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Wold decomposition for operators close to isometries

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Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

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Introduction

The classical Wold decomposition theorem, which for instance can be found in [4], states that every isometry on a Hilbert space decomposes into a direct sum of a unitary operator and a unilateral shift.
More precisely, for every isometry $T \in \mathcal{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$, this Hilbert space will decompose into an orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of reducing subspaces $\mathcal{H}_0, \mathcal{H}_1 \subset \mathcal{H}$ for T such that T restricted to $\mathcal{H}_0$ is unitary and T restricted to $\mathcal{H}_1$ is unitarily equivalent to a unilateral shift. In this case, the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are uniquely determined by

$$\mathcal{H}_0 = \bigcap_{n\in\mathbb{N}} T^n\mathcal{H}$$

and

$$\mathcal{H}_1 = \bigvee_{n\in\mathbb{N}} T^n E,$$

where $E = \mathcal{H} \ominus T\mathcal{H}$. The space E is a closed subspace of $\mathcal{H}$ such that $E \perp T^n E$ holds for all $n \in \mathbb{N}^*$. We will call such spaces wandering subspaces for T.

If one drops the requirement that T is an isometry, it is still reasonable to define the spaces above. Hence for a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ we set

$$H_\infty(T) = \bigcap_{n\in\mathbb{N}} T^n\mathcal{H}$$

and call T analytic if $H_\infty(T) = \{0\}$. Also, for a subset $M \subset \mathcal{H}$ we define

$$[M]_T = \bigvee_{n\in\mathbb{N}} T^n M.$$

This is the smallest closed subspace of $\mathcal{H}$ that contains M and is invariant under T. In Theorem 2.9 we will see that if T is an analytic operator and fulfils

$$\|Tx + y\|^2 \leq 2\left(\|x\|^2 + \|Ty\|^2\right) \tag{0.1}$$

for all $x, y \in \mathcal{H}$, then T possesses the so called wandering subspace property, that is,

$$\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T.$$

In analogy to the classical Wold decomposition theorem we will say that an operator $T \in \mathcal{B}(\mathcal{H})$ admits a Wold-type decomposition if $H_\infty(T)$ is a reducing subspace for T such that the restriction

$$T|_{H_\infty(T)} \colon H_\infty(T) \to H_\infty(T)$$

is unitary and $\mathcal{H}$ admits the orthogonal decomposition

$$\mathcal{H} = H_\infty(T) \oplus [\mathcal{H} \ominus T\mathcal{H}]_T\,.$$

By definition an operator $T \in \mathcal{B}(\mathcal{H})$ is concave if

$$\left\|T^2x\right\|^2 + \|x\|^2 \leq 2\,\|Tx\|^2 \tag{0.2}$$

holds for all $x \in \mathcal{H}$. The main result of this thesis is Theorem 3.3, which is due to Shimorin [8] and states that concave operators and operators fulfilling (0.1) admit Wold-type decompositions. In particular, conditions (0.1) and (0.2) both hold for isometries, so Theorem 3.3 contains the classical Wold decomposition theorem as a special case.

In the first chapter we will introduce our definitions and notations as well as gather some general results that will be required in the later parts.

Then, in the second chapter we will prove the above-stated wandering subspace theorem, loosely following arguments found in Chapter 9.3 of [2]. In particular, we show that for an operator $T \in \mathcal{B}(\mathcal{H})$ that fulfils (0.1), its Cauchy dual $T' = T(T^*T)^{-1}$ is concave. Additionally, we will see that T admits a Wold-type decomposition if and only if T' is analytic. Hence the proof of Theorem 2.9 reduces to showing that T' is analytic.

Thereafter, we prove the already mentioned Wold-type decomposition theorem in Chapter 3, following ideas from the original proof by Shimorin given in [8]. Crucial steps are showing that a left invertible operator admits a Wold-type decomposition if and only if its Cauchy dual does so, as well as proving a wandering subspace theorem for analytic, concave operators due to Richter, see Theorem 1 in [5].

Finally, in Chapter 4 we introduce functional Hilbert spaces and then apply the developed theory to shift operators on the analytic functional Hilbert spaces $\mathcal{H}(K_\alpha)$ given by the reproducing kernels

$$K_\alpha\colon \mathbb{D}\times\mathbb{D}\to\mathbb{C},\ K_\alpha(z,w)=\left(\frac{1}{1-z\overline{w}}\right)^{\alpha+2}\quad(\alpha>-2).$$

In particular, we will see that for $-1\leq\alpha\leq 0$ the shift operator S on $\mathcal{H}(K_\alpha)$ satisfies a Beurling-type theorem, that is to say each restriction $S|_M$ of S to a closed invariant subspace M of S possesses the wandering subspace property, whereas for $\alpha>4$ this is no longer the case.
Furthermore, we consider the Dirichlet shift as an example of a concave operator.

1 Preliminaries

This chapter's purpose is to introduce relevant definitions and notations and furthermore to discuss some basic results that will be needed later on.

1.1 Terminology and notation

For the rest of this thesis, let $\mathcal{H}$ be a complex Hilbert space and, if not stated otherwise, let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on $\mathcal{H}$.

Recall that a closed linear subspace $W \subset \mathcal{H}$ is called an *invariant subspace* for T if $TW \subset W$ and a *reducing subspace* for T if W is invariant for both T and T^*.

For our purposes, the subspaces

$$H_\infty(T) = \bigcap_{n \in \mathbb{N}} T^n \mathcal{H}$$

and, for a subset $M \subset \mathcal{H}$,

$$[M]_T = \bigvee_{n \in \mathbb{N}} T^n M$$

play an important role. In particular, we are interested in operators $T \in \mathcal{B}(\mathcal{H})$ such that $H_\infty(T) = \{0\}$, since these behave especially nicely. This leads to the following definition.

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *analytic* if $H_\infty(T) = \{0\}$.

Another object we are interested in are so-called wandering subspaces.

Definition 1.2. A closed subspace $W \subset \mathcal{H}$ is called a *wandering subspace* for T, if

$$W \perp T^n W$$

holds for all $n \in \mathbb{N}^*$.

Remark 1.3. Let $T \in \mathcal{B}(\mathcal{H})$ be an isometry. Then, a closed subspace $W \subset \mathcal{H}$ is a wandering subspace for T if and only if

$$T^k W \perp T^n W$$

holds for all distinct $n, k \in \mathbb{N}$.

Proof. Suppose that W is a wandering subspace for T and let w, v be elements of W. For natural numbers $n > k$, we then have

$$\langle T^k w, T^n v\rangle = \langle T^k w, T^k(T^{n-k}v)\rangle = \langle w, T^{n-k}v\rangle = 0$$

and thus $T^k W \perp T^n W$. The converse implication is obvious. $\square$

Remark 1.4. For any subset $M \subset \mathcal{H}$, the space $[M]_T$ is the smallest closed subspace of $\mathcal{H}$ which contains M and is invariant under T.

Proof. By definition, $[M]_T$ is a closed subspace of $\mathcal{H}$ containing M. To see that $[M]_T$ is invariant under T let $m \in [M]_T$. Then there exists a sequence $(m_n)_n$ in $\mathrm{LH}\left(\bigcup_{k\in\mathbb{N}} T^k M\right)$ converging to m. Obviously, $Tm_n \in \mathrm{LH}\left(\bigcup_{k\in\mathbb{N}} T^k M\right)$ holds for all $n \in \mathbb{N}$, so

$$Tm = \lim_{n\to\infty} Tm_n \in \overline{\mathrm{LH}\left(\bigcup_{k\in\mathbb{N}} T^k M\right)} = [M]_T.$$

Now, let A be a closed subspace of $\mathcal{H}$ that contains M and is invariant under T. Then we have

$$T^k M \subset T^k A \subset A$$

for all $k \in \mathbb{N}$ and therefore

$$\bigcup_{k\in\mathbb{N}} T^k M \subset A.$$

Since A is a closed subspace, we conclude $[M]_T \subset A$. $\square$

In case T^*T is invertible, it is helpful to study the operator defined below.

Definition 1.5. If T^*T is invertible, we define the *Cauchy dual* T' of T by

$$T' = T(T^*T)^{-1}.$$

Remark 1.6. If T^*T is invertible, then T'^*T' is invertible and $T'' = T$ holds.

