Master's thesis

Factorizations Induced By Complete Nevanlinna-Pick Factors

Sebastian Toth

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advised by: Prof. Dr. Jörg Eschmeier

Fakultät 6 - Mathematik und Informatik

Universität des Saarlandes

Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

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Sebastian Toth

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Contents

Introduction

For a complex Hilbert space $\mathscr{E}$ and a set Ω, a Hilbert space $\mathscr{H} \subset \mathscr{E}^{\Omega}$ is called a functional Hilbert space if all point evaluations

$$\delta_z \colon \mathscr{H} \to \mathscr{E},\ \delta_z(f) = f(z) \quad (z \in \Omega)$$

are continuous. The map

$$k \colon \Omega \times \Omega \to L(\mathscr{E}), k(z,w) = \delta_z \delta_w^*$$

is called the reproducing kernel of $\mathscr{H}$, since it has the property that

$$\langle f(w), v\rangle_{\mathscr{E}} = \langle f, k(\cdot, w) v\rangle_{\mathscr{H}}$$

holds for all $w \in \Omega$ and $v \in \mathscr{E}$. Such a reproducing kernel is positive definite, that is, we have

$$\sum_{i,j=0}^{n} \left\langle k\left(z_j, z_i\right) v_i, v_j \right\rangle_{\mathscr{E}} \geq 0$$

for all finite sequences $(z_i)_{i=0}^n$ in Ω and $(v_i)_{i=0}^n$ in $\mathscr{E}$. Conversely one can show that, for every positive definite map $k \colon \Omega \times \Omega \to L(\mathscr{E})$, there exists a unique functional Hilbert space $\mathscr{H} \subset \mathscr{E}^{\Omega}$ with reproducing kernel k. Identifying $\mathbb{C} \cong L(\mathbb{C})$, one also calls the function $k \colon \Omega \times \Omega \to \mathbb{C}$ induced by a positive definite map $k \colon \Omega \times \Omega \to L(\mathbb{C})$ positive definite. Whenever $k \colon \Omega \times \Omega \to \mathbb{C}$ is positive definite, we denote by $\mathscr{H}_k(\mathscr{E})$ the functional Hilbert space with reproducing kernel $k \mathrm{Id}_{\mathscr{E}}$. If $\mathscr{E} = \mathbb{C}$ we write $\mathscr{H}_k = \mathscr{H}_k(\mathbb{C})$. In the following we consider kernels of the form

$$\mathfrak{s} \colon \Omega \times \Omega \to \mathbb{C},\ \mathfrak{s}(z,w) = \frac{1}{1 - \sum_{n=0}^{\infty} u_n(z) \overline{u_n(w)}},$$

where $u_n \colon \Omega \to \mathbb{C}$ $(n \in \mathbb{N})$ are functions such that there exists a $z_0 \in \Omega$ with $u_n(z_0) = 0$ for all $n \in \mathbb{N}$. Kernels of this type are called normalized complete Nevanlinna-Pick (CNP) kernels, and solve the Nevanlinna-Pick Problem (see [1]). A central role in the present thesis is taken by functional Hilbert spaces $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, whose kernels $\mathfrak{k}$ have a complete Nevanlinna-Pick factor, i.e., we have $\mathfrak{k} = \mathfrak{s}\mathfrak{g}$, where $\mathfrak{s}$ is a normalized CNP kernel and $\mathfrak{g} \colon \Omega \times \Omega \to \mathbb{C}$ is positive definite. Now let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two complex Hilbert spaces and $\mathscr{H}_i \subset \mathscr{E}_i^{\Omega}$ $(i = 1, 2)$ functional Hilbert spaces with reproducing kernels $k_i \colon \Omega \times \Omega \to L(\mathscr{E}_i)$. The elements of

$$\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2) = \{\varphi \colon \Omega \to L(\mathscr{E}_1, \mathscr{E}_2);\ \varphi \mathscr{H}_1 \subset \mathscr{H}_2\}$$

are called multipliers from $\mathscr{H}_1$ to $\mathscr{H}_2$. Here, if $f\colon \Omega \to \mathscr{E}_1$ is a function, the map $\varphi f\colon \Omega \to \mathscr{E}_2$ is defined by

$$(\varphi f)(z) = \varphi(z) f(z) \quad (z \in \Omega).$$

For $\varphi \in \operatorname{Mult}(\mathscr{H}_1, \mathscr{H}_2)$, we denote by

$$M_\varphi\colon \mathscr{H}_1 \to \mathscr{H}_2,\ M_\varphi f = \varphi f$$

the multiplication operator with symbol φ. Let $\mathscr{H} \subset \mathbb{C}^\Omega$ be a scalar-valued functional Hilbert space and $\mathscr{E}$ an arbitrary Hilbert space. We call a function $\varphi\colon \Omega \to \mathscr{E}$ a multiplier from $\mathscr{H}$ to $\mathscr{H}(\mathscr{E})$ and simply write $\varphi \in \operatorname{Mult}(\mathscr{H}, \mathscr{H}(\mathscr{E}))$ if the function

$$\varphi^{L(\mathbb{C},\mathscr{E})}\colon \Omega \to L(\mathbb{C},\mathscr{E}),\ \varphi^{L(\mathbb{C},\mathscr{E})}(z)(w) = w\varphi(z)$$

belongs to $\operatorname{Mult}(\mathscr{H}, \mathscr{H}(\mathscr{E}))$. Similarly, we call a scalar-valued function $\varphi\colon \Omega \to \mathbb{C}$ a multiplier of $\mathscr{H}$ if φ regarded as a function $\varphi\colon \Omega \to L(\mathbb{C}) \cong \mathbb{C}$ is a multiplier of $\mathscr{H}$.

From the theory of Hardy spaces it is well known that $H^2(\mathbb{D})$ is a functional Hilbert space with reproducing kernel

$$\mathfrak{s}\colon \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ \mathfrak{s}(z,w) = \frac{1}{1 - z\overline{w}}$$

the so called Szegö kernel. This map can easily be seen to be a normalized CNP kernel with $z_0 = 0$. The multipliers $\operatorname{Mult}\left(H^2(\mathbb{D})\right)$ from $H^2(\mathbb{D})$ to $H^2(\mathbb{D})$ are exactly the bounded analytic functions $H^\infty(\mathbb{D})$ on $\mathbb{D}$. A theorem going back to Nevanlinna implies in particular that, for $f \in H^2(\mathbb{D})$, there exist functions $g, h \in H^\infty(\mathbb{D})$ with $0 \notin h(\mathbb{D})$ such that $f = \frac{g}{h}$. In the first part of this thesis we give the following generalization of this theorem for kernels with a normalized complete Nevanlinna-Pick factor $\mathfrak{k} = \mathfrak{s}\mathfrak{g}$. Here we proceed exactly as in the recent paper [2] by Aleman, Hartz, McCarthy and Richter. The proof is based on an idea of Sarason [16],[17].

Theorem 0.0.1. *(i) For $F \in \mathscr{E}^\Omega$, the following are equivalent:*

(a) $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ with $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \leq 1$.

(b) There exist multipliers $\Psi\colon \Omega \to \mathbb{C}$ of $\mathscr{H}_{\mathfrak{s}}$ with $\Psi(z_0) = 0$ and $\Phi\colon \Omega \to \mathscr{E}$ from $\mathscr{H}_{\mathfrak{s}}$ to $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ such that $F = \frac{\Phi}{1-\Psi}$ and

$$\|\Psi h\|^2_{\mathscr{H}_{\mathfrak{s}}} + \|\Phi h\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \leq \|h\|^2_{\mathscr{H}_{\mathfrak{s}}}$$

for all $h \in \mathscr{H}_{\mathfrak{s}}$.

(ii) If $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ with $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ then the functions Φ and Ψ are uniquely determined. In fact, $\mathfrak{s}_z \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$, and if

$$V_F(z) = 2\langle F, \mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - 1 \quad (z \in \Omega),$$

then $\operatorname{Re} V_F \geq 0$ in Ω and (b) holds with

$$\Psi = \frac{V_F - 1}{V_F + 1}, \quad \Phi = \frac{2}{V_F + 1} F.$$

It is well known that the multiplier space of $H^2(\mathbb{D})$ is $H^\infty(\mathbb{D})$. However, in general it is a difficult problem to characterize the multipliers of a given reproducing kernel Hilbert space. In a first step, we will give a class of examples of multipliers in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. When proving Theorem 0.0.1 we will show that $\frac{1}{V_F+x-iy}F$ belongs to $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ for all $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, $x > 0$ and $y \in \mathbb{R}$. Using this, we will further show that for every positive Borel measure with compact support on $\mathbb{R}$, the weak integral

$$\Phi = \int_{\operatorname{supp}(\mu)} \frac{F}{V_F + x - iy} d\mu(y)$$

defines a multiplier from $\mathscr{H}_{\mathfrak{s}}$ to $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$.

An elementary application of Theorem 0.0.1 shows that the elements of $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ have the same zero sets as the functions in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. We extend this idea and show that the multiplier invariant space generated by an arbitrary function in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is also generated by a multiplier function.

A function $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is called extremal if

$$\langle \varphi F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = \varphi(z_0)$$

holds for all $\varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\right)$. Extremal functions play an essential role in the theory of wandering subspaces for shift-invariant subspaces. If F is an extremal function, then $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ and it is easy to see that its Sarason function V_F is given by $V_F = 1$. Thus the factorization from Theorem 0.0.1 (ii) reduces to the identity $F = \Phi$. In particular, every extremal function $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is a multiplier from $\mathscr{H}_{\mathfrak{s}}$ to $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ with multiplier norm at most 1. In the case that

$$\mathfrak{s}\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, \mathfrak{s}(z,w) = \frac{1}{1 - \langle z, w \rangle}$$

is the Drury-Arveson kernel on the open unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ and $\mathscr{H}_{\mathfrak{k}} \subset \mathscr{O}(\mathbb{B}_d)$ we show that there is a close relation between extremal functions $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and $\mathfrak{k}$-inner functions $F\colon \mathbb{B}_d \to L(\mathbb{C}, \mathscr{E})$ recently studied in [7].

As a direct application of Theorem 0.0.1 we can conclude that $F \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ whenever $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ and $V_F = 1$. Indeed it is possible to show that for a large class of reproducing kernels the weaker assumption that the real part of the Sarason function of a function $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is bounded is sufficient to conclude that $F \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. This theorem covers a large family of reproducing kernel Hilbert spaces like the Drury Arveson space, weighted Dirichlet and Bergman spaces. However, we can still not fully characterize multipliers this way, since the converse direction of the theorem is false at least in this generality. We proceed as in [2] to construct multipliers of the standard weighted Dirichlet spaces on the disc, whose Sarason functions have unbounded real part.

1 Preliminaries

1.1 Nevanlinna-Pick kernels

A positive map $\boldsymbol{s}\colon \Omega\times\Omega\to\mathbb{C}^*$ is called Nevanlinna-Pick kernel if

$$\Omega\times\Omega\to\mathbb{C},\ (z,w)\mapsto 1-\frac{1}{\boldsymbol{s}(z,w)}$$

is positive definite. A functional Hilbert space $\mathscr{H}_{\boldsymbol{s}}$ with Nevanlinna-Pick kernel is called Nevanlinna-Pick space. In the following, let $H=\mathscr{H}_{\boldsymbol{s}}$ be a Nevanlinna-Pick space with kernel $\boldsymbol{s}\colon \Omega\times\Omega\to\mathbb{C}$ and $\operatorname{Mult}(H)$ the space of multipliers of H.

Proposition 1.1.1. *We have that* $1\colon \Omega\to\mathbb{C},\ z\mapsto 1_{\mathbb{C}}$ *is in* H *with* $\|1\|\leq 1$.

Proof. The function

$$\Omega\times\Omega\to\mathbb{C},\ (z,w)\mapsto \boldsymbol{s}(z,w)-1=\boldsymbol{s}(z,w)\left(1-\frac{1}{\boldsymbol{s}(z,w)}\right)$$

is positive definite by the Lemma of Schur. It follows by [5, Satz 1.9 (b)] that $1\in H$ with $\|1\|\leq 1$. □

By [5, Satz 1.13] there is a Hilbert space $\mathscr{F}$ and a map $d\colon \Omega\to\mathscr{F}$ with

$$1-\frac{1}{\boldsymbol{s}(z,w)}=\langle d(w),d(z)\rangle_{\mathscr{F}}$$

for all $z,w\in\Omega$.

