widetilde{f} \circ \kappa_{\omega_\alpha})\right\|_2 = 0$$

if and only if

$$\widetilde{f}^\beta \circ \kappa = P(\widetilde{f}^\beta \circ \kappa).$$

Using Theorem 2.3 and the compactness of $\beta(G)^G$, one easily deduces part (i). Part (ii) follows from part (i) together with Remark 3.8. □

A result about the connection between the compactness of a Toeplitz operator and the Berezin transform of its symbol is the following theorem which was proven by Raimondo in [7] ($\Omega = \mathbb{B}_n$) and by Mitkowski and Wick in [5] ($\Omega = \mathbb{D}^n$).

Theorem 3.16. *Let $f \in L^\infty(V_\gamma)$. Then T_f is compact if and only if $\widetilde{f}(\lambda) \to 0$ as $\lambda \to \partial\Omega$.*

Chapter 4

Essential spectrum of Toeplitz operators

We first recall some definitions.

Definition 4.1. (i) We call

$$\mathcal{C}(H) = \mathfrak{L}(H)/\mathfrak{K}(H)$$

the *Calkin algebra*.

(ii) For an operator $T \in \mathfrak{L}(H)$, we define the *essential spectrum* of T by

$$\sigma_e(T) = \sigma_{\mathcal{C}(H)}(T + \mathfrak{K}(H)).$$

Remark 4.2. Since the Hilbert space will always be $L^2_a(\rho)$, we omit the underlying Hilbert space and write $\mathcal{C}$ and $\mathfrak{K}$ for $\mathcal{C}(H)$ and $\mathfrak{K}(H)$, respectively.

From now on, we consider f to be in $AQ(\rho)$.
We are interested in the properties of the essential spectrums of Toeplitz operators with symbols in $AQ(\rho)$. Hence we state the following lemma.

Lemma 4.3. *The condition $|f(w)| \geq \delta$, for some $\delta > 0$ and all $w \in G$, is sufficient for $\frac{1}{f}$ to lie in $AQ(\rho)$.*

Proof. Let $f \in AQ(\rho)$ with $|f(w)| \geq \delta$ for some $\delta > 0$ and almost all $w \in G$. If we set $H = \frac{1}{f}$, it suffices to show $\|(I - P)(H \circ \kappa_\lambda)\|_2 \to 0$ as $\lambda \to \Delta$, since Theorem 2.3 holds. Let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence in G with $\lambda_m \to \Delta$ as $m \to \infty$. Corollary 2.23 gives us

$$\left\| f \circ \kappa_{\lambda_m} - \widetilde{f} \circ \kappa_{\lambda_m} \right\|_2 \to 0$$

as $m \to \infty$. Hence, by passing to a subsequence $(\lambda_{m_k})_{k\in\mathbb{N}}$ of $(\lambda_m)_{m\in\mathbb{N}}$ we can achieve that

$$f \circ \kappa_{\lambda_{m_k}} - \widetilde{f} \circ \kappa_{\lambda_{m_k}} \to 0$$

almost everywhere on G as $k \to \infty$. Let $(\lambda_\alpha)_\alpha$ be a subnet of $(\lambda_{m_k})_{k\in\mathbb{N}}$ with $\kappa_{\lambda_\alpha} \to \kappa$ in $\beta(G)^G$ for some $\kappa \in \Psi$. By Proposition 2.22 (ii) we have $\widetilde{f} \in AQ(\rho)$ and therefore $h = \widetilde{f}^\beta \circ \kappa \in H^\infty(G)$ by Theorem 3.15 (i). Furthermore

$$\|f \circ \kappa_{\lambda_\alpha} - h\|_2 \leq \left\|f \circ \kappa_{\lambda_\alpha} - \widetilde{f} \circ \kappa_{\lambda_\alpha}\right\|_2 + \left\|\widetilde{f} \circ \kappa_{\lambda_\alpha} - h\right\|_2 \overset{\alpha}{\to} 0,$$

where the second term tends to 0 by Corollary 3.14. Hence

$$\begin{aligned} |h(w)| &\geq |f \circ \kappa_{\lambda_\alpha}| - \left|\widetilde{f} \circ \kappa_{\lambda_\alpha}(w) - f \circ \kappa_{\lambda_\alpha}(w)\right| - \left|\widetilde{f}^\beta \circ \kappa(w) - \widetilde{f} \circ \kappa_{\lambda_\alpha}(w)\right| \\ &\geq \delta - \left|\widetilde{f} \circ \kappa_{\lambda_\alpha}(w) - f \circ \kappa_{\lambda_\alpha}(w)\right| - \left|\widetilde{f}^\beta \circ \kappa(w) - \widetilde{f} \circ \kappa_{\lambda_\alpha}(w)\right| \\ &\overset{\alpha}{\to} \delta, \end{aligned}$$

