Universität des Saarlandes
Naturwissenschaftlich-Technische Fakultät I
Fachrichtung Mathematik

Bachelorarbeit

Operators of Cowen-Douglas class

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August 2012

Angefertigt unter Betreuung von
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Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit vollständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, den   Johannes Schmitt

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Acknowledgements

At this point I want to thank the numerous people who contributed to this thesis. First to Professor Dr. Jörg Eschmeier who chose the subject of the thesis and who advised me during my studies. He encouraged me to develop my own proofs and invested much time in proofreading the first draft of this text.
Secondly to all the other lecturers who lead me through my studies during the last years, especially to Prof. Dr. Frank-Olaf Schreyer, who also gave me advice concerning the complex geometry section of the thesis. The study of Cowen-Douglas operators touches upon many areas of mathematics and without a broad mathematical education it would have been very difficult to grasp.
Then of course I want to thank the participants of my Bachelor seminar who gave very interesting presentations during the seminar and who listened patiently to my presentation as well.
I also want to thank my family, who supported me during my studies, emotionally and financially.
Last but certainly not least I want to thank my partner Johannes Lengler. He told me how to use BibTeX, he was always there when I needed him and our discussions greatly helped me with this thesis. I had the best ideas while I was near him and he encouraged me to write them down.

1 Introduction

In classical operator theory, many operators of interest have a relatively “thin” spectrum. For finite dimensional vector spaces H, the spectrum $\sigma(T)$ of an operator $T : H \to H$ consists of finitely many points and the corresponding normal form of T is the Jordan form. For infinite dimensions there are for instance the compact operators, for which $\sigma(T)\backslash\{0\}$ is a countable discrete set. The normal compact operators T can be written as a (possibly infinite) sum

$$T = \sum_{\lambda \in \sigma(T)\backslash\{0\}} \lambda P_{\ker(T-\lambda)}.$$

Other examples of operators with a thin spectrum include self-adjoint and unitary operators, for which the spectrum is contained in $\mathbb{R}$ and $\partial D_1(0)$, respectively. For all the operators mentioned above, the spectrum is a zero set with respect to the Lebesgue measure on $\mathbb{C}$ and those classes of operators are understood very well.
However, there are some operators, like the left shift L on $l^2(\mathbb{N})$ defined by

$$L(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots) \quad \text{for } (x_i)_{i\in\mathbb{N}} \in l^2(\mathbb{N}),$$

which have a much richer spectrum. For this particular example there is even a nonempty open set $\Omega \subset \mathbb{C}$ with

$$\Omega \subset \sigma_p(L) = \{\omega \in \mathbb{C};\ L - \omega \text{ not injective}\}.$$

In fact $\sigma_p(L) = D_1(0)$ and $\ker(L - \omega) = \langle (1, \omega, \omega^2, \omega^3, \ldots) \rangle$ for all $\omega \in D_1(0)$. So in this case, the dimension of $\ker(L - \omega)$ is constant and equal to one on $D_1(0)$.
Operators of this type were studied more closely by Cowen and Douglas in [CD78] and they are called operators of Cowen-Douglas class or simply Cowen-Douglas operators. The formal definition is as follows.

Definition. *Let H be a Hilbert space, $\Omega \subset \mathbb{C}$ an open set and n a positive integer. Then we define the Cowen-Douglas class $B_n(\Omega)$ as the set of all bounded linear operators T on H such that*

1. $\text{Ran}(T - \omega) = H \quad \text{for } \omega \in \Omega$,
2. $\dim\ker(T - \omega) = n \quad \text{for } \omega \in \Omega$,
3. $\overline{\text{LH}(\bigcup_{\omega\in\Omega} \ker(T - \omega))} = H$.

Cowen and Douglas used complex geometry to classify the operators in $B_n(\Omega)$. To illustrate this, we note that the operator L defined above is an element of $B_1(D_1(0))$.

The map

$$\gamma : D_1(0) \to l^2(\mathbb{N}),\ \omega \mapsto (1, \omega, \omega^2, \omega^3, \ldots). \tag{1.1}$$

is holomorphic (for the exact definition see Definition 2.10) and $\ker(L - \omega) = \langle \gamma(\omega) \rangle$. This can be used to show that the map $\omega \mapsto \ker(L-\omega)$ induces a hermitian holomorphic vector bundle E_T over $\Omega = D_1(0)$. In fact, this is true for arbitrary Cowen-Douglas operators T and the properties of this bundle translate into properties of T. For instance it was shown in [CD78] that, for Ω a domain and $T \in B_1(\Omega)$, the curvature of the bundle E_T is a complete unitary invariant for T.
In the following thesis we will consider a related approach first presented by Kehe Zhu in [Zhu00]. For a bounded linear operator T on H with $T \in B_n(\Omega)$, Kehe Zhu defines the notion of a spanning holomorphic cross-section. This is a holomorphic map $\gamma : \Omega \to H$ such that $\gamma(\omega) \in \ker(T - \omega)$ for all $\omega \in \Omega$ and

$$\mathrm{LH}(\{\gamma(\omega);\ \omega \in \Omega\}) \subset H \text{ is dense.} \tag{1.2}$$

The map γ defined in (1.1) is an example. For $n = 1$, we see that condition (1.2) automatically follows from the Definition of Cowen-Douglas operators, at least when γ has no zero. For $n \geq 2$, this is no longer clear and in fact it is easy to provide counterexamples. However, Zhu was able to show that there always exists a spanning holomorphic cross-section γ if Ω is a domain. This made it possible to simplify the proofs of some of the results obtained in the Cowen-Douglas theory.
Additional to the classical Cowen-Douglas theory, we also consider a generalization first proposed by Cowen and Douglas in [CD83] and later expanded upon for instance in [CS84]. Here instead of single operators, entire tuples of d operators on a common Hilbert space are considered. Thus Ω is no longer a subset of $\mathbb{C}$, but rather of $\mathbb{C}^d$. In this thesis it is proven that virtually all results from [Zhu00] for the case $d = 1$ can be generalized to arbitrary d if the set Ω meets certain reasonable conditions. The crucial point in the chain of proofs is the existence of suitable uniqueness sets for Banach spaces of holomorphic functions.
We now give an outline of the following thesis.
In Chapter 2 we provide basic notation and introduce some mathematical concepts used later, like functional Hilbert spaces and vector bundles.
In Chapter 3 we prove the existence of discrete uniqueness sets for Banach spaces of holomorphic functions with continuous point evaluations.
Chapter 4 deals with the vector bundle associated with an operator $T \in B_n(\Omega)$ and the existence of global holomorphic frames is proven.
These preparations are used in Chapter 5 to prove the existence of spanning holomorphic cross-sections.
Finally in Chapters 6-8 this existence theorem is used to classify operators which are unitary equivalent to Cowen-Douglas operators. Furthermore, we determine the similarity orbit and the commutant of a Cowen-Douglas operator tuple.

2 Basic definitions and results

In the following chapter some notation is introduced and some definitions and first elementary results are presented. Presumably many of those will be known to the reader and this section has rather the purpose of giving a reference, if a statement in the latter sections is not obvious right away. The proofs given here are standard and can be found in any textbook dealing with the respective subjects. Some special propositions are taken from [Zhu00] and the proofs below follow the ones in this paper. Finally note that we will closely follow the notation used in [Zhu00].

2.1 Cowen-Douglas operators

In the following thesis we consider a generalization of the definition of Cowen-Douglas operators, which we gave in the introduction. This generalization was already examined more closely in [CS84]. The fact that this new definition contains the classical definition as a special case is established in Remark 5.8.

Definition 2.1. *Let H be a Hilbert space, $d \in \mathbb{N}^*$ and let $T_1, \ldots, T_d \in B(H)$ operators. For $n \in \mathbb{N}^*$, $\Omega \subset \mathbb{C}^d$ open, an element $T = (T_1, \ldots, T_d) \in B(H)^d$ is called a Cowen-Douglas operator tuple of degree n on Ω if the following conditions hold for the map $T_\omega : H \to H^d,\ x \mapsto ((T_i - \omega_i)x)_{i=1}^d$:*

1. *T_ω has closed range for all $\omega \in \Omega$,*
2. *$\dim \ker(T_\omega) = n$ for all $\omega \in \Omega$,*
3. *$\overline{\mathrm{LH}(\bigcup(\ker(T_\omega); \omega \in \Omega))} = H$.*

We denote by $B_n(\Omega)$ the class of such operator tuples.

Remark 2.2. *Note that for T a Cowen-Douglas operator tuple as above, the operators $T_1, \ldots, T_d$ commute: For $\omega \in \Omega$, $x \in \ker(T_\omega)$ and $i, j \in \mathbb{N}_d$ we see*

$$T_i T_j x = T_i \omega_j x = \omega_i \omega_j x = \omega_i T_j x = T_j T_i x.$$

Thus $T_1, \ldots, T_d$ commute on $\bigcup(\ker(T_\omega); \omega \in \Omega)$ and hence by Condition 3 in Definition 2.1 these operators commute on all of H.

2.2 Linear Algebra

Proposition 2.3. *Let H be a complex vector space with inner product $\langle\cdot,\cdot\rangle$ and let $v_1,\dots,v_n \in H$ be linearly independent. Then the matrix*

$$A = (\langle v_i, v_j\rangle)_{i,j=1}^n$$

is invertible.

Proof. Assume that there is a linear combination $0 = \sum_{i=1}^n \alpha_i((\langle v_i, v_j\rangle))_{j=1}^n$ of the row vectors of A with $\alpha_1,\dots,\alpha_n \in \mathbb{C}$. Then

$$\sum_{i=1}^n \alpha_i \langle v_i, v_j\rangle = \langle \sum_{i=1}^n \alpha_i v_i, v_j\rangle = 0 \quad \text{for } j \in \mathbb{N}_n.$$

This implies

$$\sum_{j=1}^n \overline{\alpha_j} \sum_{i=1}^n \alpha_i \langle v_i, v_j\rangle = \langle \sum_{i=1}^n \alpha_i v_i, \sum_{j=1}^n \alpha_j v_j\rangle = 0.$$

Thus $\sum_{i=1}^n \alpha_i v_i = 0$, and as the vectors v_i are linearly independent, we have $\alpha_1 = \ldots = \alpha_n = 0$. Hence the row vectors of A are linearly independent and A is invertible. □

Lemma 2.4. *Let k be a field, V, W be vector spaces over k, $M \subset V$ a subset and $f : M \to W$ a map. Then there exists a linear map $F : \mathrm{LH}(M) \to W$ extending f if and only if for all $n \in \mathbb{N}^*$, $c_1,\dots,c_n \in k$ and $v_1,\dots,v_n \in M$ we have*

$$\sum_{i=1}^n c_i v_i = 0 \implies \sum_{i=1}^n c_i f(v_i) = 0. \tag{2.1}$$

In this case, F is uniquely determined by f.

Proof. The given condition is certainly necessary, because for $\sum_{i=1}^n c_i v_i = 0$ with notation as above, we see that

$$\sum_{i=1}^n c_i f(v_i) = \sum_{i=1}^n c_i F(v_i) = F(\sum_{i=1}^n c_i v_i) = F(0) = 0,$$

if F is a linear expansion of f.
Now we show that the condition (2.1) is also sufficient. If an element of $\mathrm{LH}(M)$ has two representations

$$\sum_{k=1}^n c_k v_k = \sum_{k=1}^{n'} c'_k v'_k,$$

then

$$\sum_{k=1}^{n} c_k v_k + \sum_{k=1}^{n'} (-c'_k) v'_k = 0$$

and applying condition (2.1) we see that

$$0 = \sum_{k=1}^{n} c_k f(v_k) + \sum_{k=1}^{n'} (-c'_k) f(v'_k).$$

Hence we may define $F : \mathrm{LH}(M) \to W$ by

$$F(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i f(v_i)$$

for $n \in \mathbb{N}^*$, $c_1, \dots, c_n \in k$ and $v_1, \dots, v_n \in M$. It is obvious from the definition that F is linear and extends f. The uniqueness of F follows from its linearity. □

Corollary 2.5. *Let k be a field, V, W be vector spaces over k, M a set and let $\gamma_1 : M \to V$, $\gamma_2 : M \to W$ be arbitrary maps. Then if, for all $n \in \mathbb{N}^*$, $c_1, \dots, c_n \in k$ and $z_1, \dots, z_n \in M$, we have*

$$\sum_{i=1}^{n} c_i \gamma_1(z_i) = 0 \implies \sum_{i=1}^{n} c_i \gamma_2(z_i) = 0 \tag{2.2}$$

then there is a unique linear map $A : \mathrm{LH}(\gamma_1(M)) \to W$ with $A\gamma_1(z) = \gamma_2(z)$ for all $z \in M$.

Proof. The map $f : \gamma_1(M) \to W,\ \gamma_1(z) \mapsto \gamma_2(z)$ is well-defined: Let $z_1, z_2 \in M$ with $\gamma_1(z_1) = \gamma_1(z_2)$. Then $\gamma_1(z_1) - \gamma_1(z_2) = 0$ and thus by (2.2) we have $\gamma_2(z_1) = \gamma_2(z_2)$. Applying Lemma 2.4 to f yields the desired linear map A. □

2.3 Function theory

Definition 2.6. *Let $d \in \mathbb{N}^*$ and let $\Omega \subset \mathbb{C}^d$ be an open set. Then Ω is called a domain of holomorphy if there exist no non-empty open sets $U \subset \Omega$ and $V \subset \mathbb{C}^d$ with V connected, $V \not\subset \Omega$ and $U \subset \Omega \cap V$ such that for every $f \in \mathcal{O}(\Omega)$ there exists $g \in \mathcal{O}(V)$ with $f|_U = g|_U$.*

Remark 2.7. *Without proof we give the following examples of open subsets $\Omega \subset \mathbb{C}^d$ which are domains of holomorphy:*

1. *For $d = 1$ every open set $\Omega \subset \mathbb{C}$ is a domain of holomorphy.*

2. *$\Omega = \mathbb{C}^d,\ B_r(a),\ P_r(a)$ for $a \in \mathbb{C}^d$, $r > 0$.*

3. More generally: Ω *a convex set.*

For more details see for instance [Hör90].

Proposition 2.8. *Let* $d \in \mathbb{N}^*$ *and let* $f \in \mathcal{O}(\Omega)$ *be an analytic function on an open set* $\Omega \subset \mathbb{C}^d$*. Then the function*

$$\tilde{f} : \Omega^* \to \mathbb{C}, \; z \mapsto \overline{f(\overline{z})}$$

is holomorphic again.

Proof. For $j \in \mathbb{N}_d$, $z \in \Omega^*$, we compute the limit

$$\lim_{h \to 0} \frac{\tilde{f}(z + he_j) - \tilde{f}(z)}{h} = \lim_{h \to 0} \frac{\overline{f(\overline{z + he_j})} - \overline{f(\overline{z})}}{h} = \lim_{h \to 0} \overline{\left(\frac{f(\overline{z} + \overline{he_j}) - f(\overline{z})}{\overline{h}} \right)}$$
$$= \overline{\frac{\partial}{\partial z_j} f(\overline{z})}.$$

As this limit exists everywhere, the function $\tilde{f}$ is holomorphic. □

Theorem 2.9. *Let* $f \in \mathcal{O}(\Omega)$ *be an analytic function on an open set* $\Omega \subset \mathbb{C}^d$ *and let* $K \subset \Omega$ *be a given compact set. Then the continuous function*

$$|f| : \Omega \to \mathbb{R}, \; z \mapsto |f(z)|$$

achieves its maximum m *on the compact set* K *on the boundary* ∂K *of* K*.*

Proof. If $\mathrm{Int}(K) = \emptyset$, then $\partial K = K$ and the statement is trivial. So first assume we have $\omega_0 \in \mathrm{Int}(K)$ with $|f(\omega_0)| = m$, then let $D \subset \mathbb{C}^d$ be the connected component of Ω containing ω_0. Then obviously $|f(\omega)| \leq |f(\omega_0)|$ for all $\omega \in \mathrm{Int}(K)$. By Theorem 4 on page 6 in [Gun90] we have $f|_D = f(\omega_0)$ is constant on D. We claim that $(\partial K) \cap D \neq \emptyset$. Otherwise the equality $D = (D \cap \mathrm{Int}(K)) \cup (D \cap K^c)$ would imply that $D \subset \mathrm{Int}(K)$ and hence that $\partial D \subset \partial \Omega \cap K = \emptyset$. Thus $D = \mathbb{C}^d$ and therefore $(\partial K) \cap D = \partial K \neq \emptyset$. So the function $|f|$ achieves its maximum m on K in every point of D, so especially in a point of $\partial K \cap D \subset \partial K$.
If on the other hand, for all $\omega_0 \in \mathrm{Int}(K)$, we have $|f(\omega_0)| < m$, choose a point $\omega \in K$ with $|f(\omega)| = m$ (this is always possible as $|f|$ is continuous and K is compact). Then $\omega \notin \mathrm{Int}(K)$ and thus $\omega \in \partial K = K \backslash \mathrm{Int}(K)$. This completes the proof. □

2.4 Banach space valued holomorphic functions

In order to use results from complex geometry and from function theory in the study of Cowen-Douglas operators, we need to extend the concept of holomorphy to functions with values in Banach spaces.

Definition 2.10. *Let $\Omega \subset \mathbb{C}^d$ be open and let V be a complex Banach space. Then $f : \Omega \to V$ is called holomorphic (or analytic) if it is continuous and the limit*

$$\frac{\partial}{\partial z_j} f(z) = \lim_{h \to 0} \frac{f(z + he_j) - f(z)}{h}$$

exists in V for all $z \in \Omega$ and $j \in \mathbb{N}_d$.

It is easy to see from this definition that scalar multiples and sums of holomorphic functions remain holomorphic. We now verify that our new concept of holomorphy behaves well under other standard operations like bounded linear mappings, multiplication with scalar holomorphic functions and scalar products.

Proposition 2.11. *Let V, W be complex Banach spaces and let $A \in B(V, W)$ be a bounded linear operator. If $f : \Omega \to V$ is an analytic function on an open set $\Omega \subset \mathbb{C}^d$, then the function*

$$Af : \Omega \to W, \ z \mapsto Af(z)$$

is holomorphic.