Lemma 1.1.2. *For* $f\in\mathscr{F}$, *the function*

$$d_f\colon \Omega\to\mathbb{C},\ d_f(z)=\langle f,d(z)\rangle$$

defines a multiplier $d_f\in\operatorname{Mult}(H)$ *with* $\left\|M_{d_f}\right\|_{L(H)}\leq\|f\|$.

Proof. Without loss of generality let $\|f\|=1$. Let $(f_i)_{i\in I}$ be an orthonormal basis of $\mathscr{F}$, which contains f. Then we have

$$\begin{aligned}
&\mathsf{s}(z,w)\left(1-d_f(z)\overline{d_f(w)}\right)\\
&=\mathsf{s}(z,w)\left(1-\langle f,d(z)\rangle\langle d(w),f\rangle\right)\\
&=\mathsf{s}(z,w)\left(1-\sum_{i\in I}\langle d(w),f_i\rangle\langle f_i,d(z)\rangle\right)\\
&\quad+\mathsf{s}(z,w)\sum_{i\in I,f_i\neq f}\langle d(w),f_i\rangle\langle f_i,d(z)\rangle\\
&=\mathsf{s}(z,w)\left(1-\langle d(w),d(z)\rangle+\sum_{i\in I,f_i\neq f}\langle f_i,d(z)\rangle\overline{\langle f_i,d(w)\rangle}\right)\\
&=1+\mathsf{s}(z,w)\sum_{i\in I,f_i\neq f}\langle f_i,d(z)\rangle\overline{\langle f_i,d(w)\rangle}
\end{aligned}$$

for $z,w\in\Omega$. The last expression defines a positive definite map by the Lemma of Schur, because pointwise converging sums of positive definite maps are positive definite. By [5, Satz 1.11 (a)] we have $d_f\in\operatorname{Mult}(H)$ with $\left\|M_{d_f}\right\|_{L(H)}\leq 1$. □

Theorem 1.1.3. *For $w\in\Omega$, we have $\mathsf{s}_w=\mathsf{s}(\cdot,w)\in\operatorname{Mult}(H)$.*

Proof. For $d\colon\Omega\to\mathscr{F}$ and $z\in\Omega$ as in the remarks before Lemma 1.1.2, we have

$$\|d(z)\|^2=1-\frac{1}{\mathsf{s}(z,z)}<1.$$

Hence we conclude

$$\begin{aligned}
\mathsf{s}(z,w)&=\frac{1}{1-\langle d(w),d(z)\rangle}\\
&=\sum_{k=0}^{\infty}\langle d(w),d(z)\rangle^k\\
&=\sum_{k=0}^{\infty}\left(d_{d(w)}(z)\right)^k
\end{aligned}$$

for all $z,w\in\Omega$. Because of Lemma 1.1.2, we have $d_{d(w)}\in\operatorname{Mult}(H)$ with $\left\|M_{d_{d(w)}}\right\|_{L(H)}\leq\|d(w)\|<1$ for all $w\in\Omega$. Since $\operatorname{Mult}(H)$ with the multiplier norm

$$\|\cdot\|_{\operatorname{Mult}(H)}:\ \operatorname{Mult}(H)\to\mathbb{R}_{\geq 0},\ \|f\|_{\operatorname{Mult}(H)}=\left\|M_f\right\|_{L(H)}$$

is a Banach algebra with pointwise composition as multiplication, we get

$$\left\|\left(d_{d(w)}\right)^k\right\|_{\operatorname{Mult}(H)}\leq\left\|d_{d(w)}\right\|^k_{\operatorname{Mult}(H)}\leq\|d(w)\|^k$$

Hence the sum

$$\sum_{k=0}^{\infty} \left(d_{d(w)}\right)^k$$

converges absolutely in $\mathrm{Mult}(H)$. Since the inclusion map $\mathrm{Mult}(H) \hookrightarrow H$ is well defined by Proposition 1.1.1, continuous and linear, convergence in $\mathrm{Mult}(H)$ yields pointwise convergence. Hence

$$\mathfrak{s}(\cdot, w) = \sum_{k=0}^{\infty} \left(d_{d(w)}\right)^k \in \mathrm{Mult}(H)$$

holds. $\square$

2 Factorization Theorem

In the following let $\mathscr{E}$ be a separable Hilbert space and $\boldsymbol{s}\colon \Omega \times \Omega \to \mathbb{C}$ a normalized complete Nevanlinna-Pick (CNP) kernel with

$$\boldsymbol{s}_w(z) = \frac{1}{1 - \sum_{n=0}^{\infty} u_n(z)\overline{u_n(w)}} \qquad (z, w \in \Omega)$$

where $u_n\colon \Omega \to \mathbb{C}$ $(n \in \mathbb{N})$ are functions such that $\left\|(u_n(z))_{n\in\mathbb{N}}\right\|_{l^2} < 1$ for all $z \in \Omega$ and there is a $z_0 \in \Omega$ such that $u_n(z_0) = 0$ for all $n \in \mathbb{N}$.

For a map $k\colon \Omega \times \Omega \to \mathbb{C}$, we write $k_w = k(\cdot, w)$ for all $w \in \Omega$.

2.1 The main theorem

In this chapter we want to show that every element of a reproducing kernel Hilbert space with normalized complete Nevanlinna-Pick factor can be written as a quotient of two multipliers. For more details see Theorem 0.0.1. The proof is based on an idea of Sarason [16],[17]. In the case of the Szegö kernel one direction of the proof of our main theorem is easy to see. We therefore give a sketch of Sarason's proof in the special case $\Omega = \mathbb{D}$, $\mathscr{E} = \mathbb{C}$ and $\mathfrak{k} = \boldsymbol{s}1$ where

$$\boldsymbol{s}\colon \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ \boldsymbol{s}(z, w) = \frac{1}{1 - z\overline{w}},$$

$z_0 = 0$ and $\mathscr{H}_{\boldsymbol{s}} = \mathscr{H}_{\mathfrak{k}} = H^2(\mathbb{D})$, as motivation for the general approach. Since $\boldsymbol{s}_z \in \operatorname{Mult}\left(H^2(\mathbb{D})\right)$ for every $z \in \mathbb{D}$, the following function, the so called Sarason function

$$V_F\colon \mathbb{D} \to \mathbb{C},\ V_F(z) = 2\langle F, \boldsymbol{s}_z F\rangle_{H^2(\mathbb{D})} - \|F\|^2_{H^2(\mathbb{D})}$$

is well defined for every $F \in H^2(\mathbb{D})$. Let $H^2(\mathbb{D}) \to H^2(\mathbb{T}), f \mapsto f^*$ be the canonical isometric isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$. We write dm for the normalized arc length measure on the unit circle $\mathbb{T}$. Then we have

$$\begin{aligned} V_F(z) &= \langle F, (2\boldsymbol{s}_z - 1) F\rangle_{H^2(\mathbb{D})} \\ &= \left\langle F^*, \left(2\boldsymbol{s}_z^* - 1\right) F^*\right\rangle_{H^2(\mathbb{T})} \\ &= \int_{\mathbb{T}} |F^*(\xi)|^2 \overline{\left(\frac{2}{1 - \xi\overline{z}} - 1\right)} dm(\xi) \\ &= \int_{\mathbb{T}} |F^*(\xi)|^2 \frac{\xi + z}{\xi - z} dm(\xi), \end{aligned}$$

for all $z \in \mathbb{D}$. By a standard result about parameter-dependent integrals V_F is analytic. Further, we see that $\operatorname{Re} V_F$ is the Poisson integral $P\left[|F^*|^2\right]$ of $|F^*|^2$ since $\operatorname{Re}\left(\frac{\xi+z}{\xi-z}\right) = \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2}$ for all $z \in \mathbb{D}$ and $\xi \in \mathbb{T}$. As the Poisson transform is the inverse of the canonical isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ we can use the Hölder inequality to deduce that

$$\begin{aligned}
&|P[F^*](z)|^2 \\
&= \left(\int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} |F^*(\xi)|\, dm(\xi)\right)^2 \\
&\leq \int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} dm(\xi) \int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} |F^*(\xi)|^2\, dm(\xi) \\
&= P\left[|F^*|^2\right](z)
\end{aligned}$$

and, thus

$$0 \leq |F(z)|^2 = |P[F^*](z)|^2 \leq P\left[|F^*|^2\right](z) = \operatorname{Re} V_F(z) \tag{2.1}$$

for all $z \in \mathbb{D}$. We now assume that $\|F\|_{H^2(\mathbb{D})} = 1$ and set

$$\Psi = \frac{V_F - 1}{V_F + 1} \text{ and } \Phi = \frac{2}{V_F + 1} F.$$

Then we have $\Psi(0) = 0$, $F = \frac{1}{1-\Psi}\Phi$ and a calculation using inequality (2.1) shows

$$|\Psi(z)|^2 + |\Phi(z)|^2 = \frac{|V_F(z)|^2 - 2\operatorname{Re} V_F(z) + 1 + 4|F(z)|^2}{|V_F(z) + 1|^2} \leq 1.$$

Therefore $\Phi, \Psi \in H^\infty(\mathbb{D}) = \operatorname{Mult}\left(H^2(\mathbb{D})\right)$ and the direction from (a) to (b) in Theorem 0.0.1 follows in our special case. Since we can not do these point wise estimations in the general case, we have to proceed as in [2].

Let K be a Hilbert space and $x \in K$. Then the adjoint of the operator $\langle \cdot, x \rangle_K : K \to \mathbb{C}$ is easily seen to be the operator

$$\mathbb{C} \to K, \alpha \to \alpha x.$$

If $f\colon \Omega \to K$ is a function and $\mathscr{H}_k \subset \mathbb{C}^\Omega$ is a functional Hilbert space given by a reproducing kernel $k\colon \Omega \times \Omega \to \mathbb{C}$, then the induced mapping

$$F\colon \Omega \to K',\ F(z) = \langle \cdot, f(z) \rangle_K$$

defines a multiplier $F \in \operatorname{Mult}(\mathscr{H}_k(K), \mathscr{H}_k)$ with $\|M_F\|_{L(\mathscr{H}_k(K),\mathscr{H}_k)} \le c$ if and only the mapping

$$\Omega \times \Omega \to \mathbb{C},\ (z,w) \mapsto k(z,w)\left(c^2 - \langle f(w), f(z)\rangle_K\right)$$

is positive definite. This follows from [5, Satz 1.11 (b)] using the canonical identification $L(\mathbb{C}) \cong \mathbb{C}$.

Lemma 2.1.1. *With the notations from the beginning of Chapter 2, let*

$$\mathfrak{U}\colon \Omega \to \left(\mathfrak{l}^2\right)',\ \mathfrak{U}(z)\left((x_n)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty} u_n(z)\, x_n.$$

Then we have

(i) $\mathfrak{U} \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}\left(\mathfrak{l}^2\right), \mathscr{H}_{\mathfrak{s}}\right)$ *with* $\|M_{\mathfrak{U}}\|_{L(\mathscr{H}_{\mathfrak{s}}(\mathfrak{l}^2),\mathscr{H}_{\mathfrak{s}})} \le 1$. *In particular* $u_n \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$ *for all* $n \in \mathbb{N}$ *and we have*

$$\left\| \sum_{n=0}^{\infty} u_n h_n \right\|_{\mathscr{H}_{\mathfrak{s}}}^2 \le \sum_{n=0}^{\infty} \|h_n\|_{\mathscr{H}_{\mathfrak{s}}}^2$$

for all $(h_n)_{n\in\mathbb{N}} \in \mathscr{H}_{\mathfrak{s}}\left(\mathfrak{l}^2\right)$.

(ii) the map $\mathrm{Id}_{\mathscr{H}_{\mathfrak{s}}} - M_{\mathfrak{U}} M_{\mathfrak{U}}^*$ *is the orthogonal projection* $P_{\mathscr{H}_0}$ *onto* $\mathscr{H}_0 = \operatorname{span}\{\mathfrak{s}_{z_0}\} \cong \mathbb{C}$ *and satisfies* $P_{\mathscr{H}_0} h \equiv h(z_0)$ *for all* $h \in \mathscr{H}_{\mathfrak{s}}$.