Proof. A simple computation shows that

$$T'^*T' = \left(T(T^*T)^{-1}\right)^* \left(T(T^*T)^{-1}\right) = (T^*T)^{-1}.$$

So T'^*T' is invertible and

$$T'' = T'(T'^*T')^{-1} = T(T^*T)^{-1}T^*T = T.$$

□

We end this section by introducing the properties that we are primarily interested in, namely what it means for an operator $T \in \mathcal{B}(\mathcal{H})$ to admit a Wold-type decomposition or to possess the wandering subspace property.

Definition 1.7. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to admit a *Wold-type decomposition* if $H_\infty(T)$ is a reducing subspace for T, the restriction

$$T|_{H_\infty(T)} \colon H_\infty(T) \to H_\infty(T)$$

is unitary and $\mathcal{H}$ admits the orthogonal decomposition

$$\mathcal{H} = H_\infty(T) \oplus [\mathcal{H} \ominus T\mathcal{H}]_T\,.$$

Definition 1.8. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ possesses the *wandering subspace property* if

$$\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ satisfies a *Beurling-type theorem* if each restriction $T|_M$ of T to a closed invariant subspace M of T possesses the wandering subspace property.

Remark 1.9. An operator $T \in \mathcal{B}(\mathcal{H})$ possesses the wandering subspace property if and only if there is a wandering subspace W for T such that $\mathcal{H} = [W]_T$. In this case, $W = \mathcal{H} \ominus T\mathcal{H}$.

Proof. Let $W \subset \mathcal{H}$ be a wandering subspace for T such that $\mathcal{H} = [W]_T$. Then we have

$$\mathcal{H} \ominus T\mathcal{H} = [W]_T \ominus T[W]_T = \left(\bigvee_{n\in\mathbb{N}} T^n W\right) \ominus \left(\bigvee_{n\in\mathbb{N}^*} T^n W\right) = W.$$

Hence $\mathcal{H} = [W]_T = [\mathcal{H} \ominus T\mathcal{H}]_T$, so T possesses the wandering subspace property.

On the other hand, if T possesses the wandering subspace property, then the closed linear subspace $W = \mathcal{H} \ominus T\mathcal{H}$ obviously is a wandering subspace for T such that $\mathcal{H} = [W]_T$. □

Remark 1.10. An analytic operator that possesses the wandering subspace property obviously admits a Wold-type decomposition.

1.2 Basic results

In this section we gather some results that will be needed throughout the thesis. We start with the following proposition.

Proposition 1.11. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) T is injective with closed range.

(ii) T^*T is invertible.

(iii) T is left invertible.

In this case, T'^n is injective with closed range for all $n \in \mathbb{N}$.

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be injective with closed range. For $x \in \ker T^*T$, we have

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = 0$$

and hence $x = 0$, since T is injective. Therefore T^*T is injective as well. Furthermore, for $y \in (\operatorname{Im} T^*T)^\perp$ we have

$$\|Ty\|^2 = \langle T^*Ty, y\rangle = 0,$$

hence $y = 0$. We conclude that

$$(\operatorname{Im} T^*T)^\perp = \{0\}$$

holds. Consequently, $\operatorname{Im} T^*T \subset \mathcal{H}$ is dense. By the closed range theorem, T^* has closed range, whence

$$T^*T\mathcal{H} = T^*(T\mathcal{H} + \ker T^*) = T^*(T\mathcal{H} + (T\mathcal{H})^\perp) = T^*\mathcal{H}$$

is closed. Thus T^*T is surjective and hence invertible.

Next, suppose that T^*T is invertible. Then $(T^*T)^{-1}T^* \in \mathcal{B}(\mathcal{H})$ obviously is a left inverse for T.

Finally, suppose that T is left invertible with left inverse $S \in \mathcal{B}(\mathcal{H})$. Then T obviously is injective. From

$$T^*S^* = (ST)^* = I,$$

we conclude that T^* is right invertible and hence surjective. Therefore, T has closed range.

In this situation, T'^*T' is invertible by Remark 1.6, hence T' is injective with closed range and thus the same is true for T'^n for any $n \in \mathbb{N}$. □

The next proposition describes the orthogonal complement of an intersection of closed subspaces.

Proposition 1.12. *Let $(W_j)_{j\in J}$ be a family of closed subspaces $W_j \subset \mathcal{H}$. Then,*

$$\left(\bigcap_{j\in J} W_j\right)^\perp = \bigvee_{j\in J} W_j^\perp$$

holds.

Proof. Since the W_j are closed subspaces, $W_j = W_j^{\perp\perp}$ holds for all $j \in J$. Thus, for

$$x \in \left(\bigcup_{j\in J} W_j^\perp\right)^\perp,$$

we have

$$x \in \left(W_j^\perp\right)^\perp = W_j$$

for all $j \in J$ and therefore

$$x \in \bigcap_{j\in J} W_j.$$

Conversely, for

$$x \in \bigcap_{j\in J} W_j,$$

obviously $x \in W_j = W_j^{\perp\perp}$ holds for all $j \in J$, so

$$x \perp \bigcup_{j\in J} W_j^\perp,$$

which implies

$$x \in \left(\bigcup_{j \in J} W_j^{\perp} \right)^{\perp}.$$

This shows

$$\bigcap_{j \in J} W_j = \left(\bigcup_{j \in J} W_j^{\perp} \right)^{\perp}.$$

Taking orthogonal complements we obtain

$$\left(\bigcap_{j \in J} W_j \right)^{\perp} = \left(\bigcup_{j \in J} W_j^{\perp} \right)^{\perp\perp} = \overline{\mathrm{LH}\left(\bigcup_{j \in J} W_j^{\perp} \right)} = \bigvee_{j \in J} W_j^{\perp}$$

as claimed. □

We conclude this chapter with the result below.

Proposition 1.13. *For $T \in \mathcal{B}(\mathcal{H})$, the relation*

$$\mathcal{H} \ominus T\mathcal{H} = H_\infty(T)^{\perp} \cap \left[T \left(H_\infty(T)^{\perp} \right) \right]^{\perp}$$

holds.

Proof. Obviously,

$$\mathcal{H} \ominus T\mathcal{H} \subset H_\infty(T)^{\perp} \cap \left[T \left(H_\infty(T)^{\perp} \right) \right]^{\perp}$$

holds. Conversely, let $h \in H_\infty(T)^{\perp} \cap \left[T \left(H_\infty(T)^{\perp} \right) \right]^{\perp}$ and let $k \in \mathcal{H}$ be arbitrary. We may write

$$k = k_\infty + k_\perp$$

for some $k_\infty \in \overline{H_\infty(T)}$ and $k_\perp \in H_\infty(T)^{\perp}$. Since $T\overline{H_\infty(T)} \subset \overline{H_\infty(T)}$, this implies

$$\langle h, Tk \rangle = \langle h, Tk_\infty \rangle + \langle h, Tk_\perp \rangle = 0.$$

Hence, we have $h \in \mathcal{H} \ominus T\mathcal{H}$. □

2 A wandering subspace theorem

In this chapter, we will show that an analytic operator $T \in \mathcal{B}(\mathcal{H})$, which fulfils

$$\|Tx + y\|^2 \leq 2 \left(\|x\|^2 + \|Ty\|^2 \right) \tag{2.1}$$

for all $x, y \in \mathcal{H}$, possesses the wandering subspace property. This result for instance appears in Chapter 9 of [2]. We first note some basic properties of such operators.

Proposition 2.1. *Let T be a linear operator on $\mathcal{H}$ for which (2.1) holds. Then*

(i) T is bounded, i.e. $T \in \mathcal{B}(\mathcal{H})$,

(ii) T is injective and has closed range.

In particular, the Cauchy dual T' of T exists.

Proof. By setting $y = 0$, condition (2.1) yields

$$\|Tx\|^2 \leq 2 \|x\|^2$$

for $x \in \mathcal{H}$ and thus (i). Analogously, setting $x = 0$ yields

$$\|y\|^2 \leq 2 \|Ty\|^2$$

for $y \in \mathcal{H}$, so T is bounded below and thus injective with closed range. In particular, by Proposition 1.11, the Cauchy dual T' of T exists. $\square$

Next, we want to show that for an operator $T \in \mathcal{B}(\mathcal{H})$ fulfilling (2.1), the Cauchy dual T' is concave in the sense of the following definition.