Proof. For $j \in \mathbb{N}_d$, $z \in \Omega$ we calculate the limit

$$\lim_{h \to 0} \frac{Af(z + he_j) - Af(z)}{h} = A \lim_{h \to 0} \frac{f(z + he_j) - f(z)}{h} = A \frac{\partial}{\partial z_j} f(z).$$

This limit exists everywhere and so Af is holomorphic. □

Lemma 2.12. *Let V be a Banach space and let $f \in \mathcal{O}(\Omega)$, $\gamma : \Omega \to V$ be holomorphic functions on an open set $\Omega \subset \mathbb{C}^d$. Then the function*

$$f\gamma : \Omega \to V, \ z \mapsto f(z)\gamma(z)$$

is holomorphic on Ω.

Proof. For $j \in \mathbb{N}_d$, $z \in \Omega$ we calculate the limit

$$\begin{aligned}
&\lim_{h \to 0} \frac{(f\gamma)(z + he_j) - (f\gamma)(z)}{h} \\
=& \lim_{h \to 0} \frac{f(z + he_j)\gamma(z + he_j) - f(z + he_j)\gamma(z) + f(z + he_j)\gamma(z) - f(z)\gamma(z)}{h} \\
=& \lim_{h \to 0} f(z + he_j) \lim_{h \to 0} \frac{\gamma(z + he_j) - \gamma(z)}{h} + \lim_{h \to 0} \frac{f(z + he_j) - f(z)}{h} \gamma(z) \\
=& f(z) \frac{\partial}{\partial z_j} \gamma(z) + (\frac{\partial}{\partial z_j} f(z))\gamma(z).
\end{aligned}$$

This limit exists everywhere and thus $f\gamma$ is holomorphic. □

Proposition 2.13. *Let H be a Hilbert space and let $\gamma : \Omega \to H$ be a holomorphic function on an open set $\Omega \subset \mathbb{C}^d$. Then for all $x \in H$ the function*

$$f : \Omega \to \mathbb{C},\ z \mapsto \langle \gamma(z), x \rangle$$

is holomorphic.

Proof. This follows from Proposition 2.11 by choosing $A = \langle \cdot, x \rangle$. □

Before the next Corollary we recall that a function $f : \Omega \to \mathbb{C}$ on an open set $\Omega \subset \mathbb{C}^d$ is called anti-holomorphic, if the function

$$\overline{f} : \Omega \to \mathbb{C},\ z \mapsto \overline{f(z)}$$

is holomorphic.

Corollary 2.14. *Let H be a Hilbert space and let $\gamma : \Omega \to H$ be a holomorphic function on an open set $\Omega \subset \mathbb{C}^d$. Then the function*

$$K : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto \langle \gamma(z), \gamma(w) \rangle$$

is holomorphic in z and anti-holomorphic in w.

Proof. K is holomorphic in z by Proposition 2.13. The function $\overline{K}$ satisfies $\overline{K}(z, w) = \langle \gamma(w), \gamma(z) \rangle$ and thus is holomorphic in w. Hence K is anti-holomorphic in w. □

Lemma 2.15. *Let E be a Banach space and let $f : \Omega \to E$ be a map on an open set $\Omega \subset \mathbb{C}^d$. Then f is holomorphic if and only if it is continuous and for every $u \in E'$ the function $u \circ f : \Omega \to \mathbb{C}$ is holomorphic.*

Proof. This is Theorem 9.12 in [Cha85] □

Lemma 2.16. *Let E, F be Banach spaces and let $f : \Omega \to B(E, F)$ be a map on an open set $\Omega \subset \mathbb{C}^d$. Then f is holomorphic if and only if for every $x \in E$ the function*

$$f_x : \Omega \to F,\ z \mapsto f(z)x$$

is holomorphic.

Proof. This follows for instance from Theorems 9.13 and 14.5 in [Cha85]. □

Proposition 2.17. *Let E, F, G be Banach spaces and let $f : \Omega \to B(E, F)$, $g : \Omega \to B(F, G)$ be holomorphic maps on an open set $\Omega \subset \mathbb{C}^d$. Then the map*

$$gf : \Omega \to B(E, G),\ z \mapsto g(z)f(z)$$

is holomorphic.

Proof. For $j \in \mathbb{N}_d$, $z \in \Omega$ we calculate the limit

$$\begin{aligned}
&\lim_{h\to 0} \frac{(gf)(z+he_j)-(gf)(z)}{h} \\
=&\lim_{h\to 0} \frac{g(z+he_j)f(z+he_j)-g(z+he_j)f(z)+g(z+he_j)f(z)-g(z)f(z)}{h} \\
=&\lim_{h\to 0} g(z+he_j)\lim_{h\to 0}\frac{f(z+he_j)-f(z)}{h}+\lim_{h\to 0}\frac{g(z+he_j)-g(z)}{h}f(z) \\
=&g(z)\frac{\partial}{\partial z_j}f(z)+(\frac{\partial}{\partial z_j}g(z))f(z).
\end{aligned}$$

This limit exists everywhere and thus gf is holomorphic. □

Proposition 2.18. *Let V be a Banach space and let $(f_n : \Omega \to V)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on an open set $\Omega \subset \mathbb{C}^d$. Assume that there is a function $f : \Omega \to V$ such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly towards f on all compact subsets $C \subset \Omega$. That is, for all $C \subset \Omega$ compact, $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\|f(z) - f_n(z)\| < \epsilon \text{ for } z \in C, n \geq N.$$

Then f is holomorphic.

Proof. The compact convergence obviously implies that f is continuous. Let $u \in V'$ be a bounded linear form on V and let $C \subset \Omega$ be compact. Then for all $z \in C$, $n \in \mathbb{N}$ we have

$$|u(f(z)) - u(f_n(z))| \leq \|u\|\|f(z) - f_n(z)\|.$$

Thus the sequence of holomorphic functions $(u \circ f_n)_{n\in\mathbb{N}}$ converges uniformly towards $u \circ f$ on all compact sets $C \subset \Omega$. Hence by the theorem of Weierstrass, the function $u \circ f$ is holomorphic for all $u \in V'$ and so by Lemma 2.15, f is holomorphic. □

Theorem 2.19. *Let B be a complex Banach algebra with unit and let $f : \Omega \to B$ be a holomorphic function on an open set $\Omega \subset \mathbb{C}^d$ such that $f(\Omega) \subset B^{-1} = \{b \in B;\ b \text{ is invertible}\}$. Then the function*

$$g : \Omega \to B,\ \omega \mapsto f(\omega)^{-1}$$

is holomorphic.

Proof. As the map $B^{-1} \to B^{-1}$, $a \mapsto a^{-1}$, is continuous, the function g is continuous. Let $z_0 \in \Omega$ and choose $r > 0$ such that $D_r(z_0) \subset \Omega$ and

$$\|f(z_0)^{-1}(f(z) - f(z_0))\| < \frac{1}{2}$$

for all $z \in D_r(z_0)$. Then using the Neumann series we have

$$\begin{aligned} f(z)^{-1} &= [f(z_0)(1 + f(z_0)^{-1}(f(z) - f(z_0)))]^{-1} \\ &= (\sum_{k=0}^{\infty}(f(z_0)^{-1}(f(z_0) - f(z))^k)f(z_0)^{-1} \end{aligned}$$

for $z \in D_r(z_0)$ and the Neumann series converges uniformly on $D_r(z_0)$. All partial sums are holomorphic functions in z by Propositions 2.11 and 2.17 and thus by Proposition 2.18 the Neumann series is holomorphic in z. Again with Proposition 2.11 the result follows. □

2.5 Functional Hilbert spaces and reproducing kernels

Definition 2.20. *Let Ω be an arbitrary set. A linear subspace $\mathcal{H}$ of the complex vector space $\mathbb{C}^\Omega$ equipped with the structure of a Hilbert space is called a functional Hilbert space if all point evaluations*

$$\delta_z : \mathcal{H} \longrightarrow \mathbb{C}, \ f \mapsto f(z) \quad (z \in \Omega)$$

are continuous.

It is clear that the maps δ_λ are also linear. So by the Riesz representation theorem, any δ_z can be expressed as the scalar product with some unique vector $K_z \in \mathcal{H}$, that is,

$$\delta_z(f) = \langle f, K_z \rangle \text{ for } f \in \mathcal{H}.$$

Since K_z is a function from Ω to $\mathbb{C}$, we obtain a complex-valued function on $\Omega \times \Omega$.

Definition 2.21. *Let $\mathcal{H} \subset \mathbb{C}^\Omega$ be a functional Hilbert space. The unique function $K : \Omega \times \Omega \to \mathbb{C}$ satisfying*

1. $K(\cdot, w) \in \mathcal{H}$ for $w \in \Omega$,
2. $\langle f, K(\cdot, w) \rangle = f(w)$ for $f \in \mathcal{H}, \ w \in \Omega$

is called the reproducing kernel of $\mathcal{H}$.

While this definition of a reproducing kernel strongly depends on the structure of a corresponding functional Hilbert space $\mathcal{H}$, it is also possible to give an intrinsic criterion characterising functions $K : \Omega \times \Omega \to \mathbb{C}$ which are reproducing kernels of functional Hilbert spaces on Ω. The following result first appeared in in a paper of Aronszajn (see [Aro50]) although he attributed it to earlier work of Moore (see [MB35]).

Theorem 2.22 (Moore-Aronsajn)**.** *Let Ω be a set. A map $K : \Omega \times \Omega \to \mathbb{C}$ is the reproducing kernel of a functional Hilbert space if and only if*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \overline{c_j} K(z_j, z_i) \geq 0 \tag{2.3}$$

for all $n \in \mathbb{N}^*,\ c_1, \ldots, c_n \in \mathbb{C},\ z_1, \ldots, z_n \in \Omega.$

In general, a function $K : \Omega \times \Omega \to \mathbb{C}$ satisfying (2.3) is called positive definite. We will now collect some simple properties of positive definite functions, that will be of use later on.

Proposition 2.23. *Let* Ω *be a set and let* $K : \Omega \times \Omega \to \mathbb{C}$ *be an arbitrary map.*
(a) K *is positive definite if and only if for all* $n \in \mathbb{N}^*, z_1, \ldots, z_n \in \Omega$ *the matrix*

$$(K(z_i, z_j))_{i,j=1}^n \in M(n \times n, \mathbb{C})$$

is positive semi-definite.
(b) If K *is positive definite, then*

$$K(w, z) = \overline{K(z, w)} \text{ for all } z, w \in \Omega \tag{2.4}$$

Proof. (a) For all $n \in \mathbb{N}^*, z_1, \ldots, z_n \in \Omega, c_1, \ldots, c_n \in \mathbb{C}$ we have

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} K(z_j, z_i) = \left\langle (K(z_i, z_j))_{i,j=1}^n \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right\rangle_{\mathbb{C}^n}.$$

Thus the definition of a positive definite map is just a reformulation of the positivity of all $n \times n$-matrices $(K(z_i, z_j))_{i,j=1}^n$.
(b) This is clear from the fact that, for all $z, w \in \Omega$, the 2×2-matrix

$$\begin{pmatrix} K(z,z) & K(z,w) \\ K(w,z) & K(w,w) \end{pmatrix}$$

is positive semi-definite, hence hermitian and so $K(w, z) = \overline{K(z, w)}$. □

Proposition 2.24. *Let* Ω *be a set and let* $K : \Omega \times \Omega \to \mathbb{C}$ *be the reproducing kernel of a functional Hilbert space* H*. Then* $H_0 = \mathrm{LH}(K(\cdot, \mu); \mu \in \Omega) \subset H$ *is a dense subspace.*

Proof. Let $f \in H$ with $x \perp H_0$. Then we obtain

$$f(\mu) = \langle f, K(\cdot, \mu) \rangle = 0 \quad \text{for all } \mu \in \Omega.$$

Thus $f = 0$ and so $\overline{H_0} = H$. □

We now want to look at a special type of reproducing kernels which are fundamental for the understanding of Cowen-Douglas operators. These kernels induce functional Hilbert spaces of holomorphic functions.

Proposition 2.25. *Let* H *be a Hilbert space and let* $\gamma : \Omega \to H$ *be a holomorphic function on an open set* $\Omega \subset \mathbb{C}^d$*. Then the function*

$$K_\gamma : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto \langle \gamma(z), \gamma(w) \rangle$$

is the reproducing kernel of a Hilbert space of holomorphic functions in Ω.

Proof. We first show that K_γ is positive definite. For all $n \in \mathbb{N}^*$, $c_1, \dots, c_n \in \mathbb{C}$, $\omega_1, \dots, \omega_n \in \Omega$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}K_\gamma(\omega_j,\omega_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\langle\gamma(\omega_j),\gamma(\omega_i)\rangle = \langle\sum_{j=1}^{n}\overline{c_j}\gamma(\omega_j),\sum_{i=1}^{n}\overline{c_i}\gamma(\omega_i)\rangle \geq 0.$$

So by Theorem 2.22, K_γ is the reproducing kernel of a a functional Hilbert space V on $\mathbb{C}^\Omega$. It remains to show that V consists of holomorphic functions. By Proposition 2.13 we see that for all $w \in \Omega$ the function $K_\gamma(\cdot, w) \in V$ is holomorphic. So the linear subspace $V_0 = \mathrm{LH}(\{K_\gamma(\cdot, w); w \in \Omega\}) \subset V$ consists of holomorphic functions. By Proposition 2.24 we have that V_0 is a dense subspace of V. Let now $f \in V$ be arbitrary and let $(f_n)_{n\in\mathbb{N}}$ be an approximating sequence in V_0. It suffices to show that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly on all compact subsets of Ω, or explicitly, that for all $C \subset \Omega$ compact, $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\infty,C} < \epsilon \quad \text{for all } n, m \geq N.$$

This implies that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on all compact sets $C \subset \Omega$ to a function $\tilde{f}$ and by the theorem of Weierstrass the function $\tilde{f}$ is holomorphic. Since convergence in V implies pointwise convergence and the pointwise limit of $(f_n)_{n\in\mathbb{N}}$ is unique, we have that $f = \tilde{f}$ is holomorphic.

So let $C \subset \Omega$ be compact. Then for all $z \in C$ and $n, m \in \mathbb{N}$, we have

$$\begin{aligned}|f_n(z) - f_m(z)|^2 &= |\langle f_n, K_\gamma(\cdot, z)\rangle - \langle f_m, K_\gamma(\cdot, z)\rangle|^2\\ &= |\langle f_n - f_m, K_\gamma(\cdot, z)\rangle|^2\\ &\leq \|f_n - f_m\| \cdot \sup_{w\in C}\|K_\gamma(\cdot, w)\|\end{aligned}$$

It remains to show that $\sup_{w\in C}\|K_\gamma(\cdot, w)\|$ is finite. But this follows from

$$\sup_{w\in C}\|K_\gamma(\cdot, w)\|^2 = \sup_{w\in C}\langle K_\gamma(\cdot, w), K_\gamma(\cdot, w)\rangle = \sup_{w\in C} K_\gamma(w, w) = \sup_{w\in C}\|\gamma(w)\|^2 < \infty,$$

since γ is holomorphic and hence continuous. Thus the proof is complete. $\square$

Now, returning to the general theory, we will define a relation on the set of all reproducing kernels on a given set Ω. This relation will be crucial for identifying Cowen-Douglas operators, which are similar.

Definition 2.26. *Let* $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ *reproducing kernels on am arbitrary set* Ω. *Then we write* $K_1 \prec K_2$ *if there is a constant* $C > 0$ *such that* $CK_2 - K_1$ *is a positive definite.*
For Hilbert spaces H_1, H_2, $\Omega \subset \mathbb{C}^d$ *open and holomorphic maps* $\gamma_1 : \Omega \to H_1$, $\gamma_2 : \Omega \to H_2$, *we write* $\gamma_1 \prec \gamma_2$ *if* $K_{\gamma_1} \prec K_{\gamma_2}$ *with* K_{γ_i} *defined as in Proposition 2.25.*

Remark 2.27. *From Proposition 2.23 it is clear that if we have a constant $C > 0$ as in Definition 2.26, then for all $n \in \mathbb{N}^*$ and $z_1, \dots, z_n \in \Omega$, we have*

$$(K_1(z_i, z_j))_{i,j=1}^n \le C(K_2(z_i, z_j))_{i,j=1}^n$$

as an inequality between complex $n \times n$-matrices.

Definition 2.28. *Let Ω be a set and let $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ be reproducing kernels on Ω. Then we write $K_1 \sim K_2$ if $K_1 \prec K_2$ and $K_2 \prec K_1$.*
For Hilbert spaces H_1, H_2, $\Omega \subset \mathbb{C}^d$ open and holomorphic maps $\gamma_1 : \Omega \to H_1$, $\gamma_2 : \Omega \to H_2$ we write $\gamma_1 \sim \gamma_2$ if $\gamma_1 \prec \gamma_2$ and $\gamma_2 \prec \gamma_1$.

Remark 2.29. *It is easy to show that $\prec$ defines a reflexive, transitive relation on the set of all reproducing kernels on a given set Ω and $\sim$ defines an equivalence relation on this set.*

We will now prove a property of these relations which will be useful in the classification of the similarity classes of Cowen-Douglas operators.

Proposition 2.30. *Let H be a Hilbert space and let $\gamma : \Omega \to H$ be a holomorphic function on an open set $\Omega \subset \mathbb{C}^d$. Let $P \in B(H)$ be a bounded positive operator on H. Define*

$$K_1 : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto \langle \gamma(z), \gamma(w) \rangle$$
$$K_2 : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto \langle P\gamma(z), \gamma(w) \rangle.$$

Then K_1, K_2 are reproducing kernels and $K_2 \prec K_1$. If P is invertible, then $K_1 \sim K_2$.