Proof. Since $\mathfrak{U}(z) = \left\langle \cdot, \left(\overline{u_n(z)}\right)_{n\in\mathbb{N}} \right\rangle_{\mathfrak{l}^2}$ and since the function

$$\begin{aligned}\Omega \times \Omega \to \mathbb{C},\ (z,w) \mapsto & \mathfrak{s}(z,w)\left(1 - \left\langle \left(\overline{u_n(w)}\right)_{n\in\mathbb{N}}, \left(\overline{u_n(z)}\right)_{n\in\mathbb{N}} \right\rangle_{\mathfrak{l}^2}\right) \\ &= \mathfrak{s}(z,w)\left(1 - \sum_{n=0}^{\infty} u_n(z)\,\overline{u_n(w)}\right) \\ &= 1\end{aligned}$$

is positive definite, the remarks preceding Lemma 2.1.1 show that $\mathfrak{U}$ is a multiplier with $\|M_{\mathfrak{U}}\|_{L(\mathscr{H}_{\mathfrak{s}}(\mathfrak{l}^2),\mathscr{H}_{\mathfrak{s}})} \le 1$. Hence

$$\begin{aligned}\left\| \sum_{n=0}^{\infty} u_n h_n \right\|_{\mathscr{H}_{\mathfrak{s}}}^2 &= \|M_{\mathfrak{U}} h\|_{\mathscr{H}_{\mathfrak{s}}}^2 \\ &\le \|h\|_{\mathscr{H}_{\mathfrak{s}}(\mathfrak{l}^2)}^2 \\ &= \sum_{n=0}^{\infty} \|h_n\|_{\mathscr{H}_{\mathfrak{s}}}^2\end{aligned}$$

for all $h = (h_n)_{n\in\mathbb{N}} \in \mathscr{H}_{\mathfrak{s}}\left(\mathfrak{l}^2\right)$ by Corollary 4.0.6. To show that u_n is in $\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$, let $n \in \mathbb{N}$ and observe that

$$\begin{aligned} u_n(z)\,h(z) &= \sum_{k=0}^{\infty} u_k(z)\,h(z)\,\delta_{kn} \\ &= \mathfrak{U}(z)\,(h(z)\,e_n) \\ &= (M_{\mathfrak{U}}(he_n))\,(z) \end{aligned}$$

for all $h \in \mathscr{H}_{\mathfrak{s}}$ and $z \in \Omega$. Hence we conclude $u_n h = M_{\mathfrak{U}}(he_n) \in \mathscr{H}_{\mathfrak{s}}$.

To prove (ii) first note that $P_{\mathscr{H}_0} h = \langle h, \mathfrak{s}_{z_0}\rangle\, \mathfrak{s}_{z_0} \equiv h(z_0)$ for $h \in \mathscr{H}_{\mathfrak{s}}$. Using $\mathfrak{U}(z)^* w = \left(\overline{u_n(z)} w\right)_{n\in\mathbb{N}}$ for all $z \in \Omega$ and $w \in \mathbb{C}$ an easy calculation shows that

$$M_{\mathfrak{U}}^* \mathfrak{s}_z = \mathfrak{U}(z)^*\,(1_{\mathbb{C}})\,\mathfrak{s}_z = \left(\overline{u_n(z)} \mathfrak{s}_z\right)_{n\in\mathbb{N}}$$

for all $z \in \Omega$. Hence

$$\left(\left(\mathrm{Id}_{\mathscr{H}_{\mathfrak{s}}} - M_{\mathfrak{U}} M_{\mathfrak{U}}^*\right) \mathfrak{s}_w\right)(z) = \left(1 - \sum_{n=0}^{\infty} u_n(z)\,\overline{u_n(w)}\right) \mathfrak{s}_w(z) = 1 = \mathfrak{s}_w(z_0)$$

for all $z, w \in \Omega$. Since $M_{\mathfrak{U}}$ and $M_{\mathfrak{U}}^*$ are continuous and linear and since $\mathscr{H}_{\mathfrak{s}} = \bigvee\{\mathfrak{s}_z;\ z \in \Omega\}$ it follows that $\left(\mathrm{Id}_{\mathscr{H}_{\mathfrak{s}}} - M_{\mathfrak{U}} M_{\mathfrak{U}}^*\right) h \equiv h(z_0)$ for all $h \in \mathscr{H}_{\mathfrak{s}}$. Hence the claim holds. □

Corollary 2.1.2. *Let $\mathfrak{U}$ be the mapping from Lemma 2.1.1. Then we have*

$$M_{\mathfrak{U}}^* \mathfrak{s}_z = \left(\overline{u_n(z)} \mathfrak{s}_z\right)_{n\in\mathbb{N}}$$

for all $z \in \Omega$.

We give a condition for kernels to have a complete Nevanlinna-Pick factor.

Lemma 2.1.3. *Let*

$$\mathfrak{U}\colon \Omega \to \left(\mathfrak{l}^2\right)',\ \mathfrak{U}(z)\left((x_n)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty} u_n(z)\,x_n.$$

be as before. Then a given reproducing kernel $\mathfrak{k}\colon \Omega \times \Omega \to \mathbb{C}$ can be written as a product $\mathfrak{k} = \mathfrak{s}\mathfrak{g}$ with $\mathfrak{g}\colon \Omega \times \Omega \to \mathbb{C}$ positive definite if and only if

$$\mathfrak{U} \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2\right), \mathscr{H}_{\mathfrak{k}}\right) \ \textit{with}\ \|M_{\mathfrak{U}}\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right)} \le 1.$$

In this case we have $\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}}(\mathscr{E})) \subset \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ *and* $\left\|M_\varphi\right\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \le \left\|M_\varphi\right\|_{L(\mathscr{H}_{\mathfrak{s}}(\mathscr{E}))}$ *for all* $\varphi \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}}(\mathscr{E}))$ *and any Hilbert space* $\mathscr{E}$.

Proof. Exactly as in the proof of Lemma 2.1.1 it follows that

$$\mathfrak{U}\colon \Omega \to \left(\mathfrak{l}^2\right)', \, \mathfrak{U}(z) = \left\langle \cdot, \left(\overline{u_n(z)}\right)_{n\in\mathbb{N}} \right\rangle_{\mathfrak{l}^2}$$

defines a multiplier $\mathfrak{U} \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2\right), \mathscr{H}_{\mathfrak{k}}\right)$ with multiplier norm $\|M_{\mathfrak{U}}\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right)} \leq 1$ if and only if the function

$$\begin{aligned} \mathfrak{g}\colon \Omega \times \Omega \to \mathbb{C}, \, \mathfrak{g}(z,w) &= \mathfrak{k}(z,w)\left(1 - \sum_{n=0}^{\infty} u_n(z)\overline{u_n(w)}\right) \\ &= \frac{\mathfrak{k}_w(z)}{\mathfrak{s}_w(z)} \end{aligned}$$

is positive definite. This yields the claimed equivalence. For the second part, let $\varphi \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}}(\mathscr{E}))$ be a multiplier with $c = \left\|M_\varphi\right\|_{L(\mathscr{H}_{\mathfrak{s}}(\mathscr{E}))} \neq 0$. By [5, Satz 1.11 (b)] the function

$$\mathfrak{a}\colon \Omega \times \Omega \to L(\mathscr{E}), \, \mathfrak{a}(z,w) = \mathfrak{s}(z,w)\left(c^2 \operatorname{Id}_{\mathscr{E}} - \varphi(z)\varphi(w)^*\right)$$

is positive definite. By the Schur-Product Lemma the function

$$\begin{aligned} \tilde{\mathfrak{a}}\colon \Omega \times \Omega \to L(\mathscr{E}), \, \tilde{\mathfrak{a}}(z,w) &= \mathfrak{g}(z,w)\mathfrak{a}(z,w) \\ &= \mathfrak{k}(z,w)\left(c^2 \operatorname{Id}_{\mathscr{E}} - \varphi(z)\varphi(w)^*\right) \end{aligned}$$

is positive definite. Another application of [5, Satz 1.11 (b)] completes the proof. □

In the following let $\mathfrak{g}\colon \Omega \times \Omega \to \mathbb{C}$ be a positive definite function and $\mathfrak{k} = \mathfrak{s}\mathfrak{g}$.

Remark 2.1.4. $\mathfrak{k}$ is positive definite by the Schur-Product Lemma.

The following function plays an important role in our further estimations.

Definition 2.1.5. For $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, we define the Sarason function V_F by

$$V_F\colon \Omega \to \mathbb{C}, \, V_F(z) = 2\left\langle F, \mathfrak{s}_z F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}$$

Remark 2.1.6. Since the functions $\mathfrak{s}_z$ $(z \in \Omega)$ are multipliers of $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, it easily follows that $\left(V_{F_n}\right)_{n\in\mathbb{N}}$ converges pointwise to V_F on Ω, whenever $(F_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ with limit F.

Lemma 2.1.7. *Let $\mathscr{H}_i \subset \mathscr{E}_i^{\Omega}$ $(i = 1,2)$ be functional Hilbert spaces. Let $\varphi\colon \Omega \to L(\mathscr{E}_1, \mathscr{E}_2)$ be a map such that there are a continuous linear operator $T\colon \mathscr{H}_1 \to \mathscr{H}_2$ and a total subset $M \subset \mathscr{H}_1$ with $Tf = \varphi f$ for all $f \in M$. Then $\varphi \in \operatorname{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ and $T = M_\varphi$.*

Proof. Each function $f \in \mathscr{H}_1$ is the limit

$$f = \lim_{n\to\infty} \sum_{i=1}^{r_n} \alpha_i^{(n)} f_i^{(n)}$$

of a sequence of linear combinations of functions $f_i^{(n)} \in M$. Since point evaluations are continuous on the spaces $\mathscr{H}_i$, it follows that

$$\begin{aligned}
(Tf)(\lambda) &= \lim_{n\to\infty} \sum_{i=1}^{r_n} \alpha_i^{(n)} \left(T f_i^{(n)}\right)(\lambda) \\
&= \lim_{n\to\infty} \sum_{i=1}^{r_n} \alpha_i^{(n)} \varphi(\lambda) f_i^{(n)}(\lambda) \\
&= \lim_{n\to\infty} \varphi(\lambda) \sum_{i=1}^{r_n} \alpha_i^{(n)} f_i^{(n)}(\lambda) \\
&= \varphi(\lambda) f(\lambda)
\end{aligned}$$

for all $\lambda \in \Omega$. □

The next lemma contains some important estimations and properties for vector valued functional Hilbert spaces with complete Nevanlinna-Pick
factors.

Lemma 2.1.8. *Let $h = \sum_{i=0}^{k} a_i s_{z_i}$ with $z_i \in \Omega$ and $a_i \in \mathbb{C}$ for $i = 1, \cdots, k$. For $n \in \mathbb{N}$, let $h_n = \sum_{i=0}^{k} a_i \overline{u_n(z_i)} s_{z_i}$. Setting*

$$\mathfrak{H}_{\mathscr{E}} \colon \Omega \to L\left(\mathscr{E}, \mathfrak{l}^2(\mathscr{E})\right),\ \mathfrak{H}_{\mathscr{E}}(\lambda)(v) = (h_n(\lambda) v)_{n\in\mathbb{N}},$$

$$\mathfrak{U}_{\mathscr{E}} \colon \Omega \to L\left(\mathfrak{l}^2(\mathscr{E}), \mathscr{E}\right),\ \mathfrak{U}_{\mathscr{E}}(\lambda)\left((v_n)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty} u_n(\lambda) v_n,$$

we have $\mathfrak{H}_{\mathscr{E}} \in \mathrm{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E}), \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right)\right)$ and

$$\mathfrak{U}_{\mathscr{E}} \in \mathrm{Mult}\left(\mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right), \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right) \text{ with } \left\|M_{\mathfrak{U}_{\mathscr{E}}}\right\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E})), \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \le 1.$$

In particular we have $(h - h(z_0))F = M_{\mathfrak{U}_{\mathscr{E}}} M_{\mathfrak{H}_{\mathscr{E}}} F$ and hence

$$\|(h - h(z_0))F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \le \left\|M_{\mathfrak{H}_{\mathscr{E}}} F\right\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E}))} = \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}$$

for all $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$.