for almost every $w \in G$, and since $h \in H^\infty(G)$, we conclude that $|h| \geq \delta$ on G. Thus $\frac{1}{h} \in H^\infty(G)$ and we obtain

$$\begin{aligned} \left\|H \circ \kappa_{\lambda_\alpha} - \frac{1}{h}\right\|_2 &= \left\|\frac{1}{(f \circ \kappa_{\lambda_\alpha})h}(h - f \circ \kappa_{\lambda_\alpha})\right\|_2 \\ &\leq \frac{1}{\delta^2}\|f \circ \kappa_{\lambda_\alpha} - h\|_2 \overset{\alpha}{\to} 0. \end{aligned}$$

as well as

$$\left\|P(H \circ \kappa_{\lambda_\alpha}) - \frac{1}{h}\right\|_2 \overset{\alpha}{\to} 0.$$

Hence

$$\begin{aligned} \|H \circ \kappa_{\lambda_\alpha} - P(H \circ \kappa_{\lambda_\alpha})\|_2 &\leq \left\|H \circ \kappa_{\lambda_\alpha} - \frac{1}{h}\right\|_2 + \left\|P(H \circ \kappa_{\lambda_\alpha}) - \frac{1}{h}\right\|_2 \\ &\overset{\alpha}{\to} 0. \end{aligned}$$

This finishes the proof. □

Notation 4.4. Let R be $(0, 1)$ for the Bergman space and $(0, \infty)$ for the Fock space. For $r \in R$, we write

$$G_r = \begin{cases} r\mathbb{B}_n, & G = \mathbb{B}_n \text{ or } G = \mathbb{C}^n, \\ r\mathbb{D}^n, & G = \mathbb{D}^n. \end{cases}$$

The property we are interested in is the connectedness of the essential spectrum. The following theorem gives us an explicit representation of the essential spectrum from which the connectedness will follow at once.

Theorem 4.5. *We have*

$$\sigma_e(T_f) = \bigcap_{r\in R} \mathrm{cl}(\widetilde{f}(G \setminus G_r)).$$

Proof. (i) We first show that $\sigma_e(T_f) \subset \bigcap_{r\in R} \mathrm{cl}(f(G \setminus G_r))$.

Suppose therefore $\zeta \notin \mathrm{cl}(f(G \setminus G_r))$ for some $r \in R$. Define

$$g\colon G \to \mathbb{C},\ z \mapsto \begin{cases} (f(z)-\zeta)^{-1}, & z \in G \setminus G_r, \\ 1, & z \in G_r. \end{cases}$$

Then $g \in L^\infty(\rho)$ and

$$g(f-\zeta) = \chi_{G\setminus G_r} + (f-\zeta)\chi_{G_r} = \chi_G + (f-\zeta-1)\chi_{G_r}.$$

With Proposition 1.29 we obtain that

$$\begin{aligned} T_g T_{f-\zeta} &= T_{g(f-\zeta)} - H_{\overline{g}}^* H_{f-\zeta} \\ &= I + T_{(f-\zeta-1)\chi_{G_r}} - H_{\overline{g}}^* H_f. \end{aligned}$$

Since $(f-\zeta-1)\chi_{G_r} \in L^\infty(\rho)$ vanishes outside of a compact subset in G, we have by Corollary 2.19 that $H_f, T_{(f-\zeta-1)\chi_{G_r}} \in \mathfrak{K}$, so that T_g is the left inverse of $T_{f-\zeta}$ modulo $\mathfrak{K}$. Furthermore (observe $H_{\overline{\zeta}}^* = 0$)

$$T_{f-\zeta} T_g = I + T_{(f-\zeta-1)\chi_{G_r}} - H_{\overline{f}}^* H_g.$$

If we set $F = (f-\zeta)\chi_{G\setminus G_r} + \chi_{G_r}$, we see that $F = 1/g$. Since

$$H_f = H_{f-\zeta} = H_{(f-\zeta)\chi_{G\setminus G_r}} + H_{(f-\zeta)\chi_{G_r}}$$

is compact, it follows from Corollary 2.19 that $H_{(f-\zeta)\chi_{G\setminus G_r}}$ and hence also $H_F = H_{(f-\zeta)\chi_{G\setminus G_r}} + H_{\chi_{G_r}}$ is compact. By the assumptions on ζ there also exists a $\delta > 0$ such that $|F(w)| \geq \delta$ for all $w \in G$. With Lemma 4.3 we conclude that $H_g \in \mathfrak{K}$ and therefore $T_{f-\zeta}$ is also right-invertible in $\mathcal{C}$. Hence

$$T_f - \zeta + \mathfrak{K} = T_{f-\zeta} + \mathfrak{K}$$

is invertible in $\mathcal{C}$, i.e., $\zeta \notin \sigma_e(T_f)$. So we obtain the desired result.