Proof. By Proposition 2.25 the function K_1 is a reproducing kernel. To see that K_2 is also a reproducing kernel, let $n \in \mathbb{N}^*$, $c_1, \dots, c_n \in \mathbb{C}$ and $z_1, \dots, z_n \in \Omega$. Then we see

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} K_2(z_i, z_j) = \left\langle P \sum_{i=1}^n c_i \gamma(z_i), \sum_{j=1}^n c_j \gamma(z_j) \right\rangle \ge 0.$$

So K_2 is positive definite and hence a reproducing kernel by Theorem 2.22. To see that $K_2 \prec K_1$ we observe that the operator $Q_1 = \|P\|I - P$ is positive because its spectrum is contained in R_+^0. Therefore the map $K : \Omega \times \Omega \to \mathbb{C}$ with

$$\begin{aligned} K(z, w) &= \|P\|K_1(z, w) - K_2(z, w) = \|P\|\langle \gamma(z), \gamma(w) \rangle - \langle P\gamma(z), \gamma(w) \rangle \\ &= \langle (\|P\|I - P)\gamma(z), \gamma(w) \rangle = \langle Q_1\gamma(z), \gamma(w) \rangle \quad \text{for } z, w \in \Omega \end{aligned}$$

is a reproducing kernel as we have seen in the proof above. Thus we get $K_2 \prec K_1$.
Assume now that P is also invertible. Then $0 \in \sigma(P)^c$ and, as this set is open, there is $r > 0$ such that $D_r(0) \subset \sigma(P)^c$. Then for $C = \frac{1}{r}$ the operator $Q_2 = CP - I$ is positive due to the spectral mapping theorem. This implies that the map $K' : \Omega \times \Omega \to \mathbb{C}$ with

$$K'(z, w) = \langle Q_2\gamma(z), \gamma(w) \rangle = CK_2(z, w) - K_1(z, w) \quad \text{for } z, w \in \Omega$$

is a reproducing kernel. Therefore $K_1 \prec K_2$ and hence $K_1 \sim K_2$. □

2.6 Multipliers

We now introduce the concept of multipliers and multiplication operators. The latter are maps between functional Hilbert spaces on the same set Ω, which are induced by pointwise multiplication with a complex function on Ω:

Definition 2.31. *Let Ω be a set and let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{C}^\Omega$ be two functional Hilbert spaces. For functions $f, g : \Omega \to \mathbb{C}$ the product $fg : \Omega \to \mathbb{C}$ is defined in the natural way: $(fg)(z) = f(z)g(z)$ for all $z \in \Omega$. The elements of*

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) := \{\phi : \Omega \to \mathbb{C}; \phi\mathcal{H}_1 \subset \mathcal{H}_2\}$$

are called multipliers from $\mathcal{H}_1$ to $\mathcal{H}_2$.
For $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, we call

$$M_\phi : \mathcal{H}_1 \to \mathcal{H}_2, \ f \mapsto \phi f$$

the multiplication operator with symbol ϕ.

Lemma 2.32. *Let Ω be a set and let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{C}^\Omega$ be two functional Hilbert spaces with reproducing kernels K_1 and K_2, respectively. Then for $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, we have*
(a) $M_\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded and linear.
(b) $M_\phi^* K_2(\cdot, w) = \overline{\phi(w)} K_1(\cdot, w)$ for all $w \in \Omega$

Proof. (a) The linearity of M_ϕ is clear. We show the continuity with the closed graph theorem. Take a convergent sequence $(f_n)_{n\in\mathbb{N}} \to f$ in $\mathcal{H}_1$ with $M_\phi f_n \to g$ in $\mathcal{H}_2$. Then because convergence in $\mathcal{H}_1$ implies pointwise convergence:

$$(M_\phi f_n)(z) = \phi(z) f_n(z) \to \phi(z) f(z) = (M_\phi f)(z) \text{ as } n \to \infty.$$

On the other hand, we get

$$(M_\phi f_n)(z) \to g(z) \text{ as } n \to \infty.$$

As limits in $\mathbb{C}$ are unique, we have $(M_\phi f)(z) = g(z)$ for all $z \in \Omega$. Thus M_ϕ is continuous by the closed graph theorem.
(b) Let $f \in \mathcal{H}_1, w \in \Omega$ be arbitrary. Then

$$\begin{aligned} \langle f, M_\phi^* K_2(\cdot, w)\rangle &= \langle M_\phi f, K_2(\cdot, w)\rangle = (M_\phi f)(w) = \phi(w) f(w) \\ &= \phi(w)\langle f, K_1(\cdot, w)\rangle = \langle f, \overline{\phi(w)} K_1(\cdot, w)\rangle. \end{aligned}$$

Because f was arbitrary, we have proven that $M_\phi^* K_2(\cdot, w) = \overline{\phi(w)} K_1(\cdot, w)$. □

As with reproducing kernels, we can find a convenient criterion characterising the functions which are multipliers.

Theorem 2.33. *Let Ω be a set and let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{C}^\Omega$ be two functional Hilbert spaces with reproducing kernels K_1, K_2. Then for $\phi : \Omega \to \mathbb{C}$, the following are equivalent:*
(i) $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$,
(ii) There is a constant $c > 0$ such that the map

$$\gamma_c : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto c^2 K_2(z, w) - \phi(z) K_1(z, w) \overline{\phi(w)}$$

is positive definite (or equivalently, $\phi(z) K_1(z, w) \overline{\phi(w)} \prec K_2(z, w)$).

Proof. *(i)* $\implies$ *(ii)*: For $n \in \mathbb{N}^*, c_1, \ldots, c_n \in \mathbb{C}, z_1, \ldots, z_n \in \Omega$ and $c \geq \|M_\phi^*\| = \|M_\phi\|$, we have by Lemma 2.32:

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \gamma_c(z_j, z_i) \\
=&c^2 \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} K_2(z_j, z_i) - \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \phi(z_j) K_1(z_j, z_i) \overline{\phi(z_i)} \\
=&c^2 \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \langle K_2(\cdot, z_i), K_2(\cdot, z_j) \rangle - \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \phi(z_j) \langle K_1(\cdot, z_i), K_1(\cdot, z_j) \rangle \overline{\phi(z_i)} \\
=&c^2 \langle \sum_{i=1}^n c_i K_2(\cdot, z_i), \sum_{j=1}^n c_j K_2(\cdot, z_j) \rangle - \langle \sum_{i=1}^n c_i \overline{\phi(z_i)} K_1(\cdot, z_i), \sum_{j=1}^n c_j \overline{\phi(z_j)} K_1(\cdot, z_j) \rangle \\
=&c^2 \| \sum_{i=1}^n c_i K_2(\cdot, z_i) \|^2 - \| \sum_{i=1}^n c_i \overline{\phi(z_i)} K_1(\cdot, z_i) \|^2 \\
=&c^2 \| \sum_{i=1}^n c_i K_2(\cdot, z_i) \|^2 - \| M_\phi^* \sum_{i=1}^n c_i K_1(\cdot, z_i) \|^2 \geq 0.
\end{aligned} \tag{2.5}$$

So γ_c is positive definite for $c \geq \|M_\phi\|$.
(ii) $\implies$ *(i)*: Let $c > 0$ be a positive real number such that γ_c is positive definite. Suppose now we have $n \in \mathbb{N}^*$, $c_1, \ldots, c_n \in \mathbb{C}, z_1, \ldots, z_n \in \Omega$ such that $\sum_{i=1}^n c_i K_2(\cdot, z_i) = 0$. Then (2.5) implies

$$\| \sum_{i=1}^n c_i \overline{\phi(z_i)} K_1(\cdot, z_i) \|^2 \leq c^2 \| \sum_{i=1}^n c_i K_2(\cdot, z_i) \|^2 = 0$$

Hence Corollary 2.5 shows that there is a unique linear map

$$T_0 : \mathrm{LH}(\{K_2(\cdot, z); z \in \Omega\}) \to \mathrm{LH}(\{K_1(\cdot, z); z \in \Omega\})$$

with $T_0 K_2(\cdot, z) = \overline{\phi(z)} K_1(\cdot, z)$. Inequality (2.5) also shows that T_0 is bounded. Therefore it can be extended to a bounded linear map $T \in B(\mathcal{H}_2, \mathcal{H}_1)$. Then we find for

$f \in \mathcal{H}_1, z \in \Omega$:

$$\begin{aligned}(T^*f)(z) &= \langle T^*f, K_2(\cdot, z)\rangle = \langle f, TK_2(\cdot, z)\rangle = \langle f, \overline{\phi(z)}K_1(\cdot, z)\rangle = \langle \phi(z)f, K_1(\cdot, z)\rangle \\ &= \phi(z)f(z) = (\phi f)(z).\end{aligned}$$

Thus $\phi f = T^*f \in \mathcal{H}_2$ for all $f \in \mathcal{H}_1$, so ϕ is a multiplier from $\mathcal{H}_1$ to $\mathcal{H}_2$. □

2.7 Vector bundles

In the study of Cowen-Douglas operators, complex vector bundles arise in a natural way. In [Zhu00] Zhu uses a result from differential geometry to simplify the representation of Cowen-Douglas operators found in [CD78]. We will recall the basic definitions (restricting ourselves to complex vector bundles) and present the result used by Zhu. The following paragraph is mostly a translation of parts of Chapter 29 in [For77], but adapted to our purposes.

Definition 2.34. *Let $n \in \mathbb{N}^*$ and let $\pi : E \to X$ be a continuous map between topological spaces E and X. Let every fiber $E_x := \pi^{-1}(\{x\})$ for $x \in X$ be equipped with the structure of an n-dimensional complex vector space. Then $\pi : E \to X$ (or short: E) is called a vector bundle of rank n over X if the following condition is satisfied:*
For every $a \in X$, there exist an open neighbourhood $U \subset X$ and a homeomorphism h from $E_U = \pi^{-1}(U)$ to $U \times \mathbb{C}^n$ (equipped with the product topology) satisfying:

1. $\mathrm{pr}_1 \circ h = \pi$,
2. *for every $x \in U$, the map $h|_{E_x}$ is a vector space isomorphism from E_x to $\{x\} \times \mathbb{C}^n \cong \mathbb{C}^n$.*

In this case E is called the total space and X is called the base space of the vector bundle. The map $h : E_U \to U \times \mathbb{C}^n$ is called a linear chart of E over U.
If $(U_i)_{i \in I}$ is a family of open sets $U_i \subset X$ covering X and $h_i : E_{U_i} \to U_i \times \mathbb{C}^n$ are linear charts, then $\mathcal{A} = (h_i)_{i \in I}$ is called an atlas of E.

Definition 2.35. *A vector bundle of rank n is called trivial if (with notation as above) there exists a global linear chart $h : E \to X \times \mathbb{C}^n$.*

Definition 2.36. *Let $\pi_i : E_i \to X$, $i = 1, 2$ be two vector bundles over the same base space. A vector bundle morphism from E_1 to E_2 is a continuos map $f : E_1 \to E_2$ such that*

1. $\pi_1 = \pi_2 \circ f$
2. *for every $x \in X_1$, the map $f|_{\pi_1^{-1}(\{x\})} : \pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{x\})$ is linear.*

The vector bundles which we will use in the study of Cowen-Douglas operators will have an additional structure. In order to define it, we cite the following theorem from [For77] without proof.

Theorem 2.37. *Let $\pi : E \to X$ a vector bundle of rank n and let $(h_i)_{i\in I}$ be an atlas consisting of functions $h_i : E_{U_i} \to U_i \times \mathbb{C}^n$ for $i \in I$. Then there are uniquely determined continuous maps*

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$$

such that

$$(h_i \circ h_j^{-1})(x,t) = (x, g_{ij}(x)t) \quad \text{for all } (x,t) \in (U_i \cap U_j) \times \mathbb{C}^n.$$

Definition 2.38. *The functions g_{ij} in Theorem 2.37 are called transition functions for the atlas $(h_i)_{i\in I}$.*

Now we can define the notion of holomorphic vector bundles. In [For77] the base space of the vector bundle is a Riemann surface. We use an open subset of $\mathbb{C}^d$ instead.

Definition 2.39. *Let $X \subset \mathbb{C}^d$ be an open set equipped with the relative topology of $\mathbb{C}^d$ and let $\pi : E \to X$ a vector bundle of rank n over X. Furthermore, let*

$$\mathcal{A} = (h_i : E_{U_i} \to U_i \times \mathbb{C}^n;\ i \in I)$$

be an atlas of E. Then $\mathcal{A}$ is called holomorphic if the corresponding transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$ are holomorphic for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$.
Two holomorphic atlases $\mathcal{A}, \mathcal{A}'$ of E are called holomorphically equivalent, if $\mathcal{A} \cup \mathcal{A}'$ is a holomorphic atlas of E. It is easy to prove that this defines an equivalence relation on the set of all holomorphic atlases of E. The equivalence classes are called holomorphic linear structures.
A holomorphic vector bundle is a vector bundle $\pi : E \to X$ together with a holomorphic linear structure.
A holomorphic vector bundle $\pi : E \to X$ is called holomorphically (or analytically) trivial if its holomorphic linear structure contains an atlas consisting of a single linear chart $h : E \to X \times \mathbb{C}^n$.

Definition 2.40. *Let $\pi_i : E_i \to X$ be holomorphic vector bundles of rank $n_i \in \mathbb{N}^*$, $i = 1, 2$. Then a map $F : E_1 \to E_2$ is called a holomorphic bundle map if it is a vector bundle morphism and if, for all linear charts $h_i : \pi_i^{-1}(U_i) \to U_i \times \mathbb{C}^{n_i}$ contained in atlases representing the holomorphic linear structure of E_i for $i = 1, 2$ with $U_1 \cap U_2 \neq \emptyset$, we have $h_2 \circ F \circ h_1^{-1} : (U_1 \cap U_2) \times \mathbb{C}^{n_1} \to (U_1 \cap U_2) \times \mathbb{C}^{n_2}$ is a holomorphic map.*

Definition 2.41. *Let $n \in \mathbb{N}^*$, $\pi : E \to X$ a vector bundle of rank n and $U \subset X$ open. Then a section (or cross-section) in E over U is a continuous function $f : U \to E$ such that $\pi \circ f = \mathrm{id}_U$. If $h_i : E_{U_i} \to U_i \times \mathbb{C}^n$ is a linear chart with $U_i \cap U \neq \emptyset$, then the map $f_i : U_i \cap U \to \mathbb{C}^n$ with $f_i = \mathrm{pr}_2 \circ h_i \circ f$ is called the representation of f in the chart h_i.*

Note that the above map f_i is continuous and satisfies

$$h_i(f(x)) = (x, f_i(x)) \quad \forall x \in U_i \cap U.$$

Definition 2.42. *Let $\pi : E \to X$ be a holomorphic vector bundle of rank n and let $U \subset X$ be open. Let $(h_i : E_{U_i} \to U_i \times \mathbb{C}^n;\ i \in I)$ be an atlas representing the holomorphic linear structure of E. Then a section $f : U \to E$ in E over U is called holomorphic if its representation $f_i : U_i \cap U \to \mathbb{C}^n$ in the chart h_i is a holomorphic function for all $i \in I$ with $U_i \cap U \neq \emptyset$.*

Remark 2.43. *It is easy to show that the above definition does not depend on the choice of the atlas $(h_i; i \in I)$ in the holomorphic linear structure of E.*

Definition 2.44. *Let $n \in \mathbb{N}^*$, $\pi : E \to X$ a holomorphic vector bundle of rank n and $U \subset X$ open. Then holomorphic sections $f_1, \dots, f_n : U \to E$ are called a local holomorphic frame in E over U if $\pi^{-1}(x) = \mathrm{LH}(\{f_1(x), \dots, f_n(x)\})$ with respect to the vector space structure of $\pi^{-1}(x)$ for all $x \in U$. For $U = X$, sections with this property $f_1, \dots, f_n$ are called a global holomorphic frame.*

Lemma 2.45. *Let $\pi_i : E_i \to X$ be a holomorphic vector bundle of rank n_i for $i = 1, 2$ and let $f_1 : U \to E_1$ be a holomorphic section on an open set $U \subset X$. If $F : E_1 \to E_2$ is a holomorphic bundle map, then $f_2 = F \circ f : U \to E_2$ is a holomorphic section.*

Proof. The function f_2 is continuous as the composition of continuous functions and $\pi_2 \circ f_2 = \pi_2 \circ F \circ f_1 = \pi_1 \circ f_1 = \mathrm{id}_U$ as F is a vector bundle morphisms and f_1 a section. Thus f_2 is a section.
Let now $(h_i^j : E_{U_i^j} \to U_i^j \times \mathbb{C}^n;\ i \in I_j)$ be atlases representing the holomorphic linear structure of E_j for $j = 1, 2$. Let $i \in I_2$ such that $U \cap U_i^2 \neq \emptyset$. We have to show that the representation $f_2^i : U_i^2 \cap U \to \mathbb{C}^{n_2}$ of f_2 in the chart h_i^2 is holomorphic. Let $\omega_0 \in U_i^2 \cap U$ and $j \in I_1$ be given such that $\omega_0 \in U_j^1$. Then the representation $f_1^j : U_j^1 \cap U \to \mathbb{C}^{n_1}$ of f_1 in h_j^1 is holomorphic and as F is a holomorphic bundle map, the function $h_i^2 \circ F \circ (h_j^1)^{-1} : (U_j^1 \cap U_i^2) \times \mathbb{C}^{n_1} \to (U_j^1 \cap U_i^2) \times \mathbb{C}^{n_2}$ is holomorphic. Then in the open neighbourhood $\tilde{U} = U \cap U_j^1 \cap U_i^2$ of ω_0 the function f_2^i can be described as

$$\begin{aligned} f_2^i &= \mathrm{pr}_2 \circ h_i^2 \circ f_2 = \mathrm{pr}_2 \circ h_i^2 \circ F \circ f_1 \\ &= \underbrace{\mathrm{pr}_2}_{\text{holomorphic}} \circ \underbrace{h_i^2 \circ F \circ (h_j^1)^{-1}}_{\text{holomorphic}} \circ \underbrace{h_j^1 \circ f_1}_{=id_{\tilde{U}} \times f_1^j:\text{holomorphic}} . \end{aligned}$$

Thus f_2^i is holomorphic, which completes the proof. □

Definition 2.46. *Let $\pi : E \to X$ be a holomorphic vector bundle with representing atlas $(h_i : E_{U_i} \to U_i \times \mathbb{C}^n;\ i \in I)$. Then E is called a hermitian holomorphic vector bundle if each fibre $E_x = \pi^{-1}(\{x\})$ $(x \in X)$ is equipped with a scalar product $(\cdot, \cdot)_{E_x} : E_x \times E_x \to \mathbb{C}$ in such a way that the unique matrix-valued functions $m_i : U_i \to M(n \times n, \mathbb{C})$ with*

$$(h_i^{-1}(x, c), h_i^{-1}(x, d))_{E_x} = \langle m_i(x)c, d\rangle_{\mathbb{C}^n} \quad (x \in U_i, c, d \in \mathbb{C}^n)$$

are C^∞-functions for every $i \in I$.