Definition 2.2. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *concave* if

$$\left\|T^2 x\right\|^2 + \|x\|^2 \leq 2 \|Tx\|^2 \tag{2.2}$$

holds for all $x \in \mathcal{H}$.

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator such that (2.1) holds. Then the Cauchy dual T' of T is concave.*

Proof. By Proposition 2.1 the operator T^*T is invertible. Since

$$\langle T^*Tx, x\rangle = \|Tx\|^2 \geq 0$$

holds, T^*T is positive. So, by the continuous functional calculus, T^*T has a positive square root and we may consider the operator $(T^*T)^{-\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$. The computation

$$\begin{aligned}\left\|T(T^*T)^{-\frac{1}{2}}x\right\|^2 &= \langle T(T^*T)^{-\frac{1}{2}}x, T(T^*T)^{-\frac{1}{2}}x\rangle = \langle (T^*T)^{\frac{1}{2}}x, (T^*T)^{-\frac{1}{2}}x\rangle \\ &= \langle x, x\rangle = \|x\|^2\end{aligned}$$

shows that $T(T^*T)^{-\frac{1}{2}}$ is an isometry. Using this observation and substituting

$$y = (T^*T)^{-\frac{1}{2}}z,$$

we can rewrite condition (2.1) as

$$\left\|Tx + (T^*T)^{-\frac{1}{2}}z\right\|^2 \leq 2\left(\|x\|^2 + \|z\|^2\right)$$

for all $x, z \in \mathcal{H}$. By introducing the operator

$$L\colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{H},\ (x, z) \mapsto Tx + (T^*T)^{-\frac{1}{2}}z,$$

this takes the even more concise form

$$\|L(x, z)\|^2 \leq 2\,\|(x, z)\|^2 .$$

Thus, $\|L\| \leq \sqrt{2}$, and hence

$$\langle (2I - LL^*)x, x\rangle = 2\,\|x\|^2 - \|L^*x\|^2 \geq 2\,\|x\|^2 - \|L\|^2\,\|x\|^2 \geq 0 \quad (x \in \mathcal{H}),$$

so $LL^* \leq 2I$. Furthermore,

$$L^* = \left(T,\ (T^*T)^{-\frac{1}{2}}\right)^* = \begin{pmatrix} T^* \\ (T^*T)^{-\frac{1}{2}} \end{pmatrix}$$

yields

$$LL^* = TT^* + (T^*T)^{-1}$$

and thus

$$T'(T'^*T')^{-1}(T'^*T')^{-1}T'^* + T'^*T' = TT^* + (T^*T)^{-1} = LL^* \leq 2I$$

by Remark 1.6 and its proof. We conclude that

$$\begin{aligned} 0 &\leq T'^* \left[2I - T'(T'^*T')^{-1}(T'^*T')^{-1}T'^* + T'^*T'\right] T' \\ &= 2T'^*T' - T'^*T'(T'^*T')^{-1}(T'^*T')^{-1}T'^*T' - (T'^*)^2T'^2 \\ &= 2T'^*T' - I - (T'^*)^2T'^2, \end{aligned}$$

or equivalently,

$$2\left\|T'x\right\|^2 - \left\|x\right\|^2 - \left\|T'^2x\right\|^2 = \langle(2T'^*T' - I - (T'^*)^2T'^2)x, x\rangle \geq 0$$

for all $x \in \mathcal{H}$. So, T' is concave. □

Remark 2.4. The above proof shows that for an operator $T \in \mathcal{B}(H)$ fulfilling condition (2.1) the operator T^*T is invertible with

$$TT^* + (T^*T)^{-1} \leq 2I.$$

The reverse implication holds, too.

Proof. Let $T \in \mathcal{B}(H)$ be an operator such that T^*T is invertible with

$$TT^* + (T^*T)^{-1} \leq 2I.$$

The same arguments as above show that

$$L\colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{H},\ (x, z) \mapsto Tx + (T^*T)^{-\frac{1}{2}}z,$$

is a well-defined linear operator with

$$LL^* = TT^* + (T^*T)^{-1} \leq 2I.$$

Consequently,

$$0 \leq \langle(2I - LL^*)x, x\rangle = 2\left\|x\right\|^2 - \left\|L^*x\right\|^2$$

holds for all $x \in \mathcal{H}$ and we conclude

$$\left\|L\right\|^2 = \left\|L^*\right\|^2 \leq 2.$$

Let $x, y \in \mathcal{H}$ be arbitrary. Since T^*T is invertible, there is a unique $z \in \mathcal{H}$ such that $y = (T^*T)^{-\frac{1}{2}}z$. Furthermore, in the proof above we have shown that $T(T^*T)^{-\frac{1}{2}}$ is an isometry. Thus

$$\begin{aligned} 2\left(\|x\|^2 + \|Ty\|^2\right) - \|Tx + y\|^2 &= 2\left(\|x\|^2 + \|z\|^2\right) - \left\|Tx + (T^*T)^{-\frac{1}{2}}z\right\|^2 \\ &= 2\|(x,z)\|^2 - \|L(x,z)\|^2 \\ &\geq \left(2 - \|L\|^2\right)\|(x,z)\|^2 \\ &\geq 0. \end{aligned}$$

Hence T fulfils condition (2.1) as claimed. □

Now, we turn to the study of concave operators.

Proposition 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be concave. Then T is injective with closed range and the inequalities*

(i) $\|T^{n+1}x\| \geq \|T^n x\|$,

(ii) $\|T^n x\|^2 \leq \|x\|^2 + n\left(\|Tx\|^2 - \|x\|^2\right)$

hold for all $x \in \mathcal{H}$ and $n \in \mathbb{N}$.

Proof. Since T is concave,

$$2\|Tx\|^2 \geq \left\|T^2 x\right\|^2 + \|x\|^2 \geq \|x\|^2$$

holds for all $x \in \mathcal{H}$, so T is bounded below and hence injective with closed range. For $x \in \mathcal{H}$, $n \in \mathbb{N}$ and $y = T^n x$, the concavity property gives the inequality

$$\begin{aligned} \left\|T^{n+2}x\right\|^2 - \left\|T^{n+1}x\right\|^2 &= \left\|T^2 y\right\|^2 - \|Ty\|^2 \\ &\leq \|Ty\|^2 - \|y\|^2 \\ &= \left\|T^{n+1}x\right\|^2 - \|T^n x\|^2. \end{aligned}$$

Assume there were $k \in \mathbb{N}$ and $x \in \mathcal{H}$ such that

$$\left\|T^{k+1}x\right\| < \left\|T^k x\right\|.$$

In that case, by the previous inequality,

$$\left\|T^{n+1}x\right\|^2 - \|T^n x\|^2 \leq \left\|T^{k+1}x\right\|^2 - \left\|T^k x\right\|^2 < 0$$

holds for all $n \geq k$. We conclude

$$0 \leq \left\|T^{n+1}x\right\|^2 < \|T^n x\|^2$$

for all $n \geq k$, so $\left(\|T^n x\|^2\right)_{n\geq k}$ is a bounded, decreasing sequence in $\mathbb{R}$ and thus convergent. This leads to the contradiction

$$0 = \lim_{n\to\infty} \left(\left\|T^{n+1}x\right\|^2 - \|T^n x\|^2\right) \leq \left\|T^{k+1}x\right\|^2 - \left\|T^k x\right\|^2 < 0.$$

Thus, the assumption was false and claim (i) follows.

Since T is concave, the first part of the proof shows that

$$\|T^n x\|^2 - \|x\|^2 = \sum_{k=0}^{n-1} \left(\left\|T^{k+1}x\right\|^2 - \left\|T^k x\right\|^2\right) \leq \sum_{k=0}^{n-1} \left(\|Tx\|^2 - \|x\|^2\right)$$

holds for all $x \in \mathcal{H}$ and $n \in \mathbb{N}$, which clearly implies (ii). □

Consequently, for a concave operator $T \in \mathcal{B}(\mathcal{H})$, the intersection

$$H_\infty(T) = \bigcap_{n\in\mathbb{N}} T^n \mathcal{H}$$

is a closed invariant subspace for T. The next lemma describes how the restriction

$$T|_{H_\infty(T)} \colon H_\infty(T) \to H_\infty(T)$$

behaves.