Proof. The map $\mathfrak{H}_{\mathscr{E}}$ is well defined, since for $\lambda \in \Omega$ and $v \in \mathscr{E}$, we have

$$\begin{aligned}
\left\|(h_n(\lambda)v)_{n\in\mathbb{N}}\right\|^2_{\mathfrak{l}^2(\mathscr{E})} &= \sum_{n=0}^{\infty} |h_n(\lambda)|^2 \|v\|^2_{\mathscr{E}} \\
&= \|v\|^2_{\mathscr{E}} \sum_{n=0}^{\infty} \left| \sum_{i=0}^{k} a_i \overline{u_n(z_i)} \mathfrak{s}_{z_i}(\lambda) \right|^2 \\
&\leq \|v\|^2_{\mathscr{E}} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{k} |a_i \mathfrak{s}_{z_i}(\lambda)|^2 \sum_{j=0}^{k} \left|u_n(z_j)\right|^2 \right) \\
&= \|v\|^2_{\mathscr{E}} \sum_{i=0}^{k} |a_i \mathfrak{s}_{z_i}(\lambda)|^2 \sum_{j=0}^{k} \left(\sum_{n=0}^{\infty} \left|u_n(z_j)\right|^2 \right) \\
&= \|v\|^2_{\mathscr{E}} \sum_{i=0}^{k} |a_i \mathfrak{s}_{z_i}(\lambda)|^2 \sum_{j=0}^{k} \left\|\left(u_n(z_j)\right)_{n\in\mathbb{N}}\right\|^2_{\mathfrak{l}^2}.
\end{aligned}$$

Without loss of generality we can now assume that $h = \mathfrak{s}_z$ and $h_n = \overline{u_n(z)}\mathfrak{s}_z$ for $z \in \Omega$ and $n \in \mathbb{N}$. Consider the continuous linear operator

$$T\colon \mathscr{H}_{\mathfrak{k}}(\mathscr{E}) \cong \mathscr{H}_{\mathfrak{k}} \otimes \mathscr{E} \overset{M_{\mathfrak{s}_z}\otimes S}{\longrightarrow} \mathscr{H}_{\mathfrak{k}} \otimes \mathfrak{l}^2(\mathscr{E}) \cong \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right)$$

with $S\colon \mathscr{E} \to \mathfrak{l}^2(\mathscr{E})$, $Sx = \left(\overline{u_n(z)}x\right)_{n\in\mathbb{N}}$. For $f \in \mathscr{H}_{\mathfrak{k}}$ and $x \in \mathscr{E}$ we have

$$T(fx) = (\mathfrak{s}_z f)Sx = \mathfrak{H}_{\mathscr{E}}(fx).$$

Since the set of elementary tensors $\left\{fx;\ f \in \mathscr{H}_{\mathfrak{k}}, x \in \mathscr{E}\right\} \subset \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is total we conclude with Lemma 2.1.7 that $\mathfrak{H}_{\mathscr{E}} \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E}), \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right)\right)$. In the following we use a similar argument to show that

$$\mathfrak{U}_{\mathscr{E}} \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right), \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right) \text{ with } \left\|M_{\mathfrak{U}_{\mathscr{E}}}\right\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E})), \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \leq 1$$

holds. Let $f \in \mathscr{H}_{\mathfrak{k}}$, $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{l}^2$, $x \in \mathscr{E}$ and $\mathfrak{U}$ be the mapping from Lemma 2.1.1. The continuous linear operator

$$\tilde{T}\colon \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right) \cong \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2\right) \otimes \mathscr{E} \overset{M_{\mathfrak{U}}\otimes \mathrm{Id}_{\mathscr{E}}}{\longrightarrow} \mathscr{H}_{\mathfrak{k}} \otimes \mathscr{E} \cong \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$$

acts on the elementary tensor $f(\alpha_n x)_{n\in\mathbb{N}}$ as

$$\tilde{T}\left(f(\alpha_n x)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty} f\alpha_n u_n x = \mathfrak{U}_{\mathscr{E}}\left(f(\alpha_n x)_{n\in\mathbb{N}}\right).$$

Because the set

$$\left\{(\alpha_n x)_{n\in\mathbb{N}};\ (\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{l}^2, x \in \mathscr{E}\right\} \subset \mathfrak{l}^2(\mathscr{E}).$$

is total, we can apply Lemma 2.1.7 to conclude $\mathfrak{U}_{\mathscr{E}}$ is in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right), \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ and

$$\begin{aligned}\|M_{\mathfrak{U}_{\mathscr{E}}}\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E})),\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} &= \|\tilde{T}\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E})),\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \\ &= \|M_{\mathfrak{U}} \otimes \mathrm{Id}_{\mathscr{E}}\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2)\otimes\mathscr{E},\mathscr{H}_{\mathfrak{k}}\otimes\mathscr{E}\right)} \\ &= \|M_{\mathfrak{U}}\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2),\mathscr{H}_{\mathfrak{k}}\right)} \\ &\leq 1.\end{aligned}$$

For the final part of the claim observe that

$$\begin{aligned}(h - h(z_0) f x)(\lambda) &= \left(s_z\left(1 - \frac{1}{s_z}\right) f x\right)(\lambda) \\ &= \left(\sum_{n=0}^{\infty} s_z(\lambda)\overline{u_n(z)} u_n(\lambda)\right) f(\lambda) x \\ &= (\mathfrak{U}_{\mathscr{E}} \mathfrak{H}_{\mathscr{E}} f x)(\lambda).\end{aligned}$$

holds for all $f \in \mathscr{H}_{\mathfrak{k}}$, $x \in \mathscr{E}$ and $\lambda \in \Omega$. Since $h - h(z_0)$ and $\mathfrak{U}_{\mathscr{E}}\mathfrak{H}_{\mathscr{E}}$ are in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ the equality $h - h(z_0) = \mathfrak{U}_{\mathscr{E}}\mathfrak{H}_{\mathscr{E}}$ holds. Due to $\mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right) \cong \mathfrak{l}^2 \otimes \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ as Hilbert spaces (cf. Corollary 4.0.6) we conclude that

$$\|M_{\mathfrak{H}_{\mathscr{E}}} F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E}))} = \|(h_n F)_{n\in\mathbb{N}}\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2(\mathscr{E}))} = \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}.$$

□

Now, we can show that the real part of the Sarason function is always positive.

Corollary 2.1.9. *Let $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, then we have* $\operatorname{Re} V_F \geq 0$.

Proof. Since

$$\frac{1}{1 - \sum_{n=0}^{\infty} |u_n(z)|^2} = \frac{1}{1 - \sum_{n=0}^{\infty} u_n(z)\overline{u_n(z)}} = s(z,z) > 0$$

for all $z \in \Omega$, we have

$$\sum_{n=0}^{\infty} |u_n(z)|^2 < 1$$

for all $z \in \Omega$. If we set $h = \mathsf{s}_z$ and $h_n = \overline{u_n(z)}\mathsf{s}_z$ for $z \in \Omega$ and all $n \in \mathbb{N}$, we get by Lemma 2.1.8

$$\begin{aligned}
\|(\mathsf{s}_z - 1)F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \|(\mathsf{s}_z - \mathsf{s}_z(z_0))F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \|(h - h(z_0))F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\leq \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \sum_{n=0}^{\infty} \left\| \overline{u_n(z)} s_z F \right\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \sum_{n=0}^{\infty} \left(|u_n(z)|^2 \|s_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \right) \\
&= \left(\sum_{n=0}^{\infty} |u_n(z)|^2 \right) \|s_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\leq \|s_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}
\end{aligned}$$

for all $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. Hence we conclude

$$\begin{aligned}
&\operatorname{Re} V_F(z) \\
&= \frac{V_F(z) + \overline{V_F(z)}}{2} \\
&= \langle \mathsf{s}_z F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} + \langle F, \mathsf{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \langle F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \langle \mathsf{s}_z F, \mathsf{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\quad - \left(\langle \mathsf{s}_z F, \mathsf{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \langle \mathsf{s}_z F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \langle F, \mathsf{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} + \langle F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \right) \\
&= \|\mathsf{s}_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|(\mathsf{s}_z - 1)F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\geq 0
\end{aligned}$$

for all $z \in \Omega$ and $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. □

Our next aim is to prove a complete generalization of inequality (2.1).

Lemma 2.1.10. *For $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ the map $\mathfrak{v}\colon \Omega \times \Omega \to \mathbb{C}$,*

$$\mathfrak{v}(z,w) = \langle \mathsf{s}_w F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} + \langle F, \mathsf{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \frac{\langle \mathsf{s}_w F, \mathsf{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathsf{s}_w(z)}$$

is positive definite.

Proof. For $w \in \Omega$, let $u_w \colon \Omega \to \mathbb{C}$, $u_w(z) = \sum_{n=0}^{\infty} \overline{u_n(w)} u_n(z)$. Then $\frac{1}{\mathfrak{s}_w(z)} = 1 - u_w(z)$ for all $z, w \in \Omega$ and hence

$$\begin{aligned}\mathfrak{v}(z,w) &= \langle \mathfrak{s}_w F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} + \langle F, \mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \frac{\langle \mathfrak{s}_w F, \mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_w(z)} \\ &= u_w(z) \langle \mathfrak{s}_w F, \mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \langle (\mathfrak{s}_w - 1) F, (\mathfrak{s}_z - 1) F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}.\end{aligned}$$

Let $k \in \mathbb{N}$, $z_0, \cdots, z_k \in \Omega$ and $a_0, \cdots, a_k \in \mathbb{C}$. We define $h = \sum_{i=0}^{k} a_i \mathfrak{s}_{z_i}$ and $h_n = \sum_{i=0}^{k} a_i \overline{u_n(z_i)} \mathfrak{s}_{z_i}$ for all $n \in \mathbb{N}$. Then we obtain

$$\sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = \sum_{i,j=0}^{k} \overline{a_i} a_j \left\langle \mathfrak{s}_{z_j} F, \mathfrak{s}_{z_i} F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \sum_{n=0}^{\infty} \overline{u_n(z_j)} u_n(z_i)$$

From Lemma 2.1.8 we know that

$$\|(h - h(z_0)) F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \leq \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}.$$

Finally the claim follows from

$$\begin{aligned}&\sum_{i,j=0}^{k} \mathfrak{v}(z_i, z_j) a_j \overline{a_i} \\ &= \sum_{i,j=0}^{k} \overline{a_i} a_j \left(u_{z_j}(z_i) \left\langle \mathfrak{s}_{z_j} F, \mathfrak{s}_{z_i} F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \left\langle \left(\mathfrak{s}_{z_j} - 1\right) F, \left(\mathfrak{s}_{z_i} - 1\right) F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \right) \\ &= \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \left\langle \left(\sum_{j=0}^{k} a_j \left(\mathfrak{s}_{z_j} - 1\right)\right) F, \left(\sum_{i=0}^{k} a_i \left(\mathfrak{s}_{z_i} - 1\right)\right) F \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\ &= \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\ &\quad - \left\langle \left(\sum_{i=0}^{k} a_j \left(\mathfrak{s}_{z_j} - \mathfrak{s}_{z_j}(z_0)\right)\right) F, \left(\sum_{i=0}^{k} a_i \left(\mathfrak{s}_{z_i} - \mathfrak{s}_{z_i}(z_0)\right)\right) F \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\ &= \sum_{n=0}^{\infty} \|h_n F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|(h - h(z_0)) F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\ &\geq 0.\end{aligned}$$

□

In the following we can use the positivity of the map from Lemma 2.1.10 to generalize the inequality (2.1).