Remark 2.47. *(a) In the above setting the matrices $m_i(x) \in M(n \times n, \mathbb{C})$ are positive and invertible.*
(b) The definition does not depend on the choice of the representing atlas. More precisely, if $h_i : E_{U_i} \to U_i \times \mathbb{C}^n$ and $h_j : E_{U_j} \to U_j \times \mathbb{C}^n$ $(i, j \in I)$ are two charts in a representing atlas with $U = U_i \cap U_j \neq \emptyset$ and if $g_{ij} : U \to \mathrm{GL}(n, \mathbb{C})$ is the associated transition function, that is,

$$h_i \circ h_j^{-1}(x, c) = (x, g_{ij}(x)c) \text{ for } (x, c) \in U \times \mathbb{C}^n,$$

then by definition

$$\begin{aligned} \langle m_i(x) g_{ij}(x) c, g_{ij}(x) d \rangle_{\mathbb{C}^n} &= (h_i^{-1}(x, g_{ij}(x)c), h_i^{-1}(x, g_{ij}(x)d))_{E_x} \\ &= (h_j^{-1}(x, c), h_j^{-1}(x, d))_{E_x} = \langle m_j(x) c, d \rangle_{\mathbb{C}^n} \end{aligned}$$

for $x \in U$, $c, d \in \mathbb{C}^n$. Therefore the relation

$$g_{ij}(x)^* m_i(x) g_{i,j}(x) = m_j(x)$$

holds for all $x \in U$. As the function g_{ij} is holomorphic and thus C^∞, the map m_i is C^∞ on U if and only if the map m_j is C^∞ on U.

Definition 2.48. *Two hermitian holomorphic vector bundles $\pi_i : E_i \to X$, $i = 1, 2$ are called equivalent if there is a holomorphic bundle map $F : E_1 \to E_2$ such that*

$$F|_{\pi_1^{-1}(\{x\})} : \pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{x\})$$

defines a unitary map for all $x \in X$.

3 Uniqueness sets

In order to be able to prove the existence of spanning holomorphic cross-sections, we need to construct uniqueness sets for Banach spaces of holomorphic functions. We recall the definition of a uniqueness set.

Definition 3.1. *Let Ω be a set, $X \subset \mathbb{C}^\Omega$ a set of functions on Ω. A subset $A \subset \Omega$ is called a uniqueness set for X if, for all $f \in X$, we have that $f|_A = 0$ already implies $f = 0$.*

As a first step we will construct a uniqueness set for a special situation and then conclude that we actually covered a very broad range of spaces.

Theorem 3.2. *Let $\Omega \subset \mathbb{C}^d$ be open and let $\gamma : \Omega \to X'$ be a holomorphic function into the topological dual of a Banach space X. Consider the set $X_\gamma = \{\widehat{x}; x \in X\}$ where $\widehat{x} : \Omega \to \mathbb{C}$ is defined by*

$$\widehat{x}(z) := \langle x, \gamma(z)\rangle.$$

for $x \in X$. Then there exists a countable subset $A \subset \Omega$ such that

1. *A has no accumulation point in Ω,*
2. *A is a uniqueness set for X_γ.*

Proof. Let $K \subset \Omega$ be compact. Then the function $\gamma|_K : K \to X'$ is uniformly continuous due to the Heine-Cantor theorem. Now consider an exhaustion by compact sets $(K_n)_{n\in\mathbb{N}}$ of Ω, i.e., for all $n \in \mathbb{N}$ the set $K_n \subset \Omega$ is compact, $K_n \subset \mathrm{Int}(K_{n+1})$ and $\bigcup_{n\in\mathbb{N}} K_n = \Omega$. We formally define $K_{-1} = \emptyset$. Then we choose a sequence $(l_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^*$ and elements $(a_n^i)_{i=1}^{l_n}$ in Ω as follows:
Let $n \in \mathbb{N}$, then $K_n' = K_n \backslash \mathrm{Int}(K_{n-1})$ is compact and thus there exists $\delta_n > 0$ such that

$$\|\gamma(z_1) - \gamma(z_2)\| < \frac{1}{n} \quad \text{for all } z_1, z_2 \in K_n' \text{ with } |z_1 - z_2| < \delta_n. \tag{3.1}$$

We have an open covering $K_n' \subset \bigcup_{z\in K_n'} B_{\delta_n}(z)$ of the compact set K_n'. So there exist $l_n \in \mathbb{N}^*, a_n^1, \ldots, a_n^{l_n} \in K_n'$ such that $K_n' \subset \bigcup_{i=1}^{l_n} B_{\delta_n}(a_n^i)$ is a finite subcovering.
We claim that the set $A = \{a_n^i; n \in \mathbb{N},\ i \in \mathbb{N}_{l_n}\}$ has the properties we required.
(1) Let $z \in \Omega$ be arbitrary. Then there exists $n \in \mathbb{N}$ with $z \in K_{n-1} \subset \mathrm{Int}(K_n)$. Then for all $m > n$, $i \in \mathbb{N}_{l_m}$ the point $a_m^i \in K_m \backslash \mathrm{Int}(K_{m-1}) \subset K_m \backslash \mathrm{Int}(K_n)$ does not lie in $\mathrm{Int}(K_n)$. So of all the elements of A at most the finitely many points $\{a_k^i; k \in \mathbb{N}_n,\ i \in \mathbb{N}_{l_k}\}$ can belong to the open neighbourhood $\mathrm{Int}(K_n)$ of z. Hence z is not an accumulation point of A.

(2) Assume we have $f \in X_\gamma$ with $f(a) = 0$ for all $a \in A$, but $f \neq 0$. Then there is $x \in X$, $x \neq 0$, with $f(z) = \langle x, \gamma(z)\rangle$ for all $z \in \Omega$. We now show that

$$|f(z)| < \frac{\|x\|}{n} \quad \text{for } n \in \mathbb{N}^*,\ z \in K_n'.$$

Let $n \in \mathbb{N}^*,\ z \in K_n'$ be given. Then there exists $i \in \mathbb{N}_{l_n}$ such that $|z - a_n^i| < \delta_n$. By (3.1) we obtain that $\|\gamma(z) - \gamma(a_n^i)\| < \frac{1}{n}$. Then we conclude

$$\begin{aligned} |f(z)| &= |f(z) - f(a_n^i)| = |\langle x, \gamma(z)\rangle - \langle x, \gamma(a_n^i)\rangle| \\ &= |\langle x, \gamma(z) - \gamma(a_n^i)\rangle| \leq \|\gamma(z) - \gamma(a_n^i)\| \cdot \|x\| < \frac{\|x\|}{n}. \end{aligned}$$

Now by the version of the maximum principle formulated in Theorem 2.9, for $n \in \mathbb{N}$, the continuous function $|f|$ achieves its maximum on the compact set K_n on the boundary $\partial K_n \subset K_n \backslash \mathrm{Int}(K_n) \subset K_n \backslash \mathrm{Int}(K_{n-1}) = K_n'$. But this implies

$$|f(z)| < \frac{\|x\|}{n} \quad \text{for } z \in K_n.$$

Let $z \in \Omega$ be arbitary. Then there exists $n \in \mathbb{N}$ with $z \in K_m$ for all $m \geq n$. But then by the inequality above, $|f(z)|$ must be arbitarily small, so $f(z) = 0$. Thus $f = 0$, a contradiction. □

Proposition 3.3. *Let $\Omega \subset \mathbb{C}^d$ be open and let X be a Banach space of holomorphic functions on Ω with continuous point evaluations*

$$\delta_z : X \to \mathbb{C},\ f \mapsto f(z)$$

for $z \in \Omega$. Then the map

$$\delta : \Omega \to X',\ z \mapsto \delta_z$$

is analytic.

Proof. Let $g \in X$ be a function. Then $g(z) = \delta_z(g)$ for all $z \in \Omega$. Thus $\delta_z(g)$ defines a holomorphic function and so by Lemma 2.16 the function δ is holomorphic. □

Corollary 3.4. *Let $\Omega \subset \mathbb{C}^d$ be open and let X be a Banach space of holomorphic functions on Ω with continuous point evaluations δ_z for $z \in \Omega$. Then there exists a countable subset $A \subset \Omega$ such that*

1. *A has no accumulation point in Ω and*
2. *A is a uniqueness set for X_γ.*

Proof. By Proposition 3.3 the function $\delta : \Omega \to X',\ z \mapsto \delta_z$, is holomorphic. With notation as in Theorem 3.2 we have with $\gamma = \delta$:

$$\widehat{f}(z) = \langle f, \delta_z\rangle = f(z) \quad \text{for } f \in X, z \in \Omega.$$

Therefore $X_\delta = X$ as subsets of $\mathbb{C}^\Omega$. Now choose a uniqueness set A for X_δ without accumulation point in Ω as in Theorem 3.2. Let $f \in X$ with $f(a) = 0$ for all $a \in A$. Then $\widehat{f} \in X_\delta$ and $\widehat{f}(a) = f(a) = 0$ for all $a \in A$. But this implies $f(z) = \widehat{f}(z) = 0$ for all $z \in \Omega$, so $f = 0$. Thus A is also a uniqueness set for X and has no accumulation point in Ω as required. □

The proof of the above corollary shows that every Banach space X of holomorphic functions with continuous point evaluations can be identified with X_δ as constructed above. Lemma 3.6 below shows that also the converse is true, that is X_γ can be equipped with the structure of a Banach space in a natural way.
We recall now a basic result which will be used in the proof.

Proposition 3.5. *Let X be a complex Banach space and let $M \subset X'$ be a weak*-closed subspace. Then for $u \in M$, the function $\widehat{u} : X/{}^\perp M \to \mathbb{C}$, $\widehat{u}([x]) = u(x)$ is well-defined, linear and bounded and the map*

$$M \to (X/{}^\perp M)',\ u \mapsto \widehat{u}$$

is an isometric isomorphism between Banach spaces.

Lemma 3.6. *Let $\Omega \subset \mathbb{C}^d$ be open and let $\gamma : \Omega \to X'$ be a holomorphic function into the topological dual of a Banach space X. For every $x \in X$, define a holomorphic function $\widehat{x} : \Omega \to \mathbb{C}$ by*

$$\widehat{x}(z) = \langle x, \gamma(z)\rangle \quad \forall z \in \Omega.$$

Then the set $X_\gamma = \{\widehat{x}; x \in X\}$ can be equipped with the structure of a Banach space such that, for all $K \subset \Omega$ compact, there exists $C > 0$ with

$$|\widehat{x}(z)| \le C\|\widehat{x}\| \quad \text{for } x \in X \text{ and } z \in K.$$

Proof. The functions $\widehat{x}$ are holomorphic by Lemma 2.16. The set X_γ is a linear subspace of $\mathbb{C}^\Omega$ and the map $\phi_0 : X \to X_\gamma,\ x \mapsto \widehat{x}$, is linear, because

$$\alpha\widehat{x} + \beta\widehat{y} = \alpha\langle x, \gamma(\cdot)\rangle + \beta\langle y, \gamma(\cdot)\rangle = \langle \alpha x + \beta y, \gamma(\cdot)\rangle = \widehat{\alpha x + \beta} y \text{ for } \alpha, \beta \in \mathbb{C}, x, y, \in X.$$

By definition, ϕ_0 is surjective. We determine the kernel of ϕ_0:

$$\begin{aligned}\ker(\phi_0) &= \{x \in X; \langle x, \gamma(z)\rangle = 0 \text{ for every } z \in \Omega\}\\ &= {}^\perp\{\gamma(z); z \in \Omega\} = {}^\perp\overline{\mathrm{LH}(\{\gamma(z); z \in \Omega\})}^{w*}.\end{aligned}$$

So with the definition

$$X_0^* = \overline{\mathrm{LH}(\{\gamma(z); z \in \Omega\})}^{w*} \subset X'$$

we see that $\ker(\phi_0) = {}^\perp X_0^*$. But then

$$\phi : X_0 = X/{}^\perp X_0^* \to X_\gamma,\ [x] \mapsto \phi_0(x) = \widehat{x}$$

is an isomorphism of vector spaces. We now define a norm on X_γ by $\|\widehat{x}\| = \|[x]\|_{X_0}$.

As ${}^{\perp}X_0^* \subset X$ is a closed subspace and X is a Banach space, also X_0 and X_γ are Banach spaces.
Let $K \subset \Omega$ be compact. Then for all $x \in X_0, z \in K$ we have that

$$|\widehat{x}(z)| = |\langle x, \gamma(z) \rangle| = |\langle [x], \widehat{\gamma(z)} \rangle| \leq \|[x]\| \|\widehat{\gamma(z)}\| = \|\widehat{x}\| \|\gamma(z)\|,$$

where we have used Proposition 3.5. As γ is holomorphic, it is continuous and so $\|\gamma(z)\|$ is bounded on the compact set K. This completes the proof. □

4 Vector bundles associated with Cowen-Douglas operators

In the following section we will show that every Cowen-Douglas operator tuple in $B_n(\Omega)$ gives rise to a canonical hermitian holomorphic vector bundle of rank n over Ω. This was already shown in [CD78] for single Cowen-Douglas operators (i.e. $\Omega \subset \mathbb{C}$).

Proposition 4.1. *Let H_1, H_2 be Hilbert spaces, let $T \in B(H_1, H_2)$ be an operator with closed range and let $P \in B(H_1)$ be the orthogonal projection on* $\mathrm{ker}T$*. Then there exists an operator $S \in B(H_2, H_1)$ such that $ST = I - P$.*

Proof. The linear mapping

$$\tilde{T} : \ker(T)^{\perp} \to \mathrm{Ran}(T),\ x \mapsto Tx$$

is a bijection between Banach spaces and hence the inverse operator $\tilde{S} : \mathrm{Ran}(T) \to \ker(T)^{\perp}$ is continuous. Now choose any operator $S \in B(H_2, H_1)$ extending $\tilde{S}$, for instance by defining $S|_{\mathrm{Ran}(T)^{\perp}} = 0$. Then $ST = I - P$ is the orthogonal projection on $\ker(T)^{\perp}$ □

Theorem 4.2. *Let $T = (T_i)_{i=1}^d \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple with notation as in Definition 2.1. Then for each $\omega_0 \in \Omega$, there exist an open neighbourhood $V \subset \Omega$ and holomorphic functions $\gamma_1, \ldots, \gamma_n : V \to H$ such that*

$$\mathrm{LH}(\{\gamma_1(\omega), \ldots, \gamma_n(\omega)\}) = \mathrm{ker}T_{\omega} = \bigcap_{i=1}^{d} \ker(T_i - \omega_i) \quad \text{for } \omega \in V.$$

Proof. Without loss of generality we may assume $\omega_0 = 0$ (otherwise replace Ω by $\Omega - \omega_0$ and T by $T - \omega_0 = (T_1 - \omega_{0,1}, \ldots, T_d - \omega_{0,d})$). Then for $T_0 \in B(H, H^d)$, choose $S \in B(H^d, H)$ as in Proposition 4.1 such that $ST_0 = I - P$, where P is the orthogonal projection on $\mathrm{ker}T_0 = \bigcap_{i=1}^{d} \ker(T_i)$. Now we decompose S in the following way: for $k \in \mathbb{N}_d$ we define

$$S_k : H \to H,\ x \mapsto Si_k(x),$$

where $i_k : H \to H^d$ is the inclusion in the k-th component of H^d. Then we have

$$S(y_1, \ldots, y_d) = S_1 y_1 + \ldots + S_d y_d \quad \text{for } y_1, \ldots, y_d \in H.$$

Now for $M = d \max_{i=1,\ldots,d} \|S_i\|$, we define $R = M^{-1}$ for $M > 0$ and $R = \infty$ for M=0 and set $V = P_R(0) \cap \Omega$. Then with the Neumann series, the operator $B(\omega) =$

$I - \omega_1 S_1 - \ldots - \omega_d S_d$ has a continuous inverse $A(\omega)$ for $\omega \in V$, because for $M > 0$ we have

$$\|\omega_1 S_1 + \ldots + \omega_d S_d\| \leq |\omega_1|\|S_1\| + \ldots + |\omega_d|\|S_d\| < dM^{-1} \max_{i=1,\ldots,d} \|S_i\| = 1$$

and for $M = 0$ it follows that $S_1 = \ldots = S_d = 0$, hence the statement is trivial. Consider the functions $Q : V \to B(H^d, H)$, $P : V \to B(H)$ defined by

$$\begin{aligned} Q(\omega) &= A(\omega)S, \\ P(\omega) &= A(\omega)P \end{aligned}$$

for $\omega \in V$. As $\dim \ker T_0 = n$ and $A(\omega)$ is an isomorphism, $\dim \operatorname{Ran}(P(\omega)) = n$. For $\omega \in V$, we show that $Q(\omega)T_\omega = I - P(\omega)$: Let $x \in H$ be arbitrary. Then

$$\begin{aligned} (I - P(\omega))x &= A(\omega)((I - P) - \omega_1 S_1 - \ldots - \omega_d S_d)x \\ &= A(\omega)(ST_0 - \omega_1 S_1 - \ldots - \omega_d S_d)x \\ &= A(\omega)(S(T_1 x, \ldots, T_d x) - \omega_1 S_1 x - \ldots - \omega_d S_d x) \\ &= A(\omega)(S_1(T_1 - \omega_1)x + \ldots + S_d(T_d - \omega_d)x) \\ &= A(\omega)S((T_1 - \omega_1)x, \ldots, (T_d - \omega_d)x) \\ &= Q(\omega)T_\omega x. \end{aligned}$$