Lemma 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be concave. Then we have*

$$TH_\infty(T) = T^* H_\infty(T) = H_\infty(T),$$

and the restriction

$$T|_{H_\infty(T)} \colon H_\infty(T) \to H_\infty(T)$$

is unitary. In particular, $H_\infty(T)$ is a reducing subspace for T.

Proof. By Proposition 2.5, T is injective, hence

$$TH_\infty(T) = T \bigcap_{n\in\mathbb{N}} T^n\mathcal{H} = \bigcap_{n\geq 1} T^n\mathcal{H} = H_\infty(T)$$

holds. Therefore,

$$S\colon H_\infty(T) \to H_\infty(T),\, x \mapsto Tx$$

is well-defined and bijective, hence invertible.
Let $y \in H_\infty(T)$ be arbitrary and set $x = Sy \in H_\infty(T)$. For all $n \in \mathbb{N}$, there is a $y_n \in \mathcal{H}$ such that $x = T^n y_n$. We have

$$T^{n-1}(Ty_n) = x = T^{n-1}y_{n-1} \quad (n \in \mathbb{N}^*)$$

and since T is injective, we conclude $Ty_n = y_{n-1}$ for all $n \in \mathbb{N}^*$ and $y_1 = y$.
The concavity of T implies

$$\|y_n\|^2 + \|y_{n+2}\|^2 \leq 2\,\|y_{n+1}\|^2$$

for all $n \in \mathbb{N}$. As in the proof of the last proposition, it follows that

$$\|y_{n+1}\| \geq \|y_n\|$$

for all $n \in \mathbb{N}$. Using Proposition 2.5 we obtain that

$$\|y_n\| = \|Ty_{n+1}\| \geq \|y_{n+1}\|$$

holds for all $n \in \mathbb{N}$, whence the sequence $(\|y_n\|)_{n\in\mathbb{N}}$ is constant. We obtain

$$\|Sy\| = \|x\| = \|y_0\| = \|y_1\| = \|y\|$$

and conclude that the surjective operator S is a unitary.

By the previous proposition, T is norm-increasing, so

$$\langle (T^*T - I)x, x\rangle = \|Tx\|^2 - \|x\|^2 \geq 0$$

holds for all $x \in \mathcal{H}$. This shows that $T^*T - I$ is positive, and therefore has a square root $(T^*T - I)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$. For $x \in H_\infty(T)$, we have

$$\left\|(T^*T - I)^{\frac{1}{2}}x\right\|^2 = \langle (T^*T - I)x, x\rangle = \|Tx\|^2 - \|x\|^2 = 0.$$

This yields $(T^*T - I)^{\frac{1}{2}}x = 0$ and consequently $T^*Tx = x$. By

$$T^*H_\infty(T) = T^*TH_\infty(T) = H_\infty(T)$$

the claim follows. $\square$

Lastly, for a left invertible operator $T \in \mathcal{B}(\mathcal{H})$, we need to express the kernel of T'^n differently.

Proposition 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be left invertible. For each $n \in \mathbb{N}^*$, we have*

$$\ker T'^{*n} = E + TE + \cdots + T^{n-1}E = \bigvee_{k=0}^{n-1} T^k E,$$

where $E = \mathcal{H} \ominus T\mathcal{H} = \ker T^*$.
In particular, the identities $H_\infty(T')^\perp = [E]_T$ and $H_\infty(T)^\perp = [E]_{T'}$ hold.

Proof. First note that T is injective with closed range and that the Cauchy dual T' of T exists by Proposition 1.11. Let $k \in \{0, \ldots, n-1\}$ and $x = T^k y \in T^k E$ for an $y \in E$. Since

$$T'^* T = (T^* T)^{-1} T^* T = I$$

and $y \in \ker T^* = \ker T'^*$, we conclude

$$T'^{*n} x = T'^{*n} T^k y = T'^{*n-k} T'^{*k} T^k y = T'^{*n-k-1} T'^* y = 0.$$

Hence $x \in \ker T'^{*n}$. Therefore,

$$\bigvee_{k=0}^{n-1} T^k E \subset \ker T'^{*n}$$

holds.

To prove the reverse inclusion, we first show that

$$P = I - TT'^*$$

is the orthogonal projection onto E. Since T has closed range, we have

$$\mathcal{H} = T\mathcal{H} \oplus E.$$

Furthermore, we have

$$TT'^* Tx = Tx$$

for all $x \in \mathcal{H}$ and

$$TT'^* E = T(T^*T)^{-1} T^* E = \{0\}$$

holds, so TT'^* is the orthogonal projection onto $T\mathcal{H}$. Thus, P projects onto

$$\mathcal{H} \ominus T\mathcal{H} = E.$$

For all $n \in \mathbb{N}^*$, we have

$$I - T^n T'^{*n} = \sum_{k=0}^{n-1} \left(T^k T'^{*k} - T^{k+1} T'^{*k+1} \right) = \sum_{k=0}^{n-1} T^k P T'^{*k}.$$

Let $x \in \ker T'^{*n}$ be arbitrary. Then the claim follows from

$$x = (I - T^n T'^{*n})x = \sum_{k=0}^{n-1} T^k P T'^{*k} x \in \sum_{k=0}^{n-1} T^k E.$$

In particular, using Propositions 1.11 and 1.12, we conclude

$$\begin{aligned} H_\infty(T')^\perp &= \left(\bigcap_{n\in\mathbb{N}} T'^n \mathcal{H} \right)^\perp = \bigvee_{n\in\mathbb{N}} (T'^n \mathcal{H})^\perp = \bigvee_{n\in\mathbb{N}} \ker T'^{*n} \\ &= \bigvee_{n\in\mathbb{N}} \bigvee_{k=0}^{n-1} T^k E = \bigvee_{n\in\mathbb{N}} T^n E = [E]_T. \end{aligned}$$

Note that $\ker T'^* = \ker (T^*T)^{-1} T^* = \ker T^* = E$. Then the other identity is a consequence of Remark 1.6, since

$$H_\infty(T)^\perp = H_\infty(T'')^\perp = [\ker T'^*]_{T'} = [E]_{T'}.$$

□

Corollary 2.8. *A left invertible operator $T \in \mathcal{B}(\mathcal{H})$ possesses the wandering subspace property if and only if the Cauchy dual T' of T is analytic.*

By combining the previous results, we are now able to prove that analytic operators fulfilling (2.1) possess the wandering subspace property. In particular, this implies that such operators admit a Wold-type decomposition.

Theorem 2.9 (Wandering subspace property)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be analytic and such that*

$$\|Tx + y\|^2 \leq 2 \left(\|x\|^2 + \|Ty\|^2 \right)$$

holds for all $x, y \in \mathcal{H}$. Then T possesses the wandering subspace property.

Proof. By Proposition 2.1, the Cauchy dual T' of T is a well-defined bounded linear operator on $\mathcal{H}$ and by Proposition 2.7 it suffices to show that

$$\hat{H} = H_\infty(T') = \{0\}.$$

According to Lemma 2.3, T' is concave and thus, by Lemma 2.6,

$$T'\hat{H} = T'^*\hat{H} = \hat{H}$$

holds and $T'|_{\hat{H}} : \hat{H} \to \hat{H}$ is unitary. Hence we have

$$T\hat{H} = T'(T'^*T')^{-1}\hat{H} = T'\hat{H} = \hat{H}.$$

Consequently, for $k \in \mathbb{N}$,

$$\hat{H} = T^k\hat{H} = \bigcap_{n\in\mathbb{N}} T^n\hat{H} = \{0\}.$$

$\square$

3 Wold-type decompositions for operators close to isometries

The goal of this chapter is to prove a result of Shimorin from [8], which states that concave operators and operators fulfilling (2.1) admit Wold-type decompositions. For an isometry $T \in \mathcal{B}(\mathcal{H})$, we have

$$\left\|T^2 x\right\|^2 + \|x\|^2 = \|Tx\|^2 + \|Tx\|^2 \leq 2\|Tx\|^2$$

and

$$\|Tx + y\|^2 = \|Tx\|^2 + \|y\|^2 + 2\operatorname{Re}\langle Tx, y\rangle \leq 2\left(\|Tx\|^2 + \|y\|^2\right) = 2\left(\|x\|^2 + \|Ty\|^2\right)$$

for all $x, y \in \mathcal{H}$. Thus isometries are concave and fulfil (2.1).