Lemma 2.1.11. *Let $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and let V_F be the Sarason function. Then*

$$\mathfrak{s}_z(z) \|F(z)\|^2_{\mathscr{E}} \leq \mathfrak{k}_z(z) \operatorname{Re} V_F(z).$$

Proof. By definition we have

$$\frac{V_F(z)+\overline{V_F(w)}}{2}=\langle \mathfrak{s}_w F,F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}+\langle F,\mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}-\|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}$$

for all $z,w\in\Omega$. Due to Lemma 2.1.10 the map

$$\mathfrak{v}\colon \Omega\times\Omega\to\mathbb{C},\ \mathfrak{v}(z,w)=\frac{V_F(z)+\overline{V_F(w)}}{2}-\frac{\langle \mathfrak{s}_w F,\mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_w(z)}$$

is positive definite. Hence it follows that

$$\operatorname{Re} V_F(z)-\frac{\|\mathfrak{s}_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_z(z)}=\mathfrak{v}(z,z)\geq 0$$

for all $z\in\Omega$. Now let $z\in\Omega$. Then we have

$$\begin{aligned}\mathfrak{s}_z(z)^2\|F(z)\|^2_{\mathscr{E}}&=\|\delta_z(\mathfrak{s}_z F)\|^2_{\mathscr{E}}\\&\leq\|\delta_z\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})'}\|\mathfrak{s}_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\&=\left\|\delta_z\delta_z^*\right\|_{L(\mathbb{C})}\|\mathfrak{s}_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\&=\mathfrak{k}_z(z)\|\mathfrak{s}_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}.\end{aligned}$$

Hence

$$\mathfrak{s}_z(z)\|F(z)\|^2_{\mathscr{E}}\leq\frac{\mathfrak{k}_z(z)\|\mathfrak{s}_z F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_z(z)}\leq\mathfrak{k}_z(z)\operatorname{Re} V_F(z).$$

□

Now, we apply Lemma 2.1.10 to come closer to the multiplier estimates in Theorem 0.0.1.

Lemma 2.1.12. *For $i=1,\cdots,N$, let $f_i\in\operatorname{span}\{\mathfrak{k}_z;\ z\in\Omega\}\subset\mathscr{H}_{\mathfrak{k}}$. Let (e_n) be an orthonormal basis of $\mathscr{E}$. Set $F=\sum_{n=1}^N f_n e_n$. Then we have $F\in\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ and $V_F\in\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$ where F is regarded as a function in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. For $h\in\mathscr{H}_{\mathfrak{s}}$, we have*

$$\|hF\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\leq\operatorname{Re}\langle V_F h,h\rangle_{\mathscr{H}_{\mathfrak{s}}}\tag{2.2}$$

and if $a\in\mathbb{C}$ with $\operatorname{Re} a\geq 0$, then

$$\|(V_F-a)h\|^2_{\mathscr{H}_{\mathfrak{s}}}+4\operatorname{Re} a\|hF\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\leq\|(V_F+\overline{a})h\|^2_{\mathscr{H}_{\mathfrak{s}}}.\tag{2.3}$$

Proof. For $i = 1, \cdots, N$ let $f_i = \sum_{j=0}^{m} a_{i,j} \mathfrak{k}_{z_j}$ where $a_{i,j} \in \mathbb{C}$ and $z_j \in \Omega$ $(j = 0 \cdots, m)$. For $z \in \Omega$, we have

$$
\begin{aligned}
\langle F, \mathfrak{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \left\langle \sum_{n=1}^{N} f_n e_n, \sum_{n=1}^{N} \mathfrak{s}_z f_n e_n \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \sum_{k,l=1}^{N} \langle f_k, \mathfrak{s}_z f_l \rangle_{\mathscr{H}_{\mathfrak{k}}} \langle e_k, e_l \rangle_{\mathscr{E}} \\
&= \sum_{k=1}^{N} \langle f_k, \mathfrak{s}_z f_k \rangle_{\mathscr{H}_{\mathfrak{k}}} \\
&= \sum_{k=1}^{N} \left\langle \sum_{j=0}^{m} a_{k,j} \mathfrak{k}_{z_j}, \sum_{j=0}^{m} a_{k,j} \mathfrak{s}_z \mathfrak{k}_{z_j} \right\rangle_{\mathscr{H}_{\mathfrak{k}}} \\
&= \sum_{k=1}^{N} \sum_{i,j=0}^{m} a_{k,i} \overline{a_{k,j}} \left\langle \mathfrak{k}_{z_i}, \mathfrak{s}_z \mathfrak{k}_{z_j} \right\rangle_{\mathscr{H}_{\mathfrak{k}}} \\
&= \sum_{i,j=0}^{m} \left\langle \mathfrak{k}_{z_i}, \mathfrak{s}_z \mathfrak{k}_{z_j} \right\rangle_{\mathscr{H}_{\mathfrak{k}}} \sum_{k=1}^{N} a_{k,i} \overline{a_{k,j}} \\
&= \sum_{i,j=0}^{m} \overline{\mathfrak{s}_z (z_i) \mathfrak{k}_{z_j} (z_i)} \sum_{k=1}^{N} a_{k,i} \overline{a_{k,j}} \\
&= \sum_{i,j=0}^{m} c_{i,j} \mathfrak{k}_{z_i} (z_j) \, \mathfrak{s}_{z_i} (z)
\end{aligned}
$$

where we set $c_{i,j} = \sum_{k=1}^{N} a_{k,i} \overline{a_{k,j}}$ for $i, j = 0, \cdots, m$. Therefore we have

$$
\begin{aligned}
V_F (z) &= 2 \langle F, \mathfrak{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= 2 \sum_{i,j=0}^{m} c_{i,j} \mathfrak{k}_{z_i} (z_j) \, \mathfrak{s}_{z_i} (z) - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}
\end{aligned}
$$

for all $z \in \Omega$ and hence $V_F \in \mathrm{Mult}(\mathscr{H}_{\mathfrak{s}})$. Because of Lemma 2.1.10 the map $\mathfrak{v} \colon \Omega \times \Omega \to \mathbb{C}$,

$$
\begin{aligned}
\mathfrak{v}(z, w) &= \langle \mathfrak{s}_w F, F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} + \langle F, \mathfrak{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} - \frac{\langle \mathfrak{s}_w F, \mathfrak{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_w (z)} \\
&= \frac{V_F (z) + \overline{V_F (w)}}{2} - \frac{\langle \mathfrak{s}_w F, \mathfrak{s}_z F \rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_w (z)}
\end{aligned}
$$

is positive definite. Then $\mathfrak{a} = \mathfrak{s}\mathfrak{v}$ is positive definite by the Schur-Product lemma. Next let $w_0, \cdots, w_k \in \Omega$, $a_0, \cdots, a_k \in \mathbb{C}$ $(k \in \mathbb{N})$ and set

$h = \sum_{i=0}^{k} a_i \mathfrak{s}_{w_i}$. Then we have

$$\begin{aligned}
&\operatorname{Re}\langle V_F h, h\rangle_{\mathscr{H}_{\mathfrak{s}}} - \langle hF, hF\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \frac{1}{2}\left(\langle V_F h, h\rangle_{\mathscr{H}_{\mathfrak{s}}} + \overline{\langle V_F h, h\rangle_{\mathscr{H}_{\mathfrak{s}}}}\right) - \left\langle \sum_{i=0}^{k} a_i \mathfrak{s}_{w_i} F, \sum_{i=0}^{k} a_i \mathfrak{s}_{w_i} F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \frac{1}{2}\left(\sum_{i,j=0}^{k} a_i \overline{a_j}\left\langle V_F \mathfrak{s}_{w_i}, \mathfrak{s}_{w_j}\right\rangle_{\mathscr{H}_{\mathfrak{s}}} + \overline{\sum_{i,j=0}^{k} a_j \overline{a_i}\left\langle V_F \mathfrak{s}_{w_j}, \mathfrak{s}_{w_i}\right\rangle_{\mathscr{H}_{\mathfrak{s}}}}\right) \\
&\quad - \sum_{i,j=0}^{k} a_i \overline{a_j}\left\langle \mathfrak{s}_{w_i} F, \mathfrak{s}_{w_j} F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \sum_{i,j=0}^{k} a_i \overline{a_j}\left(\frac{\left\langle V_F \mathfrak{s}_{w_i}, \mathfrak{s}_{w_j}\right\rangle_{\mathscr{H}_{\mathfrak{s}}} + \overline{\left\langle V_F \mathfrak{s}_{w_j}, \mathfrak{s}_{w_i}\right\rangle_{\mathscr{H}_{\mathfrak{s}}}}}{2} - \left\langle \mathfrak{s}_{w_i} F, \mathfrak{s}_{w_j} F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\right) \\
&= \sum_{i,j=0}^{k} a_i \overline{a_j}\left(\frac{V_F\left(w_j\right) + \overline{V_F\left(w_i\right)}}{2} - \frac{\left\langle \mathfrak{s}_{w_i} F, \mathfrak{s}_{w_j} F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{s}_{w_i}\left(w_j\right)}\right) \mathfrak{s}_{w_i}\left(w_j\right) \\
&= \sum_{i,j=0}^{k} a_i \overline{a_j} \mathfrak{v}\left(w_j, w_i\right) \mathfrak{s}_{w_i}\left(w_j\right) \\
&= \sum_{i,j=0}^{k} a_i \overline{a_j} \mathfrak{a}\left(w_j, w_i\right) \\
&\geq 0.
\end{aligned}$$

Therefore inequality (2.2) holds for all $h \in \operatorname{span}\{\mathfrak{s}_z;\, z \in \Omega\}$. Now let $h \in \mathscr{H}_{\mathfrak{s}}$ be arbitrary. Since $\mathscr{H}_{\mathfrak{s}} = \bigvee\{\mathfrak{s}_z;\, z \in \Omega\}$ there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $\operatorname{span}(s_z;\, z \in \Omega)$ such that $\mathscr{H}_{\mathfrak{s}}\text{-}\lim_{n\to\infty} h_n = h$. Since we have established inequality (2.2) for elements of $\operatorname{span}\{s_z;\, z \in \Omega\}$ we conclude

$$\begin{aligned}
\|h_n F - h_m F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \|(h_n - h_m) F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\leq \operatorname{Re}\langle V_F (h_n - h_m), h_n - h_m\rangle_{\mathscr{H}_{\mathfrak{s}}} \\
&\leq \left|\langle V_F (h_n - h_m), h_n - h_m\rangle_{\mathscr{H}_{\mathfrak{s}}}\right| \\
&\leq \|M_{V_F}\|_{L(\mathscr{H}_{\mathfrak{s}})} \|h_n - h_m\|^2_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

for all $n, m \in \mathbb{N}$. Thus, $(h_n F)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and hence $g = \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\text{-}\lim_{n\to\infty} h_n F$ exists. Since the point evaluations on $\mathscr{H}_{\mathfrak{s}}$ and $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ are continuous we have $g(z) = \mathscr{E}\text{-}\lim_{n\to\infty}(h_n(z) F(z)) = h(z) F(z)$ for all $z \in \Omega$ and thus $hF = g \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. We conclude that

$F \in \operatorname{Mult}\left(\mathcal{H}_{\mathfrak{s}}, \mathcal{H}_{\mathfrak{k}}(\mathcal{E})\right)$ and that

$$\begin{aligned}\|hF\|^2_{\mathcal{H}_{\mathfrak{k}}(\mathcal{E})} &= \lim_{n\to\infty} \|h_nF\|^2_{\mathcal{H}_{\mathfrak{k}}(\mathcal{E})} \\ &\le \lim_{n\to\infty} \operatorname{Re}\langle V_F h_n, h_n\rangle_{\mathcal{H}_{\mathfrak{s}}} \\ &= \operatorname{Re}\langle V_F h, h\rangle_{\mathcal{H}_{\mathfrak{s}}}.\end{aligned}$$

For all $h \in \mathcal{H}_{\mathfrak{s}}$ and $a \in \mathbb{C}$, an easy calculation shows

$$\|(V_F+\overline{a})\,h\|^2_{\mathcal{H}_{\mathfrak{s}}} - \|(V_F-a)\,h\|^2_{\mathcal{H}_{\mathfrak{s}}} = 4\operatorname{Re}a\operatorname{Re}\langle V_F h, h\rangle_{\mathcal{H}_{\mathfrak{s}}} \tag{2.4}$$

Therefore inequality (2.3) easily follows from inequality (2.2), since

$$\begin{aligned}\|(V_F-a)\,h\|^2_{\mathcal{H}_{\mathfrak{s}}} + 4\operatorname{Re}a\,\|hF\|^2_{\mathcal{H}_{\mathfrak{k}}(\mathcal{E})} &\le \|(V_F-a)\,h\|^2_{\mathcal{H}_{\mathfrak{s}}} + 4\operatorname{Re}a\operatorname{Re}\langle V_F h, h\rangle_{\mathcal{H}_{\mathfrak{s}}} \\ &= \|(V_F+\overline{a})\,h\|^2_{\mathcal{H}_{\mathfrak{s}}}\end{aligned}$$

holds for all $h \in \mathcal{H}_{\mathfrak{s}}$ and $a \in \mathbb{C}$ with $\operatorname{Re}a \ge 0$. □

We can now show that each function $F \in \mathcal{H}_{\mathfrak{k}}(\mathcal{E})$ admits even a more general factorization than claimed in Theorem 0.0.1.