Now we conclude that

$$\ker T_\omega \subset \ker(Q(\omega)T_\omega) = \ker(I - P(\omega)) \subset \operatorname{Ran} P(\omega),$$

where we used that $x = P(\omega)x \in \operatorname{Ran} P(\omega)$ for all $x \in \ker(I - P(\omega))$. As

$$n = \dim \ker T_\omega = \dim \operatorname{Ran} P(\omega),$$

we must have $\ker T_\omega = \operatorname{Ran} P(\omega)$. We define functions $\gamma_1, \ldots, \gamma_n : V \to H$ by $\gamma_i(\omega) = P(\omega)e_i \in \ker T_\omega$ for $i = 1, \ldots, n$, where $(e_i)_{i=1}^n$ is a basis of $\ker T_0$. For each $\omega \in V$, the vectors $\gamma_1(\omega), \ldots, \gamma_n(\omega)$ are linearly independent as $A(\omega)$ is an isomorphism. It remains to show that they are holomorphic functions. For this it suffices to show that $A(\omega)$ is holomorphic by Lemma 2.16. It is obvious that the map

$$B : V \to B(H),\ \omega \mapsto I - \omega_1 S_1 - \ldots - \omega_d S_d$$

is continuous and partially holomorphic and thus holomorphic. By construction, $B(\omega)$ is invertible for all $\omega \in V$ with inverse $A(\omega)$. Therefore by Theorem 2.19, A is holomorphic, which finishes the proof. □

Let $T = (T_i)_{i=1}^d \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas operator tuple on an open set $\Omega \subset \mathbb{C}^d$. We equip the set

$$E_T = \{(\omega, x) \in \Omega \times H; x \in \ker T_\omega\}$$

with the relative topology of the product toplology of $\Omega \times H$. Then the map

$$\pi : E_T \to \Omega,\ (\omega, x) \mapsto \omega.$$

is continuous. For $\omega \in \Omega$, the fiber $\pi^{-1}(\{\omega\}) = \{\omega\} \times \ker T_\omega$ can be identified with $\ker T_\omega$ and thus becomes an n-dimensional complex vector space.
By Theorem 4.2, for every point $\omega_0 \in \Omega$ we can choose an open neighbourhood $V_{\omega_0} \subset \Omega$ of ω_0 and holomorphic functions $\gamma_1^{\omega_0}, \ldots, \gamma_n^{\omega_0} : V_{\omega_0} \to H$ such that the vectors $\gamma_1^{\omega_0}(\omega), \ldots, \gamma_n^{\omega_0}(\omega)$ form a basis of $\ker T_\omega$ for every point $\omega \in V_{\omega_0}$. Thus we can define bijective maps $h_{\omega_0} : \pi^{-1}(V_{\omega_0}) \to V_{\omega_0} \times \mathbb{C}^n$ by setting

$$h_{\omega_0}(\omega, x) = (\omega, (\alpha_i)_{i=1}^n) \text{ if } x = \sum_{i=1}^n \alpha_i \gamma_i^{\omega_0}(\omega).$$

Obviously the maps $(h_{\omega_0})_{\omega_0 \in \Omega}$ depend on the choices of V_{ω_0} and $\gamma_1^{\omega_0}, \ldots, \gamma_n^{\omega_0}$. For brevity of notation, we will suppress these additional parameters.

Theorem 4.3. *Let $T = (T_i)_{i=1}^d \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple on an open set $\Omega \subset \mathbb{C}^d$. Then $\pi : E_T \to \Omega$ defines a vector bundle of rank n over Ω. Any family $(h_{\omega_0})_{\omega_0 \in \Omega}$ chosen as explained above is a holomorphic atlas of E_T.*

Proof. For every $\omega_0 \in \Omega$, the map h_{ω_0} is a linear chart:
It is obvious that $\mathrm{pr}_1 \circ h_{\omega_0} = \pi|_{\pi^{-1}(V_{\omega_0})}$. For every $\omega \in V_{\omega_0}$, the map $h_{\omega_0}|_{(E_T)_\omega}$ defines an isomorphism

$$\pi^{-1}(\{\omega\}) = \{\omega\} \times \ker T_{\omega_0} \to \{\omega\} \times \mathbb{C}^n.$$

So we have to show that h_{ω_0} is a homeomorphism. First, it is bijective with inverse given by

$$\begin{aligned} h_{\omega_0}^{-1} &: V_{\omega_0} \times \mathbb{C}^n \to \pi^{-1}(V_{\omega_0}), \\ &(\omega, (\alpha_i)_{i=1}^n) \mapsto (\omega, \sum_{i=1}^n \alpha_i \gamma_i^{\omega_0}(\omega)). \end{aligned}$$

It is clear that this map is continuous as $\gamma_1^{\omega_0}, \ldots, \gamma_n^{\omega_0}$ are continuous. Now we show that h_{ω_0} is continuous. Let $(\omega, x) \in \pi^{-1}(V_{\omega_0})$. Then we want to determine $\alpha(\omega, x) = (\alpha_i(\omega, x))_{i=1}^n \in \mathbb{C}^n$ with $\sum_{i=1}^n \alpha_i(\omega, x) \gamma_i^{\omega_0}(\omega) = x$. For this we apply the linear forms $\langle \cdot, \gamma_j^{\omega_0}(\omega) \rangle$ to both sides of this equation for $j = 1, \ldots, n$. This yields the following system of linear equations:

$$(\langle \gamma_i^{\omega_0}(\omega), \gamma_j^{\omega_0}(\omega) \rangle)_{j,i=1}^n \alpha(\omega, x) = (\langle x, \gamma_j^{\omega_0}(\omega) \rangle)_{j=1}^n.$$

As $\gamma_1^{\omega_0}(\omega), \ldots, \gamma_n^{\omega_0}(\omega)$ are linearly independent, the matrix

$$A(\omega) = (\langle \gamma_i^{\omega_0}(\omega), \gamma_j^{\omega_0}(\omega) \rangle)_{j,i=1}^n$$

is invertible by Proposition 2.3. Thus the equation above determines $\alpha(\omega, x)$ uniquely:

$$\alpha(\omega, x) = A(\omega)^{-1}(\langle x, \gamma_j^{\omega_0}(\omega)\rangle)_{j=1}^n.$$

As the map $A : V_{\omega_0} \to \mathbb{C}^{n\times n},\ \omega \mapsto A(\omega)$, is continuous, $\mathrm{Ran}(A) \subset \mathrm{GL}(n, \mathbb{C})$ and the map $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C}),\ M \mapsto M^{-1}$, is continuous by Cramer's rule, the map $\alpha : \pi^{-1}(V_{\omega_0}) \to \mathbb{C}^n,\ (\omega, x) \mapsto \alpha(\omega, x)$, is continuous. But this implies that h_{ω_0} is continuous as $h_{\omega_0}(\omega, x) = (\omega, \alpha(\omega, x))$ for all $(\omega, x) \in \pi^{-1}(V_{\omega_0})$.
Now we show that the family $h = (h_{\omega_0})_{\omega_0\in\Omega}$ is a holomorphic atlas of E_T. It is clear that the h_{ω_0} are linear charts on E_T by the above argument and that $(V_{\omega_0})_{\omega_0\in\Omega}$ is an open cover of Ω. Thus h is an atlas of E_T. So let $\omega_1, \omega_2 \in \Omega$ with $V_{\omega_1} \cap V_{\omega_2} \neq \emptyset$. Then we have to show that the transition function $g : V_{\omega_1} \cap V_{\omega_2} \to \mathrm{GL}(n, \mathbb{C})$ with

$$(h_{\omega_1} \circ h_{\omega_2}^{-1})(\omega, \alpha) = (\omega, g(\omega)\alpha) \quad \text{for } \omega \in V_{\omega_1} \cap V_{\omega_2}, \alpha \in \mathbb{C}^n$$

is holomorphic.
Let $\omega_0, \omega \in V_{\omega_1} \cap V_{\omega_2}$, $\alpha \in \mathbb{C}^n$ be arbitrary. Then we have

$$(h_{\omega_1} \circ h_{\omega_2}^{-1})(\omega, \alpha) = h_{\omega_1}(\omega, \sum_{i=1}^n \alpha_i \gamma_i^{\omega_2}(\omega)) = (\omega, \beta(\omega, \alpha)),$$

where $\beta(\omega, \alpha) \in \mathbb{C}^n$ is a vector such that $\sum_{i=1}^n \beta_i(\omega, \alpha)\gamma_i^{\omega_1}(\omega) = \sum_{i=1}^n \alpha_i \gamma_i^{\omega_2}(\omega)$. Now applying the linear forms $\langle \cdot, \gamma_j^{\omega_1}(\omega_0)\rangle$, for $j = 1, \ldots, n$, to this equation gives the following system of linear equations for $\beta(\omega, \alpha)$:

$$(\langle \gamma_i^{\omega_1}(\omega), \gamma_j^{\omega_1}(\omega_0)\rangle)_{j,i=1}^n \beta(\omega, \alpha) = (\langle \gamma_i^{\omega_2}(\omega), \gamma_j^{\omega_1}(\omega_0)\rangle)_{j,i=1}^n \alpha. \tag{4.1}$$

We consider the function

$$A : V_{\omega_1} \cap V_{\omega_2} \to \mathbb{C}^{n\times n},\ \omega \mapsto (\langle \gamma_i^{\omega_1}(\omega), \gamma_j^{\omega_1}(\omega_0)\rangle)_{j,i=1}^n.$$

Again with Proposition 2.3, the matrix $A(\omega_0)$ is regular, so $\det(A(\omega_0)) \neq 0$. As the function $\det(A(\cdot))$ is continuous, there is an open neighbourhood V of ω_0 with $V \subset V_{\omega_1} \cap V_{\omega_2}$ such that $A(\omega)$ is a regular matrix for all $\omega \in V$. By Proposition 2.13 the function $A|_V$ is a holomorphic map, as all its components are holomorphic functions, with $A(V) \subset \mathrm{GL}(n, \mathbb{C})$. Thus by Theorem 2.19 the map $A(\cdot)^{-1} : V \to \mathbb{C}^{n\times n},\ \omega \to A(\omega)^{-1}$, is holomorphic. So Equation 4.1 yields

$$\beta(\omega, \alpha) = \underbrace{A(\omega)^{-1}(\langle \gamma_i^{\omega_2}(\omega), \gamma_j^{\omega_1}(\omega_0)\rangle)_{j,i=1}^n}_{g(\omega)} \alpha \quad \text{for } \omega \in V.$$

A function $V \to \mathbb{C}^{n\times n}$ is holomorphic if and only if its components are holomorphic functions. Thus by calculating the matrix multiplication above it is clear that the components of $g(\omega)$ are holomorphic as sums of products of holomorphic functions. Thus g is holomorphic around ω_0 and as $\omega_0 \in V_{\omega_1} \cap V_{\omega_2}$ was arbitrary, g is holomorphic on $V_{\omega_1} \cap V_{\omega_2}$. This completes the proof. $\square$

Remark 4.4. *The final part of the proof above also shows that, if we choose different sets $\tilde{V}_{\omega_0}$ and maps $\tilde{\gamma}_1^{\omega_0}, \ldots, \tilde{\gamma}_n^{\omega_0}$ for every $\omega_0 \in \Omega$ as described above Theorem 4.3, the resulting holomorphic atlas $(\tilde{h}_{\omega_0})_{\omega_0 \in \Omega}$ is equivalent to $(h_{\omega_0})_{\omega_0 \in \Omega}$. Therefore the following definition makes sense.*

Definition 4.5. *Let $T = (T_i)_{i=1}^d \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple on an open set $\Omega \subset \mathbb{C}^d$. Then the holomorphic vector bundle $\pi : E_T \to \Omega$ together with the equivalence class of an atlas $(h_{\omega_0})_{\omega_0 \in \Omega}$ is called the vector bundle associated with T.*

Remark 4.6. *Note that the the vector bundle associated with an operator $T \in B_n(\Omega)$ is also canonically a hermitian vector bundle. For $\omega \in \Omega$, we define the scalar product $\langle \cdot, \cdot \rangle_\omega$ on $\{\omega\} \times \ker T_\omega \cong \ker T_\omega \subset H$ as the restriction of the scalar product on H. To see that this turns E_T into a hermitian holomorphic vector bundle, it suffices to observe that, with the notation from the proof of Theorem 4.3, the identities*

$$\langle h_{\omega_0}^{-1}(\omega, (\alpha_i)_{i=1}^n), h_{\omega_0}^{-1}(\omega, (\beta_j)_{j=1}^n) \rangle = \langle \sum_{i=1}^n \alpha_i \gamma_i^{\omega_0}(\omega), \sum_{j=1}^n \beta_j \gamma_j^{\omega_0}(\omega) \rangle$$

$$= \sum_{i,j=1}^n \langle \gamma_i^{\omega_0}(\omega), \gamma_j^{\omega_0}(\omega) \rangle \alpha_i \overline{\beta_j} = \big\langle (\langle \gamma_i^{\omega_0}(\omega), \gamma_j^{\omega_0}(\omega) \rangle)_{j,i=1}^n (\alpha_i)_{i=1}^n, (\beta_j)_{j=1}^n \big\rangle_{\mathbb{C}^n}$$

hold.

The following Lemma shows that the holomorphic structure of the vector bundle E_T is compatible with the notion of holomorphic maps into Banach spaces in Definition 2.10.

Lemma 4.7. *Let $T = (T_i)_{i=1}^d \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple and let $\pi : E_T \to \Omega$ be the vector bundle associated with T. Then for $U \subset \Omega$ open, a map $f : U \to H$ with $f(\omega) \in \ker T_\omega$ for all $\omega \in U$ induces a holomorphic section*

$$\tilde{f} : U \to E_T, \ \omega \mapsto (\omega, f(\omega))$$

in E_T over U if and only if it is holomorphic in the sense of Definition 2.10.

Proof. First assume that $\tilde{f} : U \to E_T$ is a holomorphic section in E_T over U and let $\omega_0 \in U$ be arbitary. By definition the representation $\tilde{f}_{\omega_0} : V_{\omega_0} \cap U \to \mathbb{C}^n$ of $\tilde{f}$ is holomorphic and satisfies $h_{\omega_0}(\tilde{f}(\omega)) = (\omega, \tilde{f}_{\omega_0}(\omega))$ for $\omega \in V_{\omega_0} \cap U$. Applying $h_{\omega_0}^{-1}$ we obtain that

$$\tilde{f}(\omega) = (\omega, \sum_{i=1}^n \tilde{f}_{\omega_0}(\omega)_i \gamma_i^{\omega_0}(\omega)).$$

Thus the function f satisfies

$$f(\omega) = \sum_{i=1}^n \tilde{f}_{\omega_0}(\omega)_i \gamma_i^{\omega_0}(\omega)$$

and hence it is holomorphic by Proposition 2.12 and due to the fact that the components of f_{ω_0} are holomorphic functions.
Now let $f : U \to H$ be holomorphic with $f(\omega) \in \ker T_\omega$ for all $\omega \in U$. As f is continuous, the function $\tilde{f}$ it is obviously a section. Let $(h_{\omega_0})_{\omega_0 \in \Omega}$ be a holomorphic atlas of E_T as in the proof of Theorem 4.3. Fix a point $\omega_0 \in \Omega$ with $U \cap V_{\omega_0} \neq \emptyset$. Let $h_{\omega_0} : \pi^{-1}(V_{\omega_0}) \to V_{\omega_0} \times \mathbb{C}^n$ be a linear chart in the atlas chosen above such that $U \cap V_{\omega_0} \neq \emptyset$. Choose $\omega_1 \in U \cap V_{\omega_0}$. Then as in the proof of Theorem 4.3 we see that there is a neighbourhood $V \subset U \cap V_{\omega_0}$ of ω_1 such that the representation $\tilde{f}_{\omega_0}$ of $\tilde{f}$ in h_{ω_0} has the form

$$\tilde{f}_{\omega_0}(\omega) = [(\langle \gamma_i^{\omega_0}(\omega), \gamma_j^{\omega_0}(\omega_1)\rangle)_{j,i=1}^n]^{-1}(\langle f(\omega), \gamma_j^{\omega_0}(\omega_1)\rangle)_{j=1}^n$$

for $\omega \in V$. With the same arguments as above, this function is holomorphic in ω, which completes the proof. □

Definition 4.8. *Let $\Omega \subset \mathbb{C}^d$ be open and let $\gamma_1, \dots \gamma_n : \Omega \to H$ be holomorphic functions with values in a Hilbert space H. We say that $\gamma_1, \dots, \gamma_n$ span H if*

$$H = \overline{\mathrm{LH}(\{\gamma_k(z); k \in \mathbb{N}_n, z \in \Omega\})}.$$

Definition 4.9. *Let $\Omega \subset \mathbb{C}^d$ be open and let $T \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple. Then a spanning holomorphic cross-section in E_T is a holomorphic function $\gamma : \Omega \to H$ spanning H such that $\gamma(\omega) \in \ker(T_\omega)$ for all $\omega \in \Omega$.*

In our construction of spanning holomorphic cross-sections, we will use the existence of a global holomorphic frame for the vector bundle E_T. The following definition characterises the sets $\Omega \subset \mathbb{C}^d$ which assure this condition for operator tuples $T \in B_n(\Omega)$.

Definition 4.10. *Let $d \in \mathbb{N}^*$, then we call an open subset $\Omega \subset \mathbb{C}^d$ admissible if every holomorphic vector bundle over Ω is analytically trivial.*

Now we can use a result from complex geometry found by Grauert, which states that a large class of open subsets of $\mathbb{C}^d$ is admissible. The following results are Theorem 6 and 7 in [Gra58a] respectively.