3.1 Properties of concave operators

In this section, we will prove two results concerning concave operators that will be needed to show the main theorem. Note that by Proposition 2.5 concave operators are injective with closed range and hence left invertible by Proposition 1.11.

Lemma 3.1. *A left invertible operator $T \in \mathcal{B}(\mathcal{H})$ admits a Wold-type decomposition if and only if its Cauchy dual T' admits a Wold-type decomposition.*
In this case, $H_\infty(T) = H_\infty(T')$ and $[E]_T = [E]_{T'}$ hold, where $E = H \ominus T\mathcal{H}$.

Proof. Suppose that T admits a Wold-type decomposition. Then

$$T' H_\infty(T') = T' \bigcap_{n\in\mathbb{N}} T'^n \mathcal{H} = \bigcap_{n\in\mathbb{N}} T'^{n+1}\mathcal{H} = \bigcap_{n\in\mathbb{N}} T'^n \mathcal{H} = H_\infty(T')$$

and, by Proposition 2.7, we have the orthogonal decomposition

$$\mathcal{H} = H_\infty(T) \oplus [E]_T = H_\infty(T) \oplus H_\infty(T')^\perp.$$

Thus $H_\infty(T)^\perp = H_\infty(T')^\perp$. We infer that

$$H_\infty(T')^\perp = H_\infty(T)^\perp = [E]_{T'}$$

is invariant under T'. Therefore, $H_\infty(T')$ is a reducing subspace for T'. By Proposition 1.11,

$$H_\infty(T') = \bigcap_{n\in\mathbb{N}} T'^n \mathcal{H}$$

is a closed subspace of $\mathcal{H}$ and we obtain

$$H_\infty(T') = H_\infty(T')^{\perp\perp} = H_\infty(T)^{\perp\perp} = H_\infty(T).$$

An application of Proposition 2.7 yields the orthogonal decompositions

$$\begin{aligned}\mathcal{H} &= H_\infty(T) \oplus [E]_T \\ &= H_\infty(T) \oplus [E]_{T'} \\ &= H_\infty(T') \oplus [E]_{T'}.\end{aligned}$$

Since by hypothesis $H_\infty(T)$ is reducing for T and $T|_{H_\infty(T)}$ is unitary, it follows that

$$I_{H_\infty(T)} = \left(T|_{H_\infty(T)}\right)^* \left(T|_{H_\infty(T)}\right) = (TT^*)|_{H_\infty(T)}$$

and thus

$$(TT^*)^{-1}|_{H_\infty(T)} = I_{H_\infty(T)}.$$

Therefore

$$T'|_{H_\infty(T)} = \left(T(T^*T)^{-1}\right)\Big|_{H_\infty(T)} = T|_{H_\infty(T)}$$

is unitary. Hence, T' admits a Wold-type decomposition. The reverse implication follows since $T'' = T$. □

Crucial to the proof of Shimorin's theorem is the following result due to Richter, namely Theorem 1 in [5].

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an analytic, concave operator. Then T possesses the wandering subspace property.*

Proof. By part (i) of Proposition 2.5 the operator $T^*T - I$ is positive, so we may consider its positive square root $D = (T^*T - I)^{\frac{1}{2}}$. For $x \in \mathcal{H}$, we find

$$\|Dx\|^2 = \|Tx\|^2 - \|x\|^2 .$$

Hence part (ii) of Proposition 2.5 can be restated as

$$\|T^n x\|^2 - \|x\|^2 \leq n \|Dx\|^2 \tag{3.1}$$

for all $x \in \mathcal{H}$ and $n \in \mathbb{N}$.

Let $L = T'^*$. In the proof of Proposition 2.7 we have shown that

$$P = I - TL$$

is the orthogonal projection onto $E = \mathcal{H} \ominus T\mathcal{H}$ and that

$$(I - T^n L^n)x \in [E]_T$$

for all $x \in \mathcal{H}$. As a closed, convex set $[E]_T$ is weakly closed. Thus to complete the proof it suffices to show that for each $x \in \mathcal{H}$ a subsequence of $(I - T^n L^n)x$ converges weakly to x.

We will first show that

$$\|x\|^2 = \sum_{k=0}^{n-1} \left\|PL^k x\right\|^2 + \|L^n x\|^2 + \sum_{k=1}^{n} \left\|DL^k x\right\|^2 \tag{3.2}$$

holds for all $x \in \mathcal{H}$ and $n \in \mathbb{N}^*$. Let $x \in \mathcal{H}$. For $n = 1$ we find

$$\begin{aligned} \|Px\|^2 + \|Lx\|^2 + \|DLx\|^2 &= \|Px\|^2 + \|Lx\|^2 + \|TLx\|^2 - \|Lx\|^2 \\ &= \|Px\|^2 + \|(I-P)x\|^2 = \|x\|^2 , \end{aligned}$$

thus establishing the claim for $n = 1$. Substituting $x = L^n y$ in the above equation, we obtain

$$\|L^n y\|^2 = \|PL^n y\|^2 + \left\|L^{n+1} y\right\|^2 + \left\|DL^{n+1} y\right\|^2$$

for all $y \in \mathcal{H}$ and $n \in \mathbb{N}$. Using this in conjunction with the induction hypothesis for $n \in \mathbb{N}$, we find

$$\begin{aligned} \|x\|^2 &= \sum_{k=0}^{n-1} \left\|PL^k x\right\|^2 + \|L^n x\|^2 + \sum_{k=1}^{n} \left\|DL^k x\right\|^2 \\ &= \sum_{k=0}^{n} \left\|PL^k x\right\|^2 + \left\|L^{n+1} x\right\|^2 + \sum_{k=1}^{n+1} \left\|DL^k x\right\|^2 \end{aligned}$$

for all $x \in \mathcal{H}$, hence proving the claim.

Let $x \in \mathcal{H}$. We may rearrange condition (3.2) to see that

$$\begin{aligned} 0 \leq \left\| L^{n+1}x \right\|^2 &= \|x\|^2 - \sum_{k=0}^{n} \left\| PL^k x \right\|^2 - \sum_{k=1}^{n+1} \left\| DL^k x \right\|^2 \\ &\leq \|x\|^2 - \sum_{k=0}^{n-1} \left\| PL^k x \right\|^2 - \sum_{k=1}^{n} \left\| DL^k x \right\|^2 = \|L^n x\|^2 \end{aligned}$$

holds for all $n \in \mathbb{N}^*$. Thus, $\left(\|L^n x\|^2\right)_{n\geq 1}$ is a bounded, decreasing sequence in $\mathbb{R}$, hence convergent. Denote by c the limit of this sequence and, using Proposition 2.5, define

$$c_k = \inf_{n \geq k} \left(\|T^n L^n x\|^2 - \|L^n x\|^2 \right) \geq 0$$

for $k \in \mathbb{N}^*$. Conditions (3.1) and (3.2) then imply

$$\sum_{k=1}^{n} \frac{c_k}{k} \leq \sum_{k=1}^{n} \frac{1}{k} \left(\left\| T^k L^k x \right\|^2 - \left\| L^k x \right\|^2 \right) \leq \sum_{k=1}^{n} \left\| DL^k x \right\|^2 \leq \|x\|^2$$

for all $n \in \mathbb{N}^*$. Since the sequence $(c_k)_{k\geq 1}$ is increasing, it follows that $c_k = 0$ for all $k \geq 1$ or, equivalently, that there is a strictly increasing sequence $(n_k)_{k\geq 1}$ in $\mathbb{N}$ such that

$$\|T^{n_k} L^{n_k} x\|^2 - \|L^{n_k} x\|^2 \xrightarrow{(k\to\infty)} 0.$$

But then

$$\|T^{n_k} L^{n_k} x\|^2 \xrightarrow{(k\to\infty)} c.$$

In particular, $(T^n L^n x)_{n\in\mathbb{N}}$ contains a bounded subsequence and therefore a weakly convergent subsequence $(T^{n_k} L^{n_k} x)_{k\in\mathbb{N}}$ with limit $y \in \mathcal{H}$. For $N \in \mathbb{N}$, we have

$$T^{n_k} L^{n_k} x \in T^N \mathcal{H}$$

for all $k \geq N$. Since $T^N \mathcal{H}$ is weakly closed, $y \in T^N \mathcal{H}$ follows. Hence,

$$y \in \bigcap_{N\in\mathbb{N}} T^N \mathcal{H} = H_\infty(T) = \{0\}$$

and the claim follows by noting that $(I - T^{n_k} L^{n_k})x$ converges weakly to x. $\square$

3.2 Shimorin's theorem

Bringing together the previous results, it is now quite straightforward to prove the following theorem originally found by Shimorin.