Proposition 2.1.13. *Let $F \in \mathcal{H}_{\mathfrak{k}}(\mathcal{E})$ and $a \in \mathbb{C}$ with $\operatorname{Re}a > 0$. As before we write V_F for the Sarason function of F. We define functions $\Psi_a, \Phi_a \colon \Omega \to \mathbb{C}$ by*

$$\Psi_a(z) = \frac{V_F(z)-a}{V_F(z)+\overline{a}} \quad \text{and} \quad \Phi_a(z) = \frac{2}{V_F(z)+\overline{a}}F(z).$$

Then $\Psi_a \in \operatorname{Mult}(\mathcal{H}_{\mathfrak{s}})$, $\Phi_a \in \operatorname{Mult}\left(\mathcal{H}_{\mathfrak{s}}, \mathcal{H}_{\mathfrak{k}}(\mathcal{E})\right)$ and

$$\|\Psi_a h\|^2_{\mathcal{H}_{\mathfrak{s}}} + \operatorname{Re}a\,\|\Phi_a h\|^2_{\mathcal{H}_{\mathfrak{k}}(\mathcal{E})} \le \|h\|^2_{\mathcal{H}_{\mathfrak{s}}}$$

for all $h \in \mathcal{H}_{\mathfrak{s}}$.

Proof. Since $\operatorname{Re}V_F \ge 0$ and $\operatorname{Re}a > 0$, the maps Ψ_a and Φ_a are well defined. As first step we prove the assertions in the particular case that $F \in \mathcal{H}_{\mathfrak{k}}(\mathcal{E})$ is a function as in Lemma 2.1.12. For $i = 1, \cdots, N$ let $f_i \in \operatorname{span}\left\{\mathfrak{k}_z;\ z \in \Omega\right\} \subset \mathcal{H}_{\mathfrak{k}}$. Then there are $m \in \mathbb{N}$ $a_{i,j} \in \mathbb{C}$ and $z_j \in \Omega$ $(i = 1, \cdots, N,\ j = 0, \cdots, m)$ with $f_i = \sum_{j=0}^{m} a_{i,j}\mathfrak{k}_{z_j}$ for $i = 1, \cdots, N$. Let (e_n) be an orthonormal basis of $\mathcal{E}$ and set $F = \sum_{n=0}^{N} f_n e_n$. By Lemma 2.1.12 $F \in \operatorname{Mult}\left(\mathcal{H}_{\mathfrak{s}}, \mathcal{H}_{\mathfrak{k}}(\mathcal{E})\right)$ and $V_F \in \operatorname{Mult}(\mathcal{H}_{\mathfrak{s}})$. Now let $u \in \mathcal{H}_{\mathfrak{s}}$ and set $h = (V_F+\overline{a})\,u$. By inequality (2.3) it follows that

$$\begin{aligned}&\left\|\frac{V_F-a}{V_F+\overline{a}}h\right\|^2_{\mathcal{H}_{\mathfrak{s}}} + \operatorname{Re}a\left\|\frac{2}{V_F+\overline{a}}Fh\right\|^2_{\mathcal{H}_{\mathfrak{k}}(\mathcal{E})} \\ &= \|(V_F-a)\,u\|^2_{\mathcal{H}_{\mathfrak{s}}} + 4\operatorname{Re}a\,\|uF\|^2_{\mathcal{H}_{\mathfrak{k}}(\mathcal{E})} \\ &\le \|(V_F+\overline{a})\,u\|^2_{\mathcal{H}_{\mathfrak{s}}} \\ &= \|h\|^2_{\mathcal{H}_{\mathfrak{s}}}\end{aligned} \tag{2.5}$$

Thus we have proved the claimed inequality for each function
$h \in (V_F + \overline{a})\mathscr{H}_{\mathfrak{s}}$. For $h \in ((V_F + \overline{a})\mathscr{H}_{\mathfrak{s}})^{\perp}$, we can use inequality (2.2) to obtain

$$\begin{aligned}
0 &= \left| \langle (V_F + \overline{a}) h, h \rangle_{\mathscr{H}_{\mathfrak{s}}} \right| \\
&\geq \operatorname{Re} \langle (V_F + \overline{a}) h, h \rangle_{\mathscr{H}_{\mathfrak{s}}} \\
&\geq \|h\|^2_{\mathscr{H}_{\mathfrak{s}}} \operatorname{Re} a + \|hF\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\geq \|h\|^2_{\mathscr{H}_{\mathfrak{s}}} \operatorname{Re} a \\
&\geq 0.
\end{aligned}$$

Since $\operatorname{Re} a > 0$ it follows that $\|h\|^2_{\mathscr{H}_{\mathfrak{s}}} = 0$ and hence $h = 0$. This shows that $((V_F + \overline{a})\mathscr{H}_{\mathfrak{s}})^{\perp} = \{0\}$ and thus

$$\overline{(V_F + \overline{a})\mathscr{H}_{\mathfrak{s}}} = \mathscr{H}_{\mathfrak{s}}.$$

We now want to show that, for functions F as above we have $\Psi_a \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$ and $\Phi_a \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. Thus let $h \in \mathscr{H}_{\mathfrak{s}}$ and let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $(V_F + \overline{a})\mathscr{H}_{\mathfrak{s}}$ with $h = \mathscr{H}_{\mathfrak{s}}\text{-}\lim_{n\to\infty} h_n$. Then there are $u_n \in \mathscr{H}_{\mathfrak{s}}$ with $h_n = (V_F + \overline{a}) u_n$ for all $n \in \mathbb{N}$ and hence we have $\Psi_a h_n = (V_F - a) u_n \in \mathscr{H}_{\mathfrak{s}}$ and $\Phi_a h_n = 2u_n F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ all $n \in \mathbb{N}$. Further we get

$$\|\Psi_a h_n - \Psi_a h_m\|_{\mathscr{H}_{\mathfrak{s}}} = \|\Psi_a (h_n - h_m)\|_{\mathscr{H}_{\mathfrak{s}}} \overset{\text{inequality (2.5)}}{\leq} \|h_n - h_m\|_{\mathscr{H}_{\mathfrak{s}}}$$

and

$$\begin{aligned}
\|\Phi_a h_n - \Phi_a h_m\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \|\Phi_a (h_n - h_m)\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\overset{\text{inequality (2.5)}}{\leq} \frac{1}{\sqrt{\operatorname{Re} a}} \|h_n - h_m\|_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

for all $n, m \in \mathbb{N}$. Hence $(\Psi_a h_n)_{n\in\mathbb{N}}$ and $(\Phi_a h_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathscr{H}_{\mathfrak{s}}$, respectively $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. Then $\mathscr{H}_{\mathfrak{s}}\text{-}\lim_{n\to\infty} \Psi_a h_n$ and $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\text{-}\lim_{n\to\infty} \Phi_a h_n$ exist and we can use the continuity of the point evaluations on $\mathscr{H}_{\mathfrak{s}}$ and $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ to show that
$\Psi_a h = \mathscr{H}_{\mathfrak{s}}\text{-}\lim_{n\to\infty} \Psi_a h_n \in \mathscr{H}_{\mathfrak{s}}$ and $\Phi_a h = \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\text{-}\lim_{n\to\infty} \Phi_a h_n$
$\in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ hold. Therefore $\Psi_a \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$ and $\Phi_a \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. Since $\|\cdot\|_{\mathscr{H}_{\mathfrak{s}}}$,

$\|\cdot\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}$, M_{Ψ_a} and M_{Φ_a} are continuous, we obtain that:

$$\begin{aligned}
&\left\|\frac{V_F-a}{V_F+\overline{a}}h\right\|^2_{\mathscr{H}_{\mathfrak{s}}}+\operatorname{Re}a\left\|\frac{2}{V_F+\overline{a}}Fh\right\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&=\lim_{n\to\infty}\left(\|M_{\Psi_a}h_n\|^2_{\mathscr{H}_{\mathfrak{s}}}+\operatorname{Re}a\,\|M_{\Phi_a}h_n\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\right)\\
&=\lim_{n\to\infty}\left(\left\|\frac{V_F-a}{V_F+\overline{a}}h_n\right\|^2_{\mathscr{H}_{\mathfrak{s}}}+\operatorname{Re}a\left\|\frac{2}{V_F+\overline{a}}Fh_n\right\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\right)\\
&\le\lim_{n\to\infty}\|h_n\|^2_{\mathscr{H}_{\mathfrak{s}}}\\
&=\|h\|^2_{\mathscr{H}_{\mathfrak{s}}}.
\end{aligned}$$

In the second step of we prove the assertions for an arbitrary function $F\in\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. Since $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})=\bigvee\left\{fe_n;\ f\in\mathscr{H}_{\mathfrak{k}},\ n\in\mathbb{N}\right\}$ and $\mathscr{H}_{\mathfrak{k}}=\operatorname{span}\left\{\mathfrak{k}_z;\ z\in\Omega\right\}$ we can approximate F by a sequence $(F_N)_{N\in\mathbb{N}}$ in

$$\operatorname{span}\left\{fe_n;\ f\in\operatorname{span}\left\{\mathfrak{k}_z;\ z\in\Omega\right\},\ n\in\mathbb{N}\right\}.$$

We have already seen that the functions $\Psi_{a,N}\colon\Omega\to\mathbb{C}$ and $\Phi_{a,N}\colon\Omega\to\mathscr{E}$ defined by

$$\Psi_{a,N}(z)=\frac{V_{F_N}(z)-a}{V_{F_N}(z)+\overline{a}}\quad\text{and}\quad\Phi_{a,N}(z)=\frac{2}{V_{F_N}(z)+\overline{a}}F_N(z)$$

are in $\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$, respectively $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ with $\left\|M_{\Psi_{a,N}}\right\|_{L(\mathscr{H}_{\mathfrak{s}})}\le 1$ and $\left\|M_{\Phi_{a,N}}\right\|_{L\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)}\le\frac{1}{\sqrt{\operatorname{Re}a}}$. By Remark 2.1.6 the sequence $(V_{F_N})_{N\in\mathbb{N}}$ converges pointwise to V_F. Therefore $(\Psi_{a,N})_{N\in\mathbb{N}}$ and $(\Phi_{a,N})_{N\in\mathbb{N}}$ converge pointwise to Ψ_a and Φ_a and hence $\lim_{N\to\infty}(\Psi_{a,N}h)(z)=(\Psi_a h)(z)$ and $\mathscr{E}\text{-}\lim_{N\to\infty}(\Phi_{a.N}h)(z)=(\Phi_a h)(z)$ for all $z\in\Omega$ and $h\in\mathscr{H}_{\mathfrak{s}}$. By Corollary 4.0.2 we have $\Psi_a\in\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$, $\Phi_a\in\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ and $((\Psi_{a,N}h,\Phi_{a,N}h))_{N\in\mathbb{N}}$ converges weakly to $(\Psi_a h,\Phi_a h)$. Because of that we can use Proposition 4.0.3 to deduce

$$\begin{aligned}
&\|\Psi_a h\|^2_{\mathscr{H}_{\mathfrak{s}}}+\operatorname{Re}a\,\|\Phi_a h\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&\le\liminf_{N\to\infty}\|\Psi_{a,N}h\|^2_{\mathscr{H}_{\mathfrak{s}}}+\operatorname{Re}a\liminf_{N\to\infty}\|\Phi_{a,N}h\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&\le\liminf_{N\to\infty}\left(\|\Psi_{a,N}h\|^2_{\mathscr{H}_{\mathfrak{s}}}+\operatorname{Re}a\,\|\Phi_{a,N}h\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\right)\\
&\le\|h\|^2_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

for all $h\in\mathscr{H}_{\mathfrak{s}}$. $\square$

We proof the following Lemma to show the direction (b) to (a) in Theorem 0.0.1(i) and the uniqueness of the factorization in Theorem 0.0.1(ii).