Theorem 4.11. *Let Ω be a contractible, holomorphically complete complex space. Then every holomorphic fibre bundle $\pi : E_T \to \Omega$ is holomorphically trival.*

Theorem 4.12. *Let Ω be a non-compact Riemann surface and $\pi : E_T \to \Omega$ a holomorphic fibre bundle. If the structure group of E_T is a connected, complex Lie-group then E_T is holomorphically trival.*

We now have to adapt these abstract theorems to our specific situation. In order to do this in detail, it would be necessary to introduce many concepts from differential and complex geometry. This would be out of place in this Bachelor thesis and therefore we refer the interested reader to the classical literature on these subjects. All we need to know at this point is that

1. Holomorphic vector bundles of rank $n \in \mathbb{N}^*$ are holomorphic fibre bundles, where the fibre is the vector space $\mathbb{C}^n$ and the structure group is the group $\mathrm{GL}(n, \mathbb{C})$; the fibre bundle is analytically trivial if and only if it is analytically trivial as a holomorphic vector bundle.

2. An open connected subset $\Omega \subset \mathbb{C}^d$ is a holomorphically complete space if and only if it is a domain of holomorphy (cf. [Gra58b], Section 2.1).

3. Every nonempty open subset $\Omega \subset \mathbb{C}$ is a non-compact Riemann surface in a canonical way.

4. The group $\mathrm{GL}(n, \mathbb{C})$ is a connected Lie-group.

Corollary 4.13. *Let $\Omega \subset \mathbb{C}^d$ open. Then for $d = 1$ (i.e., $\Omega \subset \mathbb{C}$) or Ω a contractible domain of holomorphy we have that Ω is admissible.*

Corollary 4.14. *Let $\Omega \subset \mathbb{C}^d$ be an admissible open set and let $T \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple of degree n. Then there exist holomorphic functions $\gamma_1, \dots, \gamma_n : \Omega \to H$ such that $\ker(T_\omega) = \mathrm{LH}(\{\gamma_1(\omega), \dots, \gamma_n(\omega)\})$ for all $\omega \in \Omega$.*

Proof. By Theorem 4.11 the vector bundle E_T is trival. Let $h : E_T \to \Omega \times \mathbb{C}^n$ be a linear chart in an atlas representing the holomorphic linear structure of E_T. For $i \in \mathbb{N}_n$, we define

$$\begin{aligned} \tilde{\gamma}_i &: \Omega \to E_T, \quad \omega \mapsto h^{-1}(\omega, e_i), \\ \gamma_i &: \Omega \to H, \quad \omega \mapsto \mathrm{pr}_2 \circ \tilde{\gamma}_i. \end{aligned}$$

The representation of $\tilde{\gamma}_i$ in the chart h is obviously just the constant function e_i, which is holomorphic. Hence $\tilde{\gamma}_i$ is a holomorphic section and by Lemma 4.7 the map γ_i is a holomorphic function from Ω to H for $i \in \mathbb{N}_n$. As $(e_1, \dots, e_n)$ form a basis of $\mathbb{C}^n$ and $h(\omega, \cdot)$ is an isomorphism of vector spaces, the vectors $(\gamma_1(\omega), \dots, \gamma_n(\omega))$ form a basis of $\pi^{-1}(\{\omega\}) = \ker(T_\omega)$ as required. ☐

5 Spanning holomorphic cross-sections

Given a Hilbert space H spanned by holomorphic functions $\gamma_1, \ldots, \gamma_n : \Omega \to H$, we now want to construct a holomorphic function $\gamma : \Omega \to H$ spanning H with $\gamma(z) \in \mathrm{LH}(\{\gamma_1(z), \ldots, \gamma_n(z)\})$ for all $z \in \Omega$. By applying this result to a global holomorphic frame for the vector bundle associated with a Cowen-Douglas operator tuple one obtains a spanning holomorphic cross-section. We will inductively reduce the number of necessary functions to span H. Before we begin, we have to take care of a technical detail, which will allow us to use the results about uniqueness sets found in Section 3.

Proposition 5.1. *Let $\Omega \subset \mathbb{C}^d$ be open, and let $\gamma : \Omega \to H$ be a holomorphic function into a Hilbert space H. Then the function*

$$\tilde{\gamma} : \Omega^* \to H', \ z \mapsto \langle \cdot, \gamma(\overline{z}) \rangle$$

is holomorphic.

Proof. By Lemma 2.16 it suffices to show that for all $x \in H$ the function

$$g_x : \Omega^* \to \mathbb{C}, \ z \mapsto \tilde{\gamma}(z)(x) = \langle x, \gamma(\overline{z}) \rangle$$

is holomorphic. By Proposition 2.13 the function

$$f_x : \Omega \to \mathbb{C}, \ z \mapsto \langle \gamma(z), x \rangle$$

is holomorphic and $g_x(z) = \langle x, \gamma(\overline{z}) \rangle = \overline{\langle \gamma(\overline{z}), x \rangle} = \overline{f_x(\overline{z})}$ for all $z \in \Omega$. So by Proposition 2.8, the function g_x is holomorphic. □

Lemma 5.2. *Let $\Omega \subset \mathbb{C}^d$ be a domain of holomorphy and $A \subset \Omega$ a set without accumulation point in Ω. Then there exists a nonzero holomorphic function $f \in \mathcal{O}(\Omega) \backslash \{0\}$ such that f vanishes on A.*

Proof. In this proof we will use several notions and a result from complex geometry, which can be found in [FG02]. Since we can define the function f seperately on all connected components of Ω, it suffices to prove the statement for Ω connected. As Ω is a domain of holomorphy, it is a Stein manifold. Choose an arbitrary point $p \in \Omega \backslash A$ and define $A' = A \cup \{p\}$. Then $A' \subset \Omega$ is an analytic set. Indeed, for $z \in \Omega$, there is a constant $r > 0$ such that the sets $U = B_r(z) \cap \Omega$ and $A' \backslash \{z\}$ have no common points, because A' still has no accumulation point in Ω. For $z \notin A'$, the set $U \cap A'$ is empty and thus the zero set of the constant function 1. For $z \in A'$, we have that $U \cap A = \{z\} = \{w \in U; w_1 - z_1 = 0, \ldots, w_d - z_d = 0\}$ is the common zero set of d

holomorphic functions on U.
Now consider the function

$$f_0 : A' \to \mathbb{C},\ z \mapsto \begin{cases} 0 & z \neq p \\ 1 & z = p \end{cases}.$$

This function is holomorphic as it is locally constant. By Theorem V.1.9 in [FG02], there exists a holomorphic function $f \in \mathcal{O}(\Omega)$ with $f|_{A'} = f_0$. But then clearly $f \neq 0$ and $f|_A = 0$. □

We will try to define γ as a sum $\gamma(z) = \phi_1(z)\gamma_1(z) + \ldots + \phi_n(z)\gamma_n(z)$ with holomorphic functions $\phi_1, \ldots, \phi_n$. As we will need this in a later proof, we want to choose the functions ϕ_i simultaneously for different Hilbert spaces H_i spanned by $\gamma_1^i, \ldots, \gamma_n^i$.

Lemma 5.3. *Let $\Omega \subset \mathbb{C}^d$ be a domain of holomorphy and let $H_1, \ldots, H_m$ be Hilbert spaces. If $\gamma_1^i, \gamma_2^i : \Omega \to H_i$ are two holomorphic functions spanning H_i for $i \in \mathbb{N}_m$, then there exists a holomorphic function $\phi \in \mathcal{O}(\Omega)$ such that the functions $\gamma^i = \phi\gamma_1^i + \gamma_2^i$ also span H_i for $i \in \mathbb{N}_m$.*

Proof. We will define ϕ as a product $\phi = \prod_{i=1}^m \phi_i$, with $\phi_i \in \mathcal{O}(\Omega)$ constructed as follows:
Let $i \in \mathbb{N}_m$. If $\gamma_2^i = 0$, we set $\phi_i = 1$ and define the set $A_i = \emptyset \subset \Omega$, which is the zero set of ϕ_i. If $\gamma_2^i \neq 0$, consider the space

$$(H_i)_{\tilde{\gamma}_2^i} = \{\widehat{x}; x \in H_i\}$$

defined as in Lemma 3.6. Here $\tilde{\gamma}_2^i : \Omega^* \to H'$ is the holomorphic function associated with γ_2^i as described in Proposition 5.1. We recall that for $x \in H_i$ we defined

$$\widehat{x} : \Omega^* \to \mathbb{C},\ z \mapsto \langle x, \gamma_2^i(\overline{z})\rangle.$$

Then by Theorem 3.2, there exists a uniqueness set $B_i \subset \Omega^*$ for $(H_i)_{\tilde{\gamma}_2^i}$ without accumulation point in Ω^*. We define the set $A_i = B_i^*$ which in turn has no accumulation point in Ω. By Lemma 5.2, there exists a holomorphic function $\phi_i \in \mathcal{O}(\Omega)\backslash\{0\}$ such that ϕ_i vanishes on A_i.
We show that $\phi = \phi_1 \cdot \ldots \cdot \phi_m$ has the desired property. Let $i \in \mathbb{N}_m$ and $x \in H_i$, which is orthogonal to $\gamma^i(z) = \phi(z)\gamma_1^i(z) + \gamma_2^i(z)$ for all $z \in \Omega$. Then we see

$$\phi(z)\langle\gamma_1^i(z), x\rangle + \langle\gamma_2^i(z), x\rangle = 0 \quad \text{for } z \in \Omega.$$

If $\gamma_2^i = 0$, we have $\phi(z)\langle\gamma_1^i(z), x\rangle = 0$ for all $z \in \Omega$. Otherwise for $z \in A_i$ by definition $\phi(z) = 0$ and so $\langle\gamma_2^i(z), x\rangle = 0$. This implies $\langle x, \gamma_2^i(\overline{z})\rangle = 0$ for all $z \in A_i^* = B_i$. As B_i was a uniqueness set for $(H_i)_{\tilde{\gamma}_2^i}$, already $\langle\gamma_2^i(z), x\rangle = 0$ for all $z \in \Omega$. In any case, we get the equation

$$\phi(z)\langle\gamma_1^i(z), x\rangle = 0 \quad \text{for } z \in \Omega.$$

This implies that the function $h \in \mathcal{O}(\Omega)$ defined by $h(z) = \langle\gamma_1^i(z), x\rangle$ vanishes on $\Omega\backslash Z(\phi)$, where $Z(\phi)$ is the zero set of ϕ. But this set is dense in Ω by the identity theorem. Hence $h(z) = 0$ for all $z \in \Omega$ and thus we conclude $\langle \gamma_j^i(z), x\rangle = 0$ for $j \in \{1, 2\}, z \in \Omega$. As γ_1^i, γ_2^i span H_i, we get $x = 0$ and so γ_i spans H_i. □

Theorem 5.4. *Let $\Omega \subset \mathbb{C}^d$ be a domain of holomorphy and let $H_1, \ldots, H_m$ be Hilbert spaces. If $\gamma_1^i, \ldots, \gamma_n^i : \Omega \to H_i$ are holomorphic functions spanning H_i for $i \in \mathbb{N}_m$, then there exist holomorphic functions $\phi_1, \ldots, \phi_n \in \mathcal{O}(\Omega)$ such that $\gamma^i = \phi_1\gamma_1^i + \ldots + \phi_n\gamma_n^i$ also span H_i for $i \in \mathbb{N}_m$. If for every $i \in \mathbb{N}_m$, $z \in \Omega$, the vectors $\gamma_1^i(z), \ldots, \gamma_n^i(z)$ are linearly independent, it is possible to assure $\gamma^i(z) \neq 0$ for all $z \in \Omega$.*

Proof. The proof will be an induction on n. For $n = 1$, choose $\phi_j = 1$ for $j \in \mathbb{N}_n$. The condition $\gamma_1^i(z)$ linearly independent for all $z \in \Omega$ means that $\gamma^i(z) = \gamma_1^i(z) \neq 0$ for all $z \in \Omega$. Now let the statement be true for $n \geq 1$. For $i \in \mathbb{N}_m$, let $\gamma_1^i, \ldots, \gamma_{n+1}^i : \Omega \to H_i$ be holomorphic functions spanning H_i. Define

$$H_i' = \overline{\mathrm{LH}(\{\gamma_n^i(z); z \in \Omega\} \cup \{\gamma_{n+1}^i(z); z \in \Omega\})}.$$

By Lemma 5.3 there is a function $h \in \mathcal{O}(\Omega)$ such that $h\gamma_n^i + \gamma_{n+1}^i$ spans H_i' for $i \in \mathbb{N}_m$. This implies that H_i is spanned by the n functions

$$\gamma_1^i, \ldots, \gamma_{n-1}^i, h\gamma_n^i + \gamma_{n+1}^i.$$

If $\gamma_1^i(z), \ldots, \gamma_{n+1}^i(z)$ are linearly independent for all $z \in \Omega$, so are $\gamma_1^i(z), \ldots, \gamma_{n-1}^i(z)$, $h(z)\gamma_n^i(z) + \gamma_{n+1}^i(z)$. By the induction hypothesis there exist holomorphic functions $\phi_1', \ldots, \phi_n' \in \mathcal{O}(\Omega)$ such that the functions

$$\gamma^i = \phi_1'\gamma_1^i + \ldots + \phi_{n-1}'\gamma_{n-1}^i + \phi_n'(h\gamma_n^i + \gamma_{n+1}^i)$$

also span H_i for $i \in \mathbb{N}_m$ and if $\gamma_1^i(z), \ldots, \gamma_{n+1}^i(z)$ are linearly independent, one can achieve that $\gamma^i(z) \neq 0$ for all $z \in \Omega$. Thus the holomorphic functions

$$\phi_1 = \phi_1', \ \ldots \ , \ \phi_{n-1} = \phi_{n-1}', \ \phi_n = \phi_n' h, \ \phi_{n+1} = \phi_n'$$

satisfy the conditions of the theorem. □

Corollary 5.5. *Let H a Hilbert space, $\Omega \subset \mathbb{C}^d$ an admissible domain of holomorphy and $T \in B_n(\Omega) \subset B(H)^d$. Then there exists a spanning holomorphic cross-section $\gamma : \Omega \to H$ in E_T such that $\gamma(\omega) \neq 0$ for all $\omega \in \Omega$.*

Proof. By Corollary 4.14 there exist holomorphic functions $\gamma_1, \ldots, \gamma_n : \Omega \to H$ such that $\ker(T_\omega) = \mathrm{LH}(\{\gamma_1(\omega), \ldots, \gamma_n(\omega)\})$ for all $\omega \in \Omega$. As $\ker(T_\omega)$ is of dimension n this implies that $\gamma_1(\omega), \ldots, \gamma_n(\omega)$ are linearly independent for all $\omega \in \Omega$. By Condition 3 in the definition of Cowen-Douglas operator tuples the functions $\gamma_1, \ldots, \gamma_n$ span H. Now by Theorem 5.4 we can choose holomorphic functions $\phi_1, \ldots, \phi_n \in \mathcal{O}(\Omega)$ such that $\gamma = \phi_1\gamma_1 + \ldots + \phi_n\gamma_n$ also spans H and such that $\gamma(\omega) \neq 0$ for all $\omega \in \Omega$. The function γ is holomorphic by Lemma 2.12 and for all $\omega \in \Omega$ we have

$$\gamma(\omega) = \phi_1(\omega)\gamma_1(\omega) + \ldots + \phi_n(\omega)\gamma_n(\omega) \in \mathrm{LH}(\{\gamma_1(\omega), \ldots, \gamma_n(\omega)\}) = \ker(T_\omega).$$

Thus γ is a spanning holomorphic cross-section vanishing nowhere as desired. □

As a first consequence of the above result, we get the following corollary.