Theorem 3.3 (Wold-type decomposition for operators close to isometries)**.**
Let $T \in \mathcal{B}(\mathcal{H})$ be either concave or such that the inequality

$$\|Tx + y\|^2 \le 2\left(\|x\|^2 + \|Ty\|^2\right)$$

holds for all $x, y \in \mathcal{H}$. Then T admits a Wold-type decomposition.

Proof. First consider a concave operator T. According to Lemma 2.6, $H_\infty(T)$ is reducing for T and

$$T|_{H_\infty(T)} \colon H_\infty(T) \to H_\infty(T)$$

is unitary. Hence, it remains to show that $H_\infty(T)^\perp = [\mathcal{H} \ominus T\mathcal{H}]_T$.

Let $U = H_\infty(T)^\perp$. Since $H_\infty(T)$ is reducing for T so is U. Therefore

$$S = T|_U \colon U \to U$$

is well-defined and obviously concave. Additionally, we have

$$H_\infty(S) \subset H_\infty(T) \cap H_\infty(T)^\perp = \{0\}.$$

Consequently, S is analytic. By Lemma 3.2 applied to S and by Proposition 1.13, it follows that

$$H_\infty(T)^\perp = U = [U \ominus SU]_S = \left[U \cap (TU)^{\perp}\right]_T = [\mathcal{H} \ominus T\mathcal{H}]_T.$$

Next, assume T fulfils the inequality. By Lemma 2.3, the Cauchy dual T' is concave, so the already proven part states that T' admits a Wold-type decomposition. The claim follows from Lemma 3.1. $\square$

As an immediate consequence, we find the classical Wold-type decomposition theorem for isometries.

Corollary 3.4. *Isometries admit a Wold-type decomposition.*

4 Application to shift operators on spaces of analytic functions

In this final chapter we look at concrete examples. In order to do this, we first introduce analytic functional Hilbert spaces and then work with shift operators on those. For a special class of these spaces we will show that the shift operator satisfies a Beurling-type theorem, but we will also see that this is not the case in general.

4.1 Functional Hilbert spaces

We want to give a very brief introduction to functional Hilbert spaces and kernel functions. A more general and more exhaustive introduction to this topic can be found in Chapter 1 of [1]. We also refer to [1] for proofs of the given results.

In the following let X be an arbitrary set. We denote by $\mathbb{C}^X$ the set of all mappings from X to $\mathbb{C}$.

Definition 4.1. A Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ is called a *functional Hilbert space* if the point-evaluation functional

$$\delta_x \colon \mathcal{H} \to \mathbb{C},\ f \mapsto f(x)$$

is continuous for every $x \in X$.

Definition 4.2. Let $\mathcal{H} \subset \mathbb{C}^X$ be a Hilbert space. A map $K \colon X \times X \to \mathbb{C}$ such that

(i) $K(\cdot, x) \in \mathcal{H}$ for all $x \in X$,

(ii) $\langle f, K(\cdot, x)\rangle = f(x)$ for all $x \in X$ and $f \in \mathcal{H}$,

is called a *reproducing kernel* for $\mathcal{H}$.

Proposition 4.3. *Let $\mathcal{H} \subset \mathbb{C}^X$ be a Hilbert space. Then $\mathcal{H}$ is a functional Hilbert space if and only if there is a reproducing kernel K for $\mathcal{H}$. In this case, K is uniquely determined by $\mathcal{H}$.*

Definition 4.4. A map $K\colon X \times X \to \mathbb{C}$ is called *positive definite* if

$$\sum_{i,j=1}^{n} K(x_i, x_j) z_i \overline{z_j} \geq 0$$

holds for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $z_1, \ldots, z_n \in \mathbb{C}$.

Proposition 4.5. *Let $K\colon X \times X \to \mathbb{C}$ be positive definite. Then there is a unique functional Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ with reproducing kernel K.*

In Example 2.52 of [9], Wernet shows that for $\alpha > -2$ the map

$$K_\alpha\colon \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K_\alpha(z, w) = \left(\frac{1}{1 - z\overline{w}}\right)^{\alpha+2} = \sum_{k=0}^{\infty} a_k (z\overline{w})^k,$$

where

$$a_k = \frac{1}{k!} \frac{\Gamma(\alpha + k + 2)}{\Gamma(\alpha + 2)} > 0 \quad (k \in \mathbb{N}),$$

is positive definite. The corresponding functional Hilbert space $\mathcal{H}(K_\alpha) \subset \mathcal{O}(\mathbb{D})$ has orthonormal basis

$$b_k = \sqrt{a_k} z^k \quad (k \in \mathbb{N}).$$

This was shown for instance in Theorem 1.15 of [10].

In particular, for $\alpha = -1$ this yields the *Hardy space*

$$\mathcal{H}_{-1} = \mathcal{H}(K_{-1}) = \left\{ f \in \mathcal{O}(\mathbb{D});\ \sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^2 \,\mathrm{d}\xi < \infty \right\}$$

with $a_k = 1$ for $k \in \mathbb{N}$ and orthonormal basis

$$b_k = z^k \quad (k \in \mathbb{N}).$$

Furthermore, for $\alpha = 0$ we obtain the *Bergman space*

$$\mathcal{H}_0 = \mathcal{H}(K_0) = \left\{ f \in \mathcal{O}(\mathbb{D});\ \int_{\mathbb{D}} |f|^2 \, \mathrm{d}\lambda < \infty \right\},$$

with $a_k = k+1$ for $k \in \mathbb{N}$ and orthonormal basis

$$b_k = \sqrt{k+1} z^k \quad (k \in \mathbb{N}).$$

4.2 Weighted shifts on $\mathcal{H}(K_\alpha)$

In the following let $\alpha > -2$ be a real number and let $\mathcal{H} = \mathcal{H}_\alpha = \mathcal{H}(K_\alpha)$ be the unique functional Hilbert space with reproducing kernel K_α. A function $f \in \mathcal{H}$ is given by its Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

for all $z \in \mathbb{D}$ as well as by its representation in the orthonormal basis

$$f = \sum_{k=0}^{\infty} \tilde{f}_k b_k = \sum_{k=0}^{\infty} \left(\tilde{f}_k \sqrt{a_k} \right) z^k$$

in the Hilbert space $\mathcal{H}$. Since point-evaluation on $\mathcal{H}$ is continuous, we also have

$$f(z) = \sum_{k=0}^{\infty} \left(\tilde{f}_k \sqrt{a_k} \right) z^k$$

for all $z \in \mathbb{D}$. Hence $f_k = \tilde{f}_k \sqrt{a_k}$ holds for all $k \in \mathbb{N}$ and the Taylor expansion of f on $\mathbb{D}$ also converges to f in the Hilbert space $\mathcal{H}$.