Lemma 2.1.14. *Let $\Psi \in \mathbb{C}^{\Omega}$ and $\Phi \in \mathscr{E}^{\Omega}$ be maps with $\Psi \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$, $\Phi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ and such that*

$$\|\Psi h\|_{\mathscr{H}_{\mathfrak{s}}}^{2} + \|\Phi h\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^{2} \leq \|h\|_{\mathscr{H}_{\mathfrak{s}}}^{2} \tag{2.6}$$

holds for all $h \in \mathscr{H}_{\mathfrak{s}}$.

(i) Then, for $0 < r < 1$, the function $F_r = \frac{\Phi}{1-r\Psi}$ is well defined and $F_r \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ with

$$\|F_r h\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^{2} \leq \operatorname{Re}\left\langle \frac{1+r\Psi}{1-r\Psi} h, h \right\rangle_{\mathscr{H}_{\mathfrak{s}}} \qquad (h \in \mathscr{H}_{\mathfrak{s}}). \tag{2.7}$$

In particular, $F_r = F_r s_{z_0} \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$

(ii) If $|\psi(z_0)| < 1$ and $1 \notin \Psi(\Omega)$ then $F = \frac{\Phi}{1-\Psi} \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, $\tau_w^{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\text{-}\lim_{\substack{r \to 1 \\ 0<r<1}} F_r = F$ and the map $\mathfrak{a} \colon \Omega \times \Omega \to \mathbb{C}$ with

$$\mathfrak{a}(z,w) = \mathfrak{s}_w(z)\left(\frac{1+\psi(z)}{1-\psi(z)} + \overline{\frac{1+\psi(w)}{1-\psi(w)}}\right) - 2\left\langle \mathfrak{s}_w F, \mathfrak{s}_z F \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}$$

for all $z, w \in \Omega$ is positive definite.

Proof. (i) Fix $0 < r < 1$. By inequality (2.6) we have $\|M_\Psi\|_{L(\mathscr{H}_{\mathfrak{s}})} \leq 1$. Hence it holds that $\sigma(M_\Psi) \subset \overline{\mathbb{D}} = \{z \in \mathbb{C};\ |z| \leq 1\}$. Therefore the operator $M_{(1-r\psi)} = r\left(\frac{1}{r}\operatorname{Id}_{\mathscr{H}_{\mathfrak{s}}} - M_\Psi\right)$ is invertible for $0 < r < 1$. Since the constant function $1 \colon \Omega \to \mathbb{C}$ is in $\mathscr{H}_{\mathfrak{s}}$, there exists $g \in \mathscr{H}_{\mathfrak{s}}$ such that $M_{(1-r\psi)} g = 1$. Hence $(1-r\Psi)(z)\, g(z) = 1$ for all $z \in \Omega$ and $\frac{1}{1-r\Psi} \in \mathscr{H}_{\mathfrak{s}}$. Because $\frac{1}{1-r\Psi} h = M_{(1-r\psi)}^{-1} h$ is in $\mathscr{H}_{\mathfrak{s}}$ for all $h \in \mathscr{H}_{\mathfrak{s}}$, we indeed have $\frac{1}{1-r\Psi} \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$. Since Φ is an element of $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$, it follows that $F_r \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. Using inequality (2.6) and the fact that $0 < r < 1$ we conclude that

$$r^2 \|\Psi h\|_{\mathscr{H}_{\mathfrak{s}}}^{2} + \|\Phi h\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^{2} \leq \|h\|_{\mathscr{H}_{\mathfrak{s}}}^{2} \quad (h \in \mathscr{H}_{\mathfrak{s}}).$$

Applying this inequality to $\frac{1}{1-r\Psi} h$ we get

$$\|F_r h\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^{2} \leq \left\|\frac{1}{1-r\Psi} h\right\|_{\mathscr{H}_{\mathfrak{s}}}^{2} - \left\|\frac{r\Psi}{1-r\Psi} h\right\|_{\mathscr{H}_{\mathfrak{s}}}^{2}.$$

for all $h \in \mathscr{H}_{\mathfrak{s}}$. Using the identities

$$\frac{1+r\Psi}{1-r\Psi}-1=\frac{2r\Psi}{1-r\Psi} \text{ and } \frac{1+r\Psi}{1-r\Psi}+1=\frac{2}{1-r\Psi}$$

we obtain inequality (2.7) because

$$\begin{aligned}
&\left\|\frac{1}{1-r\Psi}h\right\|^2_{\mathscr{H}_{\mathfrak{s}}}-\left\|\frac{r\Psi}{1-r\Psi}h\right\|^2_{\mathscr{H}_{\mathfrak{s}}}\\
&=\frac{1}{4}\left(\left\|\frac{2}{1-r\Psi}h\right\|^2_{\mathscr{H}_{\mathfrak{s}}}-\left\|\frac{2r\Psi}{1-r\Psi}h\right\|^2_{\mathscr{H}_{\mathfrak{s}}}\right)\\
&=\frac{1}{4}\left\langle\left(\frac{1+r\Psi}{1-r\Psi}+1\right)h,\left(\frac{1+r\Psi}{1-r\Psi}+1\right)h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}\\
&\quad-\frac{1}{4}\left\langle\left(\frac{1+r\Psi}{1-r\Psi}-1\right)h,\left(\frac{1+r\Psi}{1-r\Psi}-1\right)h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}\\
&=\frac{1}{2}\left(\left\langle\frac{1+r\Psi}{1-r\Psi}h,h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}+\left\langle h,\frac{1+r\Psi}{1-r\Psi}h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}\right)\\
&=\operatorname{Re}\left\langle\frac{1+r\Psi}{1-r\Psi}h,h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

for all $h \in \mathscr{H}_{\mathfrak{s}}$.

(ii) Suppose that $1 \neq \Psi(\Omega)$. Then we have $\mathscr{E}\text{-}\lim_{\substack{r\to 1\\0<r<1}} F_r(z) = F(z)$ for all $z \in \Omega$, and using the identity $\mathfrak{s}_{z_0} \equiv 1$, we conclude that

$$\begin{aligned}
\|F_r\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}&=\|F_r\mathfrak{s}_{z_0}\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&\le\operatorname{Re}\left\langle\frac{1+r\Psi}{1-r\Psi}\mathfrak{s}_{z_0},\mathfrak{s}_{z_0}\right\rangle_{\mathscr{H}_{\mathfrak{s}}}\\
&=\operatorname{Re}\frac{1+r\Psi(z_0)}{1-r\Psi(z_0)}\\
&\le\frac{1+|r\Psi(z_0)|}{1-|r\Psi(z_0)|}\\
&\le\frac{1+|\Psi(z_0)|}{1-|\Psi(z_0)|}
\end{aligned}$$

for all $0<r<1$. Therefore Theorem 4.0.1 yields $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and
$\tau_w^{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\text{-}\lim_{\substack{r\to 1\\0<r<1}} F_r = F$.
We now want to show that the map $\mathfrak{a}\colon \Omega\times\Omega\to\mathbb{C}$,

$$\mathfrak{a}(z,w)=\mathfrak{s}_w(z)\left(\frac{1+\psi(z)}{1-\psi(z)}+\overline{\frac{1+\psi(w)}{1-\psi(w)}}\right)-2\langle\mathfrak{s}_wF,\mathfrak{s}_zF\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \tag{2.8}$$

is positive definite. Let $a_1,\cdots,a_n \in \mathbb{C}$, $z_1,\cdots,z_n \in \Omega$ and set $h=\sum_{i=1}^n a_i \mathfrak{s}_{z_i}$. Since $M_h \in L\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ the map $M_h\colon \mathscr{H}_{\mathfrak{k}}(\mathscr{E}) \to \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is weakly continuous. Setting $r_k = \left(1-\frac{1}{k+2}\right)$ for all $k \in \mathbb{N}$, we can use Proposition 4.0.3 and inequality (2.7) to get

$$
\begin{aligned}
&\sum_{i,j=1}^n a_i\overline{a_j}\left\langle \mathfrak{s}_{z_i}F, \mathfrak{s}_{z_j}F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&=\|hF\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&\le \liminf_{k\to\infty}\|hF_{r_k}\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\\
&\le \limsup_{n\to\infty}\operatorname{Re}\left\langle \frac{1+r_k\Psi}{1-r_k\Psi}h,h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}\\
&=\limsup_{k\to\infty}\operatorname{Re}\sum_{i,j=1}^n a_i\overline{a_j}\left\langle \frac{1+r_k\Psi}{1-r_k\Psi}\mathfrak{s}_{z_i},\mathfrak{s}_{z_j}\right\rangle_{\mathscr{H}_{\mathfrak{s}}}\\
&=\lim_{k\to\infty}\frac{1}{2}\sum_{i,j=1}^n a_i\overline{a_j}\left(\frac{1+r_k\Psi(z_j)}{1-r_k\Psi(z_j)}+\overline{\frac{1+r_k\Psi(z_i)}{1-r_k\Psi(z_i)}}\right)\mathfrak{s}_{z_i}(z_j)\\
&=\frac{1}{2}\sum_{i,j=1}^n a_i\overline{a_j}\mathfrak{s}_{z_i}(z_j)\left(\frac{1+\Psi(z_j)}{1-\Psi(z_j)}+\overline{\frac{1+\Psi(z_i)}{1-\Psi(z_i)}}\right).
\end{aligned}
$$

Therefore we have

$$\sum_{i,j=1}^n a_i\overline{a_j}\mathfrak{a}(z_i,z_j)\ge 0.$$

□

Remark 2.1.15. For a domain $\Omega\subset\mathbb{C}^n$ and a functional Hilbert space $\mathscr{H}_{\mathfrak{s}}\subset\mathscr{O}(\Omega)$ we can conclude with inequality (2.6) that $|\Psi|<1$ on Ω if $\Phi\neq 0$. This follows from the maximum principle and the inequality $\|\Psi\|_\Omega\le\|M_\Psi\|_{L(\mathscr{H}_{\mathfrak{s}})}$.

We, finally show Theorem 0.0.1.

Theorem 2.1.16. *(i) For $F\in\mathscr{E}^\Omega$, the following are equivalent:*

(a) $F\in\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ with $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\le 1$.

(b) There exist $\Psi\in\mathbb{C}^\Omega$ with $\Psi(z_0)=0$, $1\notin\Psi(\Omega)$ and $\Phi\in\mathscr{E}^\Omega$ such that $\Psi\in$ $\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}})$, $\Phi\in\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$,

$$\|\Psi h\|^2_{\mathscr{H}_{\mathfrak{s}}}+\|\Phi h\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\le\|h\|^2_{\mathscr{H}_{\mathfrak{s}}}$$

for all $h\in\mathscr{H}_{\mathfrak{s}}$ and $F=\frac{\Phi}{1-\Psi}$.

(ii) *If* $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ *with* $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ *then the factorization from (b) is uniquely given by*

$$\Psi = \frac{V_F - 1}{V_F + 1} \textit{ and } \Phi = \frac{2}{V_F + 1} F.$$

Proof. (i) (a)⇒(b) If $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ with $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$, then (b) follows from (a) by Proposition 2.1.13 with $a = 1$, $\Psi = \Psi_1$ and $\Phi = \Phi_1$. Observe that since $V_F(z_0) = 1$ we have $\Psi(z_0) = 0$. If $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} < 1$, we get (b) from (a) as follows: For $w \in \Omega$, set

$$F_w = \left(F, \sqrt{\frac{1 - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}}{\mathfrak{k}_w(w)}} \mathfrak{k}_w \right).$$

Then $F_w \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E} \oplus \mathbb{C})$ with

$$\|F_w\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E} \oplus \mathbb{C})} = \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} + \left(1 - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\right) \frac{\|\mathfrak{k}_w\|^2_{\mathscr{H}_k}}{|\mathfrak{k}_w(w)|} = 1.$$

The Sarason function of F_w is

$$V_{F_w} = V_F + \left(1 - \|F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\right)(2\mathfrak{s}_w - 1).$$

Let $\Psi_w \in \mathbb{C}^{\Omega}$, $\Phi_w \in (\mathscr{E} \oplus \mathbb{C})^{\Omega}$ be the functions corresponding to F_w from Proposition 2.1.13 with $a = 1$ and $P_{\mathscr{E}}$ be the orthogonal projection from $\mathscr{E} \oplus \mathbb{C}$ onto $\mathscr{E}$. Then the claim follows with $\Psi = \Psi_w$ and $\Phi = P_{\mathscr{E}} \Phi_w$ since $F = \frac{1}{1 - \psi_w} P_{\mathscr{E}} \Phi_w$.
(b)⇒(a) If we assume (b), then the assumptions from Lemma 2.1.14 (ii) hold. Using the notations from Lemma 2.1.14 we therefore get $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and $\tau_w^{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}\text{-}\lim_{\substack{r \to 1 \\ 0<r<1}} F_r = F$. Setting $r_k = \left(1 - \frac{1}{k+2}\right)$ $(k \in \mathbb{N})$ and applying inequality (2.7) with $h \equiv 1 \in \mathscr{H}_{\mathfrak{s}}$ we conclude

$$\begin{aligned}
\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &\leq \liminf_{k \to \infty} \|F_{r_k}\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\leq \liminf_{k \to \infty} \operatorname{Re} \left\langle \frac{1 + r_k \Psi}{1 - r_k \Psi} 1, 1 \right\rangle_{\mathscr{H}_{\mathfrak{s}}} \\
&= \liminf_{k \to \infty} \operatorname{Re} \left\langle \frac{1 + r_k \Psi}{1 - r_k \Psi} 1, s_{z_0} \right\rangle_{\mathscr{H}_{\mathfrak{s}}} \\
&= \liminf_{k \to \infty} \operatorname{Re} \frac{1 + r_k \Psi(z_0)}{1 - r_k \Psi(z_0)} \\
&= 1
\end{aligned}$$

due to Proposition 4.0.3.