Corollary 5.6. *Let $\Omega \subset \mathbb{C}^d$ be an admissible domain of holomorphy and H a Hilbert space. Then if $B_n(\Omega) \subset B(H)^d$ is not empty, H must be a seperable Hilbert space of infinite dimension.*

Proof. Let $T \in B_n(\Omega)$. It is clear that $\dim H = \infty$ because T_1 has infinitely many distinct eigenvalues. Now let $\gamma : \Omega \to H$ be a spanning holomorphic cross-section in E_T. Choose a countable dense subset $\{z_k; k \in \mathbb{N}\} \subset \Omega$ and define a countable set $M \subset H$ by

$$\bigcup_{N\in\mathbb{N}} \{\sum_{k=0}^{N} \alpha_k \gamma(z_k);\ \alpha_k \in \mathbb{Q} + i\mathbb{Q} \text{ for } k = 0, \ldots, N\}.$$

Then M is dense in H because γ spans H. □

Proposition 5.7. *Let $\Omega \subset \mathbb{C}^d$ be open, H a Hilbert space, $T = (T_1, \ldots, T_d) \in B(H)^d$ an element of $B_n(\Omega)$. If $\emptyset \neq \Omega_0 \subset \Omega$ is open and for every connected component D of Ω we have $D \cap \Omega_0 \neq \emptyset$, then $B_n(\Omega) \subset B_n(\Omega_0)$. If $\gamma : \Omega \to H$ is a spanning holomorphic cross-section in E_T, then $\gamma|_{\Omega_0}$ is a spanning holomorphic cross-section for $E_T|_{\Omega_0}$.*

Proof. To show that $T \in B_n(\Omega_0)$ it suffices to verify the third condition of Definition 2.1 for $B_n(\Omega_0)$: With $V \subset H$ defined by

$$V = \overline{\mathrm{LH}(\bigcup_{\omega\in\Omega_0} \ker(T_\omega))}$$

we must show $V = H$. Let $x \in H$ be orthogonal to V, D a connected component of Ω, $\omega_0 \in D \cap \Omega_0$ a point and let $\omega_1 \in D$ be arbitrary. As D is open and connected, it is also path-connected. Let $p : [0, 1] \to D$ be a path from ω_0 to ω_1, i.e., a continuous map with $p(0) = \omega_0$ and $p(1) = \omega_1$. Now for every $t \in [0, 1]$ choose a connected open set $V_t \subset D$ containing $p(t)$ and holomorphic functions $\gamma_1^t, \ldots, \gamma_n^t : V_t \to H$ such that

$$\mathrm{LH}(\{\gamma_1^t(\omega), \ldots, \gamma_n^t(\omega)\}) = \ker T_\omega \quad \text{for } \omega \in V_t$$

according to Theorem 4.2. Then $p([0, 1]) \subset \bigcup_{t\in[0,1]} V_t$ is an open cover of the compact set $p([0, 1])$. Thus there is a finite subcover $p([0, 1]) \subset \bigcup_{i=1}^m V_{t_i}$ with $t_1, \ldots, t_m \in [0, 1]$. We will now show that the elements of the following family of statements are correct:

$$S(i) = \text{“}x \perp \ker T_\omega \ \forall \omega \in V_{t_i}\text{''},\ i = 1, \ldots, m.$$

Let $k \in \mathbb{N}_m$ such that $\omega_0 = p(0) \in V_{t_k}$. Then $\Omega_0 \cap V_{t_k} \ni \omega_0$ is a nonempty open set and $x \perp \gamma_1^{t_k}(\omega), \ldots, x \perp \gamma_n^{t_k}(\omega)$ for all $\omega \in \Omega_0 \cap V_{t_k}$ by assumption. This means that the functions

$$f_j^{t_k} : V_{t_k} \to \mathbb{C},\ \omega \mapsto \langle \gamma_j^{t_k}(\omega), x\rangle \quad (j = 1, \ldots, n), \tag{5.1}$$

which are holomorphic by 2.13, vanish on $\Omega_0 \cap V_{t_k}$ and thus identically. This implies $S(k)$ is true. Now consider $I = \{i \in \mathbb{N}_m;\ S(i) \text{ is true}\}$. We have $k \in I$ and claim that $I = \mathbb{N}_m$. Now assume that this is not the case. Then there must be $l \in \mathbb{N}_m \backslash I$ with $V_{t_l} \cap \bigcup_{i \in I} V_{t_i} \neq \emptyset$, because otherwise we have the two disjoint open sets

$$O = \bigcup_{i \in I} V_{t_i},\ P = \bigcup_{i \in \mathbb{N}_m \backslash I} V_{t_i}$$

covering the connected set $p([0,1])$ and obviously $p(t_k) \in O \cap p([0,1])$, $p(t_r) \in P \cap p([0,1])$ for any $r \in \mathbb{N}_m \backslash I$. Thus there are $l \in \mathbb{N}_m \backslash I$, $i \in I$ with $V_{t_l} \cap V_{t_i} \neq \emptyset$. Then the functions $f_j^{t_i}$, defined as in (5.1), vanish on $V_{t_l} \cap V_{t_i}$, as $S(i)$ is true, and thus identically on V_{t_l}. So $S(l)$ is true, a contradiction to $l \notin I$. Hence $S(l)$ is true for all $l \in \mathbb{N}_m$. Now choose $s \in \mathbb{N}_m$ with $\omega_1 = p(1) \in V_{t_s}$. Then as $S(s)$ is true, we have $x \perp \ker T_{\omega_1}$. Since D was an arbitrary connected component of Ω and $\omega_1 \in D$ was arbitary, we see that $x = 0$ as $\mathrm{LH}(\bigcup_{\omega_1 \in \Omega} \ker(T_{\omega_1})) \subset H$ is dense. But this implies $V = H$ as required.
Now if $\gamma : \Omega \to H$ is a spanning holomorphic cross-section in E_T and $x \in H$ is a vector with $x \perp \gamma(\omega)$ for all $\omega \in \Omega_0$, consider the function

$$f : \Omega \to \mathbb{C},\ \omega \mapsto \langle \gamma(\omega), x \rangle,$$

which is holomorphic by Proposition 2.13. As $f|_{\Omega_0} = 0$, we already have $f = 0$ because Ω_0 is open and every connected component of Ω has a nonempty intersection with Ω_0. But then $x = 0$ as γ (defined on Ω) is a spanning holomorphic cross-section for T. This implies the desired statement. □

Remark 5.8. *We note that the definition of Cowen-Douglas operators given in the introduction is the special case of Definition 2.1 for $d = 1$. Here $T_\omega = T - \omega \in B(H)$ for all $\omega \in \Omega$. It only remains to show that the first conditions of both definitions are equivalent in the case $d = 1$. But if $T - \omega$ is surjective, it certainly has closed range. On the other hand, assume $T - \omega$ has closed range. By Proposition 5.7 we see $T \in B_n(\Omega \backslash \{0\})$, thus $V = \mathrm{LH}(\bigcup_{\omega \in \Omega \backslash \{0\}} \ker(T - \omega)) \subset H$ is a dense linear subspace. But for $\omega \in \Omega \backslash \{0\}$ and $x \in \ker(T - \omega)$ we have $T(\frac{1}{\omega} x) = x$ and so $x \in \mathrm{Ran}(T)$. Hence $V \subset \mathrm{Ran}(T)$ and as $\mathrm{Ran}(T)$ is closed, we see that T is surjective.*

Remark 5.9. *For $\Omega \subset \mathbb{C}^d$ open and connected and $T \in B_n(\Omega) \subset B(H)^d$ in general it is not clear that there is a spanning holomorphic cross-section $\gamma : \Omega \to H$. However, with $\Omega_0 \subset \Omega$ an admissible domain of holomorphy we can consider T as an element of $B_n(\Omega_0)$ by Proposition 5.7 and on Ω_0 we have a spanning holomorphic cross-section. In this way, many of the results in the following sections can be applied for general open connected sets Ω.*

The following theorem gives us a representation for Cowen-Douglas tuples as multiplication operators on a functional Hilbert space of holomorphic functions.

Theorem 5.10. *Let $\Omega \subset \mathbb{C}^d$ be an admissible domain of holomorphy, H be a Hilbert space and let $T \in B_n(\Omega) \subset B(H)^d$ be a Cowen-Douglas tuple. Then there is a* *functional Hilbert space $\widehat{H}$ of holomorphic functions on Ω^* and a unitary operator $U : H \to \widehat{H}$ such that $UT_iU^* = M_{z_i}^*$ for $i = 1, \dots, d$, where*

$$M_{z_i} : \widehat{H} \to \widehat{H}, \; f \mapsto z_i f$$

is the multiplication by the i-th coordinate function on $\widehat{H}$.

Proof. Let $\gamma : \Omega \to H$ be a spanning holomorphic cross-section for the vector bundle E_T associated with T. We now consider again Lemma 3.6 together with its proof, where we choose $X = H$, $\tilde{\gamma} : \Omega^* \to H'$ as in Proposition 5.1. With notation as in the proof of Lemma 3.6 we have

$$H_0^* = \overline{\mathrm{LH}(\{\tilde{\gamma}(z); z \in \Omega^*\})}.$$

Let $x \in {}^{\perp}H_0^*$. Then $\langle x, \gamma(z)\rangle = 0$ for all $z \in \Omega$ and thus $x = 0$ as γ spans H. As in the cited proof we see that the space

$$\widehat{H} = H_{\tilde{\gamma}} = \{\widehat{x}; x \in H\}$$

is isometrically isomorphic to $H_0 = H/{}^{\perp}H_0^* = H$ via the map

$$U : H \to \widehat{H}, \; U(x)(z) = \langle x, \gamma(\overline{z})\rangle = \widehat{x}(z),$$

if the norm on $\widehat{H}$ is defined by $\|\widehat{x}\| = \|x\|$ for $x \in H$. Then by Lemma 3.6, the space $\widehat{H}$ is a functional Hilbert space.
Now let $i \in \mathbb{N}_d$, $S_i = UT_iU^*$. We show that S_i^* is the operator of multiplication with z_i on $\widehat{H}$. For $x \in H, z \in \Omega^*$ we have

$$\begin{aligned}(S_i^* Ux)(z) &= (UT_i^* U^* Ux)(z) = (UT_i^* x)(z) \\ &= \langle T_i^* x, \gamma(\overline{z})\rangle = \langle x, T_i\gamma(\overline{z})\rangle \\ &= \langle x, \overline{z_i}\gamma(\overline{z})\rangle = z_i\langle x, \gamma(\overline{z})\rangle = z_i(Ux)(z).\end{aligned}$$

Hence, the operator T_i is unitarily equivalent to the adjoint of the operator M_{z_i} on $\widehat{H}$ via the unitary operator U. □

Remark 5.11. *It is easy to see that in the setting of Theorem 5.10 the reproducing kernel of the functional Hilbert space $\widehat{H}$ is the map*

$$K : \Omega^* \times \Omega^* \to H, \; (z, w) \mapsto \langle \gamma(\overline{w}), \gamma(\overline{z})\rangle.$$

Indeed we have $K(\cdot, w) = \widehat{\gamma(\overline{w})} \in \widehat{H}$ and, for all $\widehat{x} \in \widehat{H}$, $w \in \Omega^*$, we see that

$$\langle \widehat{x}, K(\cdot, w)\rangle_{\widehat{H}} = \langle \widehat{x}, \widehat{\gamma(\overline{w})}\rangle_{\widehat{H}} = \langle x, \gamma(\overline{w})\rangle_H = \widehat{x}(w).$$

6 Unitary equivalence

In the following section we prove a generalization of a fundamental result shown in [CD78]. This particular proof was given in the one-dimensional case by K. Zhu in [Zhu00] and it is based on the existence of a spanning holomorphic cross-section. The result characterises Cowen-Douglas operator tuples which are unitarily equivalent. Recall that for Hilbert spaces H_1, H_2 two operator tuples $S \in B(H_1)^d$, $T \in B(H_2)^d$ are called unitarily equivalent if there exists a unitary map $U : H_1 \to H_2$ such that $US_i = T_iU$ for $i = 1, \ldots, d$.

Theorem 6.1. *Let $\Omega \subset \mathbb{C}^d$ be an admissible, connected domain of holomorphy, let H_1, H_2 be Hilbert spaces and let $S \in B(H_1)^d$, $T \in B(H_2)^d$ be two operator tuples in $B_n(\Omega)$. Then the following are equivalent:*

1. *S and T are unitarily equivalent.*
2. *The hermitian holomorphic bundles E_S and E_T are equivalent.*
3. *There exist spanning holomorphic cross-sections γ_S in E_S and γ_T in E_T such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for all $z \in \Omega$.*

Proof. (1) $\implies$ (2): Let $U : H_1 \to H_2$ be a unitary operator such that $US_i = T_iU$ for all $i \in \mathbb{N}_d$. For $i \in \mathbb{N}_d$, $z \in \Omega$ and $x \in \ker(S_z)$, we get

$$T_i(Ux) = U(S_ix) = z_i(Ux),$$

so $Ux \in \ker(T_z)$. This shows that the map $f = (id_\Omega \times U)|_{E_S} : E_S \to E_T$ is well-defined and it obviously defines a vector bundle morphism. For every $z \in \Omega$, the function f restricted to the fibre $\ker(S_z) \subset E_S$ is just the map

$$id_{\{z\}} \times U|_{\ker(S_z)} : \{z\} \times \ker(S_z) \to \{z\} \times \ker(T_z).$$

This map is isometric and hence surjective as the domain of definition and the target space have the same dimension (equipping both with the natural structure of n-dimensional normed complex vector spaces), hence it is unitary.
We will now show that f is a holomorphic bundle map. For this let $h_{z_i} : \pi^{-1}(V_{z_i}) \to V_{z_i} \times \mathbb{C}^n$ be linear charts contained in atlases representing the holomorphic linear structure of E_S and E_T respectively for $i = 1, 2$ with $V_{z_1} \cap V_{z_2} \neq \emptyset$. Let $\gamma_1^{z_1}, \ldots, \gamma_n^{z_1} : V_{z_1} \to H_1$ be the corresponding local frame for h_{z_1}. Then the maps $U\gamma_1^{z_1}, \ldots, U\gamma_n^{z_1} : V_{z_1} \to H_2$ are holomorphic by Proposition 2.11 and $U\gamma_j^{z_1}(z) \in \ker(T_z)$ for all $z \in V_{z_1} \cap V_{z_2}$,

$j \in \mathbb{N}_n$. Thus by Lemma 4.7 they induce holomorphic sections

$$g_j : V_{z_1} \cap V_{z_2} \to E_T,\ z \mapsto (z, U\gamma_j^{z_1}(z)) \quad (j \in \mathbb{N}_n),$$

and their representations $g_{j,z_2} = \mathrm{pr}_2 \circ h_{z_2} \circ g_j$ are holomorphic functions. We must show that the map

$$h_{z_2} \circ f \circ h_{z_1}^{-1} : (V_{z_1} \cap V_{z_2}) \times \mathbb{C}^n \to (V_{z_1} \cap V_{z_2}) \times \mathbb{C}^n$$

is holomorphic. But this follows from the following factorization

$$(z, \alpha) \xrightarrow{h_{z_1}^{-1}} (z, \sum_{j=1}^{n} \alpha_j \gamma_j^{z_1}(z)) \xrightarrow{f} (z, \sum_{j=1}^{n} \alpha_j U\gamma_j^{z_1}(z)) \xrightarrow{h_{z_2}} (z, \sum_{j=1}^{n} \alpha_j g_{j,z_2}(z)).$$

Thus the hermitian holomorphic vector bundles E_S and E_T are equivalent.
(2) $\implies$ (3): Let E_S and E_T be equivalent as hermitian holomorphic vector bundles. Then there exists a holomorphic bundle map $f : E_S \to E_T$ which maps $\{z\} \times \ker(S_z)$ unitarily onto $\{z\} \times \ker(T_z)$ for all $z \in \Omega$. We denote by F_z the corresponding unitary map

$$F_z : \ker(S_z) \to \ker(T_z),\ x \mapsto \mathrm{pr}_2(f(z, x)).$$

Now choose a global holomorphic frame of functions $\gamma_1, \dots, \gamma_n : \Omega \to H_1$ for E_S. Then for $z \in \Omega$, the vectors $F_z\gamma_1(z), \dots, F_z\gamma_n(z)$ form a basis of $\ker(T_z)$. The functions

$$\tilde{\gamma}_i : \Omega \to H_2,\ z \mapsto F_z\gamma_i(z) \quad (i = 1, \dots, n)$$

induce holomorphic sections by Lemma 2.45 and thus are holomorphic by Lemma 4.7. Hence they are a global holomorphic frame for E_T. Now choose functions $\phi_1, \dots, \phi_n \in \mathcal{O}(\Omega)$ according to Theorem 5.4 such that

$$\gamma : \Omega \to H_1,\ z \mapsto \phi_1(z)\gamma_1(z) + \ldots + \phi_n(z)\gamma_n(z)$$
$$\tilde{\gamma} : \Omega \to H_2,\ z \mapsto \phi_1(z)F_z\gamma_1(z) + \ldots + \phi_n(z)F_z\gamma_n(z)$$

are spanning holomorphic cross-sections for E_S and E_T, respectively. But then for all $z \in \Omega$, we have $\tilde{\gamma}(z) = F_z\gamma(z)$ as F_z is linear. But F_z is also unitary and therefore $\|\tilde{\gamma}(z)\| = \|\gamma(z)\|$. So γ and $\tilde{\gamma}$ are spanning holomorphic cross-sections as required in Condition 3.
(3) $\implies$ (1): Now assume that there exist spanning holomorphic cross-sections γ_S in E_S and γ_T in E_T such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for all $z \in \Omega$. Then we define maps

$$K_S : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto \langle \gamma_S(z), \gamma_S(w) \rangle,$$
$$K_T : \Omega \times \Omega \to \mathbb{C},\ (z, w) \mapsto \langle \gamma_T(z), \gamma_T(w) \rangle.$$

By Proposition 2.14 the functions K_S and K_T are holomorphic in the first argument and anti-holomorphic in the second argument. Moreover we have for $z \in \Omega$:

$$K_S(z,z) = \|\gamma_S(z)\|^2 = \|\gamma_T(z)\|^2 = K_T(z,z).$$

Then by applying a theorem from the theory of functions of several complex variables (see [Kra82], Exercise 3 on page 326), we have $K_S = K_T$. It follows that, for $n \in \mathbb{N}^*$, $c_1, \ldots, c_n \in \mathbb{C}$ and $z_1, \ldots, z_k \in \Omega$, we have

$$\begin{aligned}
\|\sum_{k=1}^n c_k \gamma_S(z_k)\|^2 &= \left\langle \sum_{k=1}^n c_k \gamma_S(z_k), \sum_{l=1}^n c_l \gamma_S(z_l) \right\rangle \\
&= \sum_{k=1}^n \sum_{l=1}^n c_k \overline{c_l} \langle \gamma_S(z_k), \gamma_S(z_l) \rangle = \sum_{k=1}^n \sum_{l=1}^n c_k \overline{c_l} K_S(z_k, z_l) \\
&= \sum_{k=1}^n \sum_{l=1}^n c_k \overline{c_l} K_T(z_k, z_l) = \ldots = \|\sum_{k=1}^n c_k \gamma_T(z_k)\|^2.
\end{aligned}$$

This implies that

$$\sum_{k=1}^n c_k \gamma_S(z_k) = 0 \implies \sum_{k=1}^n c_k \gamma_T(z_k) = 0$$

and so by Corollary 2.5 there is a unique linear map $U_0 : \mathrm{LH}(\{\gamma_S(z); z \in \Omega\}) \to H_2$ such that $U_0 \gamma_S(z) = \gamma_T(z)$ for all $z \in \Omega$. The above calculation also shows that U_0 is isometric. Obviously $\mathrm{Ran}(U_0) = \mathrm{LH}(\{\gamma_T(z); z \in \Omega\})$ is dense in H_2. We now extend U_0 continuously on $H_1 = \overline{\mathrm{LH}(\{\gamma_S(z); z \in \Omega\})}$ and the resulting map $U : H_1 \to H_2$ is still isometric. Thus its image is closed and as $\mathrm{Ran}(U_0) \subset \mathrm{Ran}(U)$, the map U is surjective and hence unitary. Since γ_S spans H and since for all $i \in \mathbb{N}_d$ and $z \in \Omega$ we have

$$T_i U \gamma_S(z) = T_i \gamma_T(z) = z_i \gamma_T(z) = U z_i \gamma_S(z) = U S_i \gamma_S(z),$$

it follows that $T_i U = U S_i$ as required. This finishes the proof. □

As a refinement of the above result, we will derive a weaker condition for unitary equivalence. This condition was shown for $n = 1$ in [CD78] (cf. Theorem 1.17, page 195). First we give a proposition needed for the proof.