Let

$$c_k = \sqrt{\frac{a_k}{a_{k+1}}}$$

for $k \in \mathbb{N}$ and $c_{-1} = 0$. Since

$$\frac{a_k}{a_{k+1}} = (k+1) \frac{\Gamma(\alpha + k + 2)}{\Gamma(\alpha + k + 3)} = \frac{k+1}{\alpha + k + 2} \xrightarrow{(k \to \infty)} 1$$

holds, the *weighted shift* S defined by

$$S b_k = c_k b_{k+1} \quad (k \in \mathbb{N})$$

is a well-defined bounded linear operator on $\mathcal{H}$. From

$$Sz^k = \frac{1}{\sqrt{a_k}} Sb_k = \frac{1}{\sqrt{a_{k+1}}} b_{k+1} = z^{k+1} \quad (k \in \mathbb{N}),$$

it follows that

$$S = M_z \colon \mathcal{H} \to \mathcal{H},\ f \mapsto zf$$

is the multiplication operator with symbol z. The calculation

$$\langle b_n, S^* b_k \rangle = \langle S b_n, b_k \rangle = \langle c_n b_{n+1}, b_k \rangle = \langle b_n, c_{k-1} b_k \rangle \quad (n, k \in \mathbb{N})$$

shows that

$$S^* b_k = \begin{cases} c_{k-1} b_{k-1}, & k \neq 0, \\ 0, & k = 0. \end{cases}$$

Hence SS^* and S^*S are diagonal operators with

$$SS^* b_k = c_{k-1}^2 b_k, \qquad S^* S b_k = c_k^2 b_k$$

for all $k \in \mathbb{N}$. From $c_k^2 > 0$ for $k \in \mathbb{N}$ and

$$\sup_{k \in \mathbb{N}} \frac{1}{c_k^2} < \infty,$$

we infer that S^*S is invertible and its inverse is the diagonal operator given by the weight sequence $\left(c_k^{-2}\right)_{k \in \mathbb{N}}$. Since a diagonal operator is positive if and only if its defining weights are all non-negative,

$$2I - SS^* - (S^*S)^{-1} \geq 0$$

holds if and only if

$$2 - c_{k-1}^2 - \frac{1}{c_k^2} \geq 0$$

holds for all $k \in \mathbb{N}$.

Theorem 4.6. *The multiplication operator*

$$M_z \colon \mathcal{H}_\alpha \to \mathcal{H}_\alpha,\ f \mapsto zf$$

fulfils condition (2.1) *if and only if* $-1 \leq \alpha \leq 0$. *In this case,* M_z *satisfies a Beurling-type theorem.*

Proof. By Remark 2.4, $S = M_z$ fulfils condition (2.1) if and only

$$2I - SS^* - (S^*S)^{-1} \geq 0$$

holds. The remarks preceding this theorem show that this is equivalent to the condition

$$2 - c_{k-1}^2 - \frac{1}{c_k^2} \geq 0 \quad (k \in \mathbb{N}).$$

For $k = 0$ we find

$$2 - c_{-1}^2 - \frac{1}{c_0^2} = 2 - \frac{a_1}{a_0} = 2 - \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+2)} = 2 - (\alpha+2) = -\alpha \geq 0$$

if and only if $\alpha \leq 0$. Additionally,

$$\begin{aligned} 2 - c_{k-1}^2 - \frac{1}{c_k^2} &= 2 - k\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+2)} - \frac{1}{k+1}\frac{\Gamma(\alpha+k+3)}{\Gamma(\alpha+k+2)} \\ &= 2 - \frac{k}{\alpha+k+1} - \frac{\alpha+k+2}{k+1} \geq 0 \end{aligned}$$

holds for all $k \in \mathbb{N}^*$ if and only if $-1 \leq \alpha \leq 0$, as an easy computation shows.

In this case, let $M \subset \mathcal{H}$ be a closed invariant subspace for S. Since S satisfies condition (2.1) the same is true for

$$T = S|_M \colon M \to M.$$

Obviously, $H_\infty(T) \subset H_\infty(S)$. If $f \in H_\infty(S)$, then because of

$$f \in \bigcap_{k \in \mathbb{N}} z^k \mathcal{O}(\mathbb{D})$$

the function f has a zero of infinite order at $z = 0$. Thus, $H_\infty(S) = \{0\}$. By Theorem 2.9, T possesses the wandering subspace property and hence S satisfies a Beurling-type theorem. $\square$

For $\alpha > -1$ let λ_α be the weighted normalised Lebesgue measure on $\mathbb{D}$ defined by

$$\lambda_\alpha = \frac{\pi\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\left(1 - |z|^2\right)^\alpha \lambda,$$

where λ denotes the Lebesgue measure on $\mathbb{D}$. If we define the associated *weighted Bergman space* by

$$L_a^2(\mathbb{D}, \lambda_\alpha) = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D}, \lambda_\alpha),$$

then Corollary 1.19 in [6] shows that

$$\mathcal{H}(K_\alpha) = L_a^2(\mathbb{D}, \lambda_\alpha).$$

Hence the preceding considerations show that $M_z \in \mathcal{B}\left(L_a^2(\mathbb{D}, \lambda_\alpha)\right)$ satisfies a Beurling-type theorem for $-1 < \alpha \leq 0$. An application of Theorem 1.4 in [7] shows that this actually holds for $-1 < \alpha \leq 1$. Shimorin conjectured that it is false for $\alpha > 1$. At least for $\alpha > 4$ we will follow an idea from [3] to show that $M_z \in \mathcal{B}\left(L_a^2(\mathbb{D}, \lambda_\alpha)\right)$ does not satisfy a Beurling-type theorem. We start with some preparations.

Definition 4.7. Let $f \in \mathcal{H}\backslash\{0\}$. We define $\mathrm{ord}_0(0) = \infty$ and

$$\mathrm{ord}_0(f) = \inf\left\{k \in \mathbb{N};\ f^{(k)}(0) \neq 0\right\}.$$

For a closed subspace $M \subset \mathcal{H}$, let

$$\mathrm{ord}_0(M) = \inf\left\{\mathrm{ord}_0(f);\ f \in M\right\}.$$

Remark 4.8. Let $M \subset \mathcal{H}$ be a closed subspace such that $M \neq \{0\}$. Then

$$\mathrm{ord}_0(zM) = \mathrm{ord}_0(M) + 1$$

holds. Hence, if M is invariant for $S = M_z \colon \mathcal{H} \to \mathcal{H}$, it follows that

$$M \ominus zM \neq \{0\}.$$

Note that $zM \subset \mathcal{H}$ is a closed subspace again, since S is left invertible.

Proof. Obviously,

$$(zf)'(0) = f(0)$$

holds for all $f \in \mathcal{H}$. Thus,

$$\mathrm{ord}(M) = \inf\left\{\mathrm{ord}_0(f);\ f \in M\right\} = \inf\left\{\mathrm{ord}_0(zf) - 1;\ f \in M\right\} = \mathrm{ord}_0(zM) - 1.$$

$\square$

Lemma 4.9. *Let $A \subset \mathbb{D}$ be a subset with $0 \notin A$. Then*

$$M_A = \{f \in \mathcal{H};\ f|_A = 0\} \subset \mathcal{H}$$

is a closed invariant subspace for S with

$$\dim (M_A \ominus zM_A) = 1.$$

Proof. Obviously, $M_A \subset \mathcal{H}$ is a closed invariant subspace for S. This implies

$$\dim (M_A \ominus zM_A) \geq 1,$$

since $M_A \ominus zM_A \neq \{0\}$ by Remark 4.8. Choose a function $f \in (M_A \ominus zM_A) \setminus \{0\}$. Then $f(0) \neq 0$, since otherwise there would be a function $F \in \mathcal{H}$ with $f = zF$ by Theorem 2.5 in [10] and because of $a \notin A$ we would obtain the contradiction

$$f = zF \in zM_A.$$

Let $g \in M_A \ominus zM_A$ be any other function. Then

$$h = g - \frac{g(0)}{f(0)} f$$

is a function in $M_A \ominus zM_A$ with $h(0) = 0$. Hence $h = 0$ and $g \in \mathbb{C}f$. □

Proposition 4.10. *The identity*

$$\bigcup_{a \in \mathbb{D}} \left\{ \frac{1 - z\overline{a}}{1 - |a|^2};\ z \in \mathbb{D} \right\} = \left\{ w \in \mathbb{C};\ \operatorname{Re} w > \frac{1}{2} \right\}$$

holds.