(ii) By Proposition 2.22 the operators $T_{f-\widetilde{f}}$ and $H_{\widetilde{f}}$ are compact, i.e., $\sigma_e(T_f) = \sigma_e(T_{\widetilde{f}})$. With (i) we see that

$$\sigma_e(T_f) \subset \mathrm{cl}(\widetilde{f}(G \setminus G_r))$$

for all $r \in R$.

(iii) To complete the proof we fix a point $\zeta \in \bigcap_{r\in R} \mathrm{cl}(\widetilde{f}(G \setminus G_r))$. So we can find a net $(\lambda_\alpha)_\alpha$ in G with $\widetilde{f}(\lambda_\alpha) \xrightarrow{\alpha} \zeta$, $\lambda_\alpha \xrightarrow{\alpha} \Delta$ and $\kappa_{\lambda_\alpha} \xrightarrow{\alpha} \kappa$ in $(\beta(G))^G$ for some $\kappa \in \Psi$. Obviously we obtain

$$\zeta \leftarrow \widetilde{f}(\lambda_\alpha) = \widetilde{f} \circ \kappa_{\lambda_\alpha}(0) \to \widetilde{f}^\beta \circ \kappa(0),$$

i.e., $\widetilde{f}^{\beta} \circ \kappa(0) = \zeta$. Since $\widetilde{f}^{\beta} \circ \kappa \in H^{\infty}(G)$ by Theorem 3.15 (i), Proposition 1.35 gives us

$$P(\overline{\widetilde{f}}^{\beta} \circ \kappa) = P(\overline{\widetilde{f}^{\beta}} \circ \kappa) = P(\overline{\widetilde{f}^{\beta} \circ \kappa}) = \overline{\widetilde{f}^{\beta} \circ \kappa(0)} = \overline{\zeta}$$

so that $P(\overline{\widetilde{f}}^{\beta} \circ \kappa - \overline{\zeta}) \equiv 0$. As in the proof of the implication (ii) $\Rightarrow$ (iii) of Theorem 2.3 we have

$$\begin{aligned}
\left\| T_{\overline{\widetilde{f}}-\overline{\zeta}} k_{\lambda_\alpha} \right\|_2 &= \left\| P(\overline{\widetilde{f}}^{\beta} \circ \kappa_{\lambda_\alpha} - \overline{\zeta}) \right\|_2 \\
&= \left\| P(\overline{\widetilde{f}}^{\beta} \circ \kappa_{\lambda_\alpha} - \overline{\zeta} - (\overline{\widetilde{f}}^{\beta} \circ \kappa - \overline{\zeta})) \right\|_2 \\
&\leq \left\| \overline{\widetilde{f}}^{\beta} \circ \kappa_{\lambda_\alpha} - \overline{\widetilde{f}}^{\beta} \circ \kappa \right\|_2 \\
&\overset{\alpha}{\to} 0,
\end{aligned}$$

where we have used Corollary 3.14. Hence $T_{\overline{\widetilde{f}}-\overline{\zeta}} + \mathfrak{K}$ is not invertible in $\mathcal{C}$ and therefore $T_{\widetilde{f}-\zeta} + \mathfrak{K}$ is also not invertible in $\mathfrak{K}$. So we conclude that $\zeta \in \sigma_e(T_{\widetilde{f}}) = \sigma_e(T_f)$.

$\square$

Corollary 4.6. *The essential spectrum $\sigma_e(T_f)$ is connected.*

Proof. By Proposition 3.3 the function $\widetilde{f}$ is bounded and continuous on G, hence $\mathrm{cl}(\widetilde{f}(G \setminus G_r))$ is closed, bounded and connected for all $r \in R$. Therefore $\bigcap_{r \in R} \mathrm{cl}(\widetilde{f}(G \setminus G_r))$ is compact and connected and so is $\sigma_e(T_f)$ by Theorem 4.5. $\square$

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