Proposition 6.2. *Let $\Omega \subset \mathbb{C}^d$ be an open and connected set, H a Hilbert space, $T \in B_n(\Omega)$ a Cowen-Douglas tuple and let $\gamma : \Omega \to H$ be a spanning holomorphic cross-section for E_T. If $\phi \in \mathcal{O}(\Omega)$ is a holomorphic function and $\phi \neq 0$ then $\tilde{\gamma} = \phi\gamma$ is also a spanning holomorphic cross-section for E_T.*

Proof. The function $\tilde{\gamma}$ is holomorphic by Proposition 2.12. Assume that we have $x \in H$ with $\langle \phi(z)\gamma(z), x \rangle = 0$ for all $z \in \Omega$. The set $\Omega_0 = \phi^{-1}(\mathbb{C}\backslash\{0\}) \subset \Omega$ is open and nonempty by assumption and the holomorphic function

$$f : \Omega \to \mathbb{C},\ z \mapsto \langle \gamma(z), x \rangle$$

vanishes on Ω_0. Hence it vanishes identically on Ω and so $x = 0$. Thus $\tilde{\gamma}$ spans H, which concludes the proof. □

Theorem 6.3. *Let $n \in \mathbb{N}^*$, $\Omega \subset \mathbb{C}$ be a domain, H_1, H_2 Hilbert spaces and let $S \in B(H_1)$, $T \in B(H_2)$ be two operators in $B_n(\Omega)$. Then if there are spanning holomorphic cross-sections $\gamma_S : \Omega \to H_1$ and $\gamma_T : \Omega \to H_2$ which satisfy $\gamma_S(z) \neq 0, \gamma_T(z) \neq 0$ and*

$$\overline{\partial}\partial \log \|\gamma_S(z)\| = \overline{\partial}\partial \log \|\gamma_T(z)\|$$

for all $z \in \Omega$, then S and T are unitarily equivalent.

Proof. First we observe that

$$\overline{\partial}\partial(\log \|\gamma_S(z)\| - \log \|\gamma_T(z)\|) = 0$$

implies that the function $\tilde{h} : \Omega \to \mathbb{R}$ defined by

$$\tilde{h}(z) = \log \|\gamma_S(z)\| - \log \|\gamma_T(z)\|$$

is harmonic. Applying the exponential function, we get the equation

$$\|\gamma_S(z)\| = e^{\tilde{h}(z)}\|\gamma_T(z)\| = |e^{\tilde{h}(z)}|\|\gamma_T(z)\|. \tag{6.1}$$

Let $a \in \Omega$ and $r > 0$ be such that $D = D_r(a) \subset \Omega$. Then as D is simply connected, the function $h = \tilde{h}|_D$ admits a harmonic conjugate ${}^*h : D \to \mathbb{R}$ and so $f = h + i^*h \in \mathcal{O}(D)$. But $|e^{\tilde{h}(z)}| = |e^{h(z)+i^*h(z)}| = |e^{f(z)}|$ for $z \in D$ and the function $e^{f(z)}$ is holomorphic on D. So with (6.1) we see:

$$\|\gamma_S(z)\| = |e^{f(z)}|\|\gamma_T(z)\| = \|e^{f(z)}\gamma_T(z)\|.$$

Applying Proposition 5.7 we find that $S, T \in B_n(D)$ and that $\gamma_S|_D, \gamma_T|_D$ are spanning holomorphic cross-sections. By Proposition 6.2 the function

$$\tilde{\gamma}_T(z) : D \to H,\ z \mapsto e^{f(z)}\gamma_T(z)$$

is also a spanning holomorphic cross-section for E_T. Finally by Theorem 6.1, the operators S and T are unitarily equivalent. □

Remark 6.4. *It is clear from Corollary 5.5 that we can always find spanning holomorphic cross-sections $\gamma_S : \Omega \to H_1$ and $\gamma_T : \Omega \to H_2$ vanishing nowhere. In this case the function $K_i : \Omega \times \Omega \to \mathbb{C}$ defined by $K_i(z, w) = \langle \gamma_i(z), \gamma_i(w)\rangle$ is holomorphic in z and anti-holomorphic in w by Corollary 2.14. In particular, it is C^∞ considered as a function $\mathbb{R}^4 \supset \Omega \times \Omega \to \mathbb{C} \cong \mathbb{R}^2$. Thus for $z = x + iy \in \Omega$ with $x, y \in \mathbb{R}$ we have*

$$\overline{\partial}\partial \log \|\gamma_i(z)\| = \frac{1}{8}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \log K(x + iy, x + iy)$$

exists everywhere.

Remark 6.5. *For $T \in B_1(\Omega)$ and $\gamma : \Omega \to H$ a spanning holomorphic cross-section for E_T vanishing nowhere, the function $-\overline{\partial}\partial \log(\|\gamma(z)\|^2)$ is just the curvature of the vector bundle E_T (see [CD78], §2).*

7 Commutants

In this section we generalize a result from [Zhu00] which characterises the commutant of an operator from a Cowen-Douglas class in terms of a spanning holomorphic cross-section of this operator. We recall the basic definition.

Definition 7.1. *Let H be a normed vector space and $T \in B(H)$. Then we define the commutant $(T)'$ of T by*

$$(T)' = \{S \in B(H); TS = ST\}.$$

Theorem 7.2. *Let $\Omega \subset \mathbb{C}^d$ be open, let H_1, H_2 be Hilbert spaces and let $S \in B(H_1)^d$, $T \in B(H_2)^d$ be two operator tuples in $B_n(\Omega)$. Moreover let $\gamma_S : \Omega \to H_1$ be a spanning holomorphic cross-section in E_S. Then an operator $A \in B(H_1, H_2)$ intertwines S_i and T_i, i.e., $AS_i = T_iA$ for all $i \in \mathbb{N}_d$, if and only if the function $\gamma_T = A\gamma_S : \Omega \to H_2$ is a holomorphic cross-section in E_T. In this case γ_T satisfies $\gamma_T \prec \gamma_S$.*
On the other hand, for all holomorphic cross-sections γ_T in E_T with $\gamma_T \prec \gamma_S$, there exists a unique operator $A \in B(H_1, H_2)$ such that $\gamma_T(z) = A\gamma_S(z)$ and $AS_i = T_iA$ for all $i \in \mathbb{N}_d$.

Proof. First suppose that there is $A \in B(H_1, H_2)$ intertwining S_i and T_i for all $i \in \mathbb{N}_d$. By Proposition 2.11 the function $\gamma_T = A\gamma_S$ is holomorphic and for $z \in \Omega$ we have $T_i(A\gamma_S(z)) = AS_i\gamma_S(z) = z_i(A\gamma_S(z))$. Thus $\gamma_T(z) \in \ker(T_z)$ for all $z \in \Omega$. This shows that γ_T is a holomorphic cross-section in E_T. Furthermore, for $z, w \in \Omega$, we have

$$\langle \gamma_T(z), \gamma_T(w) \rangle = \langle A^*A\gamma_S(z), \gamma_S(w) \rangle$$

and A^*A is positive. Thus by Proposition 2.30 we have $\gamma_T \prec \gamma_S$.
Now let $A \in B(H_1, H_2)$ such that $\gamma_T = A\gamma_S$ is a holomorphic cross-section in E_T. Then for all $z \in \Omega$, $i \in \mathbb{N}_d$, we have

$$AS_i\gamma_S(z) = z_iA\gamma_S(z) = z_i\gamma_T(z) = T_i\gamma_T(z) = T_iA\gamma_S(z).$$

As $\mathrm{LH}(\gamma_S(\Omega)) \subset H_1$ is dense, we have $AS_i = T_iA$.
Suppose now that $\gamma_T : \Omega \to H_2$ is a holomorphic cross-section in E_T with $\gamma_T \prec \gamma_S$. Then there exists $C > 0$ such that

$$\|\sum_{k=1}^{m} c_k\gamma_T(z_k)\| \leq C\|\sum_{k=1}^{m} c_k\gamma_S(z_k)\| \tag{7.1}$$

for all $m \in \mathbb{N}^*$, $c_1, \dots, c_m \in \mathbb{C}$, $z_1, \dots, z_m \in \Omega$. Then by Corollary 2.5 there is a unique linear map $A_0 : \mathrm{LH}(\gamma_S(\Omega)) \to H_2$ with $A_0\gamma_S(z) = \gamma_T(z)$ for all $z \in \Omega$. This operator is bounded by (7.1). Thus it extends continuously to a bounded linear map $A \in B(H_1, H_2)$ still satisfying $\gamma_T = A\gamma_S$. Thus we are done by the first part of the proof. □

Theorem 7.3. *Let $\Omega \subset \mathbb{C}^d$ be open, let $T \in B_n(\Omega) \subset B(H)^d$ be an operator tuple on a Hilbert space H and let $\gamma_0 : \Omega \to H$ be a spanning holomorphic cross-section in E_T. Then the set*

$$C_T = \{\gamma : \Omega \to H; \gamma \text{ holomorphic cross-section in } E_T,\ \gamma \prec \gamma_0\}$$

is in canonical bijection to $\bigcap_{i=1}^d (T_i)'$ via the map

$$\bigcap_{i=1}^{d} (T_i)' \to C_T,\ A \mapsto A\gamma_0.$$

Proof. Applying Theorem 7.2 to $H_1 = H_2 = H$ and $S = T$ we see that an operator $A \in B(H)$ satisfies $AT_i = T_iA$ for all $i \in \mathbb{N}_d$ if and only if the function $A\gamma_0 : \Omega \to H$ is a holomorphic cross-section in E_T with $A\gamma_0 \prec \gamma_0$. Thus it suffices to show that all $\gamma \in C_T$ are of the form $\gamma = A\gamma_0$ with a suitable operator $A \in B(H)$. But this follows from the second part of Theorem 7.2. □

8 Similarity

In this section we want to characterise similar operators in the same Cowen-Douglas class. First we recall the basic definitions.

Definition 8.1. *Let H_1, H_2 be Hilbert spaces and let $S \in B(H_1)^d, T \in B(H_2)^d$ be two operator tuples.*
Then S and T are called similar if there exists a bounded invertible operator $A \in B(H_1, H_2)$ such that $AS_i = T_iA$ for $i = 1, \ldots, d$.
S and T are called quasi-similar if there exist bounded linear operators $A \in B(H_1, H_2)$ and $B \in B(H_2, H_1)$ which are injective with dense range such that $AS_i = T_iA$ and $S_iB = BT_i$ for $i = 1, \ldots, d$.

Theorem 8.2. *Let $\Omega \subset \mathbb{C}^d$ be an admissible domain of holomorphy, let H_1, H_2 be Hilbert spaces and let $S \in B(H_1)^d$, $T \in B(H_2)^d$ be two operator tuples in $B_n(\Omega)$. Then S and T are similar if and only if there exist spanning holomorphic cross-sections γ_S in E_S and γ_T in E_T such that $\gamma_S \sim \gamma_T$.*

Proof. First suppose that $A \in B(H_1, H_2)$ is invertible such that $AS_i = T_iA$ for all $i \in \mathbb{N}_d$. Let γ_S be a spanning holomorphic cross-section in E_S. Then $\gamma_T = A\gamma_S$ is a holomorphic cross-section in E_T by Theorem 7.2. Since A is onto, γ_T spans H_2:

$$\overline{\mathrm{LH}(\gamma_T(\Omega))} = \overline{\mathrm{LH}(A\gamma_S(\Omega))} \supset A\,\overline{\mathrm{LH}(\gamma_S(\Omega))} = AH_1 = H_2.$$

Thus γ_T is a spanning holomorphic cross-section for E_T. Since

$$\langle \gamma_T(z), \gamma_T(w) \rangle = \langle A^*A\gamma_S(z), \gamma_S(w) \rangle \quad \text{for } z, w \in \Omega$$

and since A^*A is positive and invertible, we have $\gamma_S \sim \gamma_T$ by Proposition 2.30.
Now let γ_S in E_S and γ_T in E_T be spanning holomorphic cross-sections such that $\gamma_S \sim \gamma_T$. Then applying Theorem 7.2 in both directions we find operators $A \in B(H_1, H_2)$, $B \in B(H_2, H_1)$ with $\gamma_T = A\gamma_S$ and $\gamma_S = B\gamma_T$ and $AS_i = T_iA$ for all $i \in \mathbb{N}_d$. But as γ_S spans H_1, γ_T spans H_2 and we have $\gamma_T = AB\gamma_T$ and $\gamma_S = BA\gamma_T$, it is clear that $AB = 1_{H_2}$ and $BA = 1_{H_1}$. Thus A is invertible as required. □

Theorem 8.3. *Let $\Omega \subset \mathbb{C}^d$ be an admissible domain of holomorphy, let H_1, H_2 be Hilbert spaces and $S \in B(H_1)^d$, $T \in B(H_2)^d$ be two operator tuples in $B_n(\Omega)$. Then there exists a bounded linear operator $A \in B(H_1, H_2)$ with dense range such that $AS_i = T_iA$ for all $i \in \mathbb{N}_d$ if and only if there exist spanning holomorphic cross-sections γ_S in E_S and γ_T in E_T such that $\gamma_T \prec \gamma_S$.*

Proof. First suppose that $A \in B(H_1, H_2)$ has dense range such that $AS_i = T_iA$ for all $i \in \mathbb{N}_d$. Let γ_S be a spanning holomorphic cross-section in E_S. Then $\gamma_T = A\gamma_S$

is a holomorphic cross-section in E_T satisfying $\gamma_T \prec \gamma_S$ by Theorem 7.2. Since A has dense range, γ_T spans H_2:

$$\overline{\mathrm{LH}(\gamma_T(\Omega))} = \overline{\mathrm{LH}(A\gamma_S(\Omega))} \supset \overline{A\,\overline{\mathrm{LH}(\gamma_S(\Omega))}} = \overline{AH_1} = H_2.$$

Thus γ_T is a spanning holomorphic cross-section for E_T.
Now let γ_S in E_S and γ_T in E_T be spanning holomorphic cross-sections such that $\gamma_T \prec \gamma_S$. Then by Theorem 7.2 there exists a unique operator $A \in B(H_1, H_2)$ such that $A\gamma_S = \gamma_T$ which also satisfies $AS_i = T_iA$ for all $i \in \mathbb{N}_d$. Finally A has dense range, since obviously $\mathrm{LH}(\gamma_T(\Omega)) \subset \mathrm{Ran}(A)$. □

Theorem 8.4. *Let $\Omega \subset \mathbb{C}^d$ be an admissible domain of holomorphy, let H_1, H_2 be Hilbert spaces and $S \in B(H_1)^d$, $T \in B(H_2)^d$ be two operator tuples in $B_n(\Omega)$. Then there exist bounded linear operators $A \in B(H_1, H_2)$ and $B \in B(H_2, H_1)$ with dense range such that $AS_i = T_iA$ and $S_iB = BT_i$ for all $i \in \mathbb{N}_d$ if and only if there exist spanning holomorphic cross-sections γ_S, γ'_S in E_S and γ_T, γ'_T in E_T such that $\gamma_S \prec \gamma_T$ and $\gamma'_T \prec \gamma'_S$.*

Proof. The theorem clearly follows by applying Theorem 8.3 in both directions. □

Remark 8.5. *In [Zhu00] it is stated that in the situation of Theorem 8.4 for $d = 1$, we already have that S and T are quasi-similar. For this we would have to show that we can choose A and B to be injective. Unfortunately the author of this thesis was not able to verify this claim.*

Conventions

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the natural numbers
$\mathbb{N}^* = \{1, 2, 3, 4, \ldots\}$, the natural numbers without zero
$\mathbb{N}_n = \{1, 2, 3, \ldots n\}$, the first n positive natural numbers
$A^B = \{f : B \to A\}$, the set of functions from B to A for sets A, B
$\mathrm{pr}_i : A_1 \times A_2 \times \ldots A_n \to A_i$, $(a_1, \ldots, a_n) \mapsto a_i$, the projection on the i-th component
$M(n \times m, R)$, the ring of $n \times m$-matrices with entries in the ring R
$\mathrm{GL}(n, k)$, the ring of invertible $n \times n$-matrices with entries in the field k
$\Omega \subset \mathbb{C}$ is called a domain if $\Omega \neq \emptyset$ is open and connected.
$D_r(a) = \{z \in \mathbb{C}; |z - a| < r\}$, the open disc of radius $r \in [0, \infty]$ around $a \in \mathbb{C}$
$B_r(a) = \{z \in \mathbb{C}^d; \sum_{i=1}^d |z_i - a_i|^2 < r^2\}$, the open ball of radius $r \in [0, \infty]$ around $a \in \mathbb{C}^d$
$P_r(a) = \{z \in \mathbb{C}^d; |z_i - a_i| < r \ \forall i \in \mathbb{N}_d\}$, the polydisc of polyradius $r \in [0, \infty]$ around $a \in \mathbb{C}^d$
For $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{C}^d$ we define $\overline{\omega} = (\overline{\omega_1}, \ldots, \overline{\omega_d})$.
For $\Omega \subset \mathbb{C}^d$ we define $\Omega^* = \{\overline{\omega};\ \omega \in \Omega\}$.
Let V, W be two normed vector spaces over $\mathbb{C}$.
$B(V, W) = \{T : V \to W; T \text{ bounded linear}\}$, the bounded linear operators from V to W
$B(V) = B(V, V)$, the bounded linear operators on V
$V' = B(V, \mathbb{C})$, the bounded linear forms on V
$\langle x, u \rangle = u(x)$ for $x \in V$, $u \in V'$
$\mathcal{O}(\Omega) = \{f : \Omega \to \mathbb{C}; f \text{ holomorphic}\}$, the space of holomorphic functions on an open set $\Omega \subset \mathbb{C}^d$

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