Proof. For $a \in \mathbb{D} \setminus \{0\}$, let

$$\Psi_a \colon \mathbb{D} \to \mathbb{C},\ z \mapsto \frac{1 - \overline{a}z}{1 - |a|^2}.$$

Obviously,

$$\Psi_a(\mathbb{D}) = D_{\frac{|a|}{1-|a|^2}} \left(\frac{1}{1 - |a|^2} \right),$$

where $D_r(w)$ denotes the open disc in $\mathbb{C}$ with centre w and radius r. Hence we find

$$\bigcup_{a\in\mathbb{D}} \left\{ \frac{1-z\overline{a}}{1-|a|^2};\ z\in\mathbb{D} \right\} = \bigcup_{a\in\mathbb{D}\setminus\{0\}} \Psi_a(\mathbb{D}) = \bigcup_{0<r<1} \Psi_r(\mathbb{D}).$$

Let $0<r<1$ and $w\in\Psi_r(\mathbb{D})$. Then

$$\operatorname{Re} w > \frac{1}{1-r^2} - \frac{r}{1-r^2} = \frac{1}{1+r} > \frac{1}{2}$$

holds and we conclude

$$\bigcup_{a\in\mathbb{D}} \left\{ \frac{1-z\overline{a}}{1-|a|^2};\ z\in\mathbb{D} \right\} \subset \left\{ w\in\mathbb{C};\ \operatorname{Re} w > \frac{1}{2} \right\}.$$

Since

$$\frac{1}{1+r} \xrightarrow{(r\uparrow 1)} \frac{1}{2},$$

a simple calculation shows that the reverse inclusion holds, too. $\square$

With this, we now are able to prove that $M_z \in \mathcal{B}\left(L_a^2(\mathbb{D},\lambda_\alpha)\right)$ does not satisfy a Beurling-type theorem for $\alpha > 4$.

Theorem 4.11. *Let $\alpha > 4$. Then the multiplication operator*

$$S = M_z \colon \mathcal{H}_\alpha \to \mathcal{H}_\alpha,\ f \mapsto zf$$

does not satisfy a Beurling-type theorem.

Proof. For $a\in\mathbb{D}\setminus\{0\}$, define

$$M_a = M_{\{a\}} = \{f\in\mathcal{H};\ f(a)=0\} \subset \mathcal{H}.$$

Then the function

$$\varphi_a = 1 - \frac{K_\alpha(\cdot,a)}{K_\alpha(a,a)}$$

belongs to M_a and satisfies

$$\langle zf, \varphi_a\rangle = \langle zf, 1\rangle - \frac{af(a)}{K_\alpha(a,a)} = 0$$

for all $f \in M_a$. Thus, by Lemma 4.9,

$$M_a \ominus zM_a = \mathbb{C}\varphi_a.$$

If $S|_{M_a} \colon M_a \to M_a$ possesses the wandering subspace property, then

$$M_a = \bigvee_{k\in\mathbb{N}} z^k \left(M_a \ominus zM_a\right) = \bigvee_{k\in\mathbb{N}} z^k \varphi_a.$$

holds by Remark 1.9. Therefore we obtain the identity

$$\{a\} = \{z \in \mathbb{D};\ f(z) = 0 \text{ for all } f \in M_a\} = \{z \in \mathbb{D};\ \varphi_a(z) = 0\} = Z(\varphi_a).$$

On the other hand, for $z \in \mathbb{D}$ an elementary computation with complex powers shows that $\varphi_a(z) = 0$ if and only if

$$\left(\frac{1 - z\overline{a}}{1 - |a|^2}\right)^{\alpha+2} = 1.$$

For $z \in \mathbb{C}$, denote by $\arg_{-\pi}(z)$ the principal value of the argument of z. Then, for $\operatorname{Re} w > \frac{1}{2}$, the identity

$$w^{\alpha+2} = e^{(\alpha+2)\log|w|} e^{i(\alpha+2)\arg_{-\pi}(w)} = 1$$

holds if and only if $|w| = 1$ and

$$(\alpha + 2)\arg_{-\pi}(w) \in 2\pi\mathbb{Z}.$$

In this case,

$$\left|\arg_{-\pi}(w)\right| < \arg_{-\pi}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3},$$

hence it follows by Proposition 4.10 that there is a point $a \in \mathbb{D}$ with $Z(\varphi_a) \neq \{a\}$ if and only if

$$(\alpha + 2)\frac{\pi}{3} > 2\pi.$$

This happens precisely if $\alpha + 2 > 6$, or equivalently, if $\alpha > 4$.

Consequently, for every $\alpha > 4$, there is a number $a \in \mathbb{D}$ such that the restriction of S to the closed invariant subspace M_a does not possess the wandering subspace property. □

Lastly, we will look to the Dirichlet shift as an example of a concave operator.

4.3 The Dirichlet shift

The functional Hilbert space $\mathcal{D} \subset \mathcal{O}(\mathbb{D})$ determined by the reproducing kernel

$$K_{\mathcal{D}} \colon \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K_{\mathcal{D}}(z,w) = \frac{1}{z\overline{w}} \log \frac{1}{1 - z\overline{w}} = \sum_{k=0}^{\infty} a_k (z\overline{w})^k,$$

where

$$a_k = \frac{1}{k+1}$$

for $k \in \mathbb{N}$, is called the *Dirichlet space* and possesses the orthonormal basis

$$b_k = \sqrt{a_k} z^k \quad (k \in \mathbb{N}).$$

If, for $k \in \mathbb{N}$, we define

$$c_k = \sqrt{\frac{a_k}{a_{k+1}}},$$

it follows that

$$0 < c_k = \sqrt{\frac{k+2}{k+1}} \xrightarrow{(k \to \infty)} 1.$$

Hence the *Dirichlet shift* $S = M_z \colon \mathcal{D} \to \mathcal{D}$ is a well-defined, left invertible operator. Since

$$S^* S b_k = c_k^2 b_k, \qquad S^{*2} S^2 b_k = c_k^2 c_{k+1}^2 b_k$$

for all $k \in \mathbb{N}$, we find

$$\begin{aligned} \left(S^{*2} S^2 - 2S^* S + I\right) b_k &= \left(c_k^2 c_{k+1}^2 - 2c_k^2 + 1\right) b_k \\ &= \left(\frac{k+2}{k+1} \frac{k+3}{k+2} - 2\frac{k+2}{k+1} + 1\right) b_k \\ &= 0. \end{aligned}$$

Thus, the Dirichlet shift $S = M_z \in \mathcal{B}(\mathcal{D})$ is a concave operator. Let M be a closed invariant subspace for S. Then the restriction

$$T = S|_M \colon M \to M$$

is a concave operator as well. Since

$$H_\infty(T) \subset H_\infty(S) \subset \bigcap_{k \in \mathbb{N}} z^k \mathcal{O}(\mathbb{D}) = \{0\},$$

it also is analytic. Applying Lemma 3.2 shows that T has the wandering subspace property. This proves the following result.

Theorem 4.12. *The Dirichlet shift*

$$M_z\colon \mathcal{D} \to \mathcal{D},\ f \mapsto zf$$

satisfies a Beurling-type theorem.

Bibliography

[1] C. Barbian. *Positivitätsbedingungen funktionaler Hilberträume und Anwendungen in der mehrdimensionalen Operatorentheorie*. Diplomarbeit. Universität des Saarlandes, 2001.

[2] P.L. Duren and A. Schuster. *Bergman Spaces*. Mathematical surveys and monographs. Amer. Math. Soc., Providence, Rhode Island, 2004.

[3] H. Hedenmalm and K. Zhu. *On the failure of optimal factorization for certain weighted bergman spaces*. Complex Variables 19 (1992), 165-176.

[4] B.S. Nagy, C. Foias, H. Bercovici, and L. Kérchy. *Harmonic Analysis of Operators on Hilbert Space*. Universitext. Springer, New York, 2010.

[5] S. Richter. *Invariant subspaces of the Dirichlet shift*. J. Reine Angew. Math. 386 (1988), 205-220.

[6] D. Schillo. *Toeplitz and Hankel operators on weighted Bergman spaces and the Fock space*. Master's thesis. Universität des Saarlandes, 2014.

[7] S. Shimorin. *On Beurling-type theorems in weighted* ℓ^2 *and Bergman spaces*. Proc. Amer. Math. Soc. 131 (2002), 1777-1787.

[8] S. Shimorin. *Wold-type decompositions and wandering subspaces for operators close to isometries*. J. Reine Angew. Math. 531 (2001), 147-189.

[9] M. Wernet. *On semi-Fredholm theory and essential normality*. Dissertation. Universität des Saarlandes, 2014.

[10] M. Wernet. *Wesentlich normale Hilbermoduln über Gebieten in* $\mathbb{C}^d$. Diplomarbeit. Universität des Saarlandes, 2008.