(ii) Let $\mathfrak{a}\colon \Omega\times\Omega\to\mathbb{C}$,

$$\mathfrak{a}(z,w)=\mathfrak{s}_w(z)\left(\frac{1+\psi(z)}{1-\psi(z)}+\overline{\frac{1+\psi(w)}{1-\psi(w)}}\right)-2\langle\mathfrak{s}_wF,\mathfrak{s}_zF\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}$$

be the positive definite map from Lemma 2.1.14. Since $\Psi(z_0)=0$ and $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=1$, we have $\mathfrak{a}(z_0,z_0)=0$ and thus

$$|\mathfrak{a}(z,z_0)|^2=\left|\langle\mathfrak{a}_z,\mathfrak{a}_{z_0}\rangle_{\mathscr{H}_{\mathfrak{a}}}\right|^2\leq\|\mathfrak{a}_z\|^2_{\mathscr{H}_{\mathfrak{a}}}\|\mathfrak{a}_{z_0}\|^2_{\mathscr{H}_{\mathfrak{a}}}=\mathfrak{a}(z_0,z_0)\,\mathfrak{a}(z,z)=0$$

for all $z\in\Omega$. We obtain $\mathfrak{a}(z,z_0)=0$ and hence

$$\frac{1+\Psi(z)}{1-\Psi(z)}=2\langle F,\mathfrak{s}_zF\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}-1=V_F(z)$$

for all $z\in\Omega$. Finally this yields

$$\Psi=\frac{V_F-1}{V_F+1}\text{ and }\Phi=\frac{2}{V_F+1}F.$$

$\square$

2.2 Applications

In general it is difficult to characterize the multipliers of a complete Nevanlinna-Pick space. We next indicate how to construct multipliers using a weak integral as explained in [15, Thm 3.17] and the Sarason function.

For a subset M of a vector space X we denote by

$$\operatorname{co}(M) = \bigcap_{M \subset C,\, C \text{ convex}} C$$

the convex hull of M.

Proposition 2.2.1. *Let H_1, H_2 be Hilbert spaces and let $C \subset L(H_1, H_2)$ be a τ_{WOT}-compact subset. Then the τ_{WOT}-closed convex hull $\overline{\operatorname{co}(C)}^{\tau_{\mathrm{WOT}}} \subset L(H_1, H_2)$ is τ_{WOT}-compact again.*

Proof. By [12, §20.6 (3)] it suffices to show that $(L(H_1, H_2), \tau_{\mathrm{WOT}})$ is quasi-complete. Let therefore $(T_\alpha)_{\alpha \in A}$ be a bounded Cauchy net in $(L(H_1, H_2), \tau_{\mathrm{WOT}})$. By the uniform boundedness principle the net $(T_\alpha)_{\alpha \in A}$ is also norm-bounded. Fix $x \in H_1$ and $y \in H_2$. Since $(\langle T_\alpha x, y \rangle)_{\alpha \in A}$ is a Cauchy net in $\mathbb{C}$, $\lim_\alpha \langle T_\alpha x, y \rangle$ exists. Now set

$$(\cdot, \cdot) : H_1 \times H_2 \to \mathbb{C},\ (x, y) = \lim_\alpha \langle T_\alpha x, y \rangle$$

Then $(\cdot, \cdot)$ is sesquilinear and continuous, since $(T_\alpha)_{\alpha \in A}$ is norm-bounded. Using the fact that, for each continuous sesquilinear form

$$(\cdot, \cdot) : H_1 \times H_2 \to \mathbb{C}$$

there is a continuous linear operator $T \in L(H_1, H_2)$ with

$$(x, y) = \langle Tx, y \rangle \quad (x \in H_1, y \in H_2),$$

it follows that $(L(H_1, H_2), \tau_{\mathrm{WOT}})$ is quasi-complete. $\square$

Theorem 2.2.2. *Let $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, $x \in (0, \infty)$ and μ a finite positive Borel-measure on $\mathbb{R}$ with compact support. Then, setting*

$$\Phi\colon \Omega \to \mathscr{E},\ \Phi(z) = \int_{\operatorname{supp}(\mu)} \frac{F(z)}{V_F(z) + x - iy} d\mu(y)$$

we have $\Phi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$.

Proof. For all $y \in \mathbb{R}$, set $\varphi_y = \frac{F}{V_F + x - iy}$. By Proposition 2.1.13 we have $\varphi_y \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ with $\left\|M_{\varphi_y}\right\|_{L\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \leq \frac{1}{\sqrt{x}}$ for all $y \in \mathbb{R}$. Set

$$f\colon \left(\operatorname{supp}(\mu), \tau_{|\cdot|}\right) \to \left(L\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right), \tau_{\mathrm{WOT}}\right),\ f(y) = M_{\varphi_y}$$

Next, we show that f is continuous. Therefore, let $y \in \operatorname{supp}(\mu)$ and $(y_\alpha)_{\alpha \in A}$ a net in $\operatorname{supp}(\mu)$ with $\mathbb{R}\text{-}\lim_\alpha y_\alpha = y$. Further, fix $h \in \mathscr{H}_{\mathfrak{s}} \setminus \{0\}$. For all $z \in \Omega$ and $v \in \mathscr{E}$, we conclude that

$$\begin{aligned}\lim_\alpha \left\langle M_{\varphi_{y_\alpha}} h, \mathfrak{k}_z v \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \lim_\alpha \left\langle \frac{F(z)\,h(z)}{V_F(z) + x - i y_\alpha}, v \right\rangle_{\mathscr{E}} \\ &= \left\langle \frac{F(z)\,h(z)}{V_F(z) + x - i y}, v \right\rangle_{\mathscr{E}} \\ &= \left\langle M_{\varphi_y} h, \mathfrak{k}_z v \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}.\end{aligned}$$

Since $\left(M_{\varphi_{y_\alpha}}\right)_{\alpha \in A}$ is norm-bounded, the set

$$\left\{ G \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E});\ \lim_\alpha \left\langle M_{\varphi_{y_\alpha}} h, G \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = \left\langle M_{\varphi_y} h, G \right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \right\} \subset \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$$

is a closed linear subspace. Thus the continuity of f follows. By [15, Thm 3.17] and Proposition 2.2.1 the weak integral

$$\int_{supp(\mu)} f\, d\mu \in \overline{\operatorname{co}\left(f\left(\operatorname{supp}(\mu)\right)\right)}^{\tau_{\mathrm{WOT}}}$$

exists. By Proposition 1.7.9 in [6] the set

$$\left\{ M_\varphi;\ \varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right) \right\} \subset L\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$$

is a τ_{WOT}-closed linear subspace. Hence there is a function $\varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ such that $\int_{supp(\mu)} f\, d\mu = M_\varphi$. Now, let $z \in \Omega$ and set

$$\varepsilon_z\colon \left(L\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right), \tau_{\mathrm{WOT}}\right) \to (\mathscr{E}, \tau_w),\ T \mapsto T(1)(z).$$

Since weak integrals are compatible with continuous linear maps, it follows that

$$\begin{aligned}\varepsilon_z\left(\int_{supp(\mu)} f\, d\mu\right) &= \int_{supp(\mu)} \varepsilon_z(f)\, d\mu \\ &= \text{weak} - \int_{supp(\mu)} \frac{F(z)}{V_F(z) + x - iy}\, d\mu(y).\end{aligned}$$

Here the last integral denotes the weak $(\mathscr{E}, \tau_w)$-valued integral from [15]. Since the $\mathscr{E}$-valued Lebesgue-integral

$$\int_{supp(\mu)} \frac{F(z)}{V_F(z) + x - iy}\, d\mu(y) = \Phi(z)$$

exists, we have

$$\varphi(z) = \varepsilon_z\left(\int_{supp(\mu)} f\, d\mu\right) = \int_{supp(\mu)} \frac{F(z)}{V_F(z) + x - iy}\, d\mu(y) = \Phi(z)$$

for all $z \in \Omega$. □

With Theorem 2.1.16 it is easy to check that the elements of $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ have the same zero sets as functions in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. We extend this idea to prove a result about multiplier invariant subspaces.

Definition 2.2.3. A closed subspace $\mathscr{M} \subset \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is called multiplier invariant if $M_\varphi \mathscr{M} \subset \mathscr{M}$ for all $\varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$.

Remark 2.2.4. Let $S \subset \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ be an arbitrary subset. Then

$$[S] = \bigvee\left(\varphi F;\ \varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right), F \in S\right)$$

is the smallest multiplier invariant subspace in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ which contains S. If $S = \{F\}$ consists of a single function, we write $[F] = [\{F\}]$.

Corollary 2.2.5. *Let $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ be a function with $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ and consider a factorization $F = \frac{\Phi}{1-\Psi}$ as in Theorem 2.1.16. Then we have $[F] = [\Phi]$. In particular, every multiplier invariant subspace of $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is generated by elements of* $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$.

Proof. Let $F = \frac{\Phi}{1-\Psi}$ be a factorization as in Theorem 2.1.16. As $(1-\Psi)\operatorname{Id}_{\mathscr{E}}$ is an element of $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ by Lemma 2.1.3, we conclude that $\Phi = (1-\Psi)F \in [F]$. For $0 < r < 1$, we have by the proof of Lemma 2.1.14 that $F^r = \frac{1}{1-r\Psi}\Phi \in [\Phi]$ and

$$\tau_w\text{-}\lim_{r\to 1, 0<r<1} F^r = F.$$

Since $[\Phi]$ is convex as a subspace, it is also τ_w-closed. Hence, we have $F \in [\Phi]$. The remaining assertion obviously follows from the first part of the corollary. ☐

In the last part of this chapter we want to consider so called extremal functions.

Definition 2.2.6. A function $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is called extremal if

$$\langle \varphi F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = \varphi(z_0)$$

holds for all $\varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\right)$.

With the aid of Theorem 2.1.16 we can show some interesting results for these functions.

Remark 2.2.7. If F is extremal in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, then $\|F\|_{\mathscr{H}_k(\mathscr{E})} = 1$ and

$$V_F(z) = 2\langle F, \mathfrak{s}_z F\rangle - 1 = 1.$$

Therefore with the notations of Theorem 2.1.16 we obtain that $\Psi = 0$ and $\Phi = F$.

Corollary 2.2.8. *Every extremal function $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ is a contractive multiplier from $\mathscr{H}_{\mathfrak{s}}$ into $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and*

$$\|F(z)\|_{\mathscr{E}}^2 \leq \frac{\mathfrak{k}_z(z)}{\mathfrak{s}_z(z)}$$

holds for all $z \in \Omega$.

Proof. The first assertion follows by Theorem 2.1.16, since $\Phi = F$. The second part follows by Lemma 2.1.11. □

Corollary 2.2.9. *For $z \in \Omega$, we denote the set*

$$\left\{ u(z)\, u(z)^*;\ u \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right), u(z_0) = 0, \|M_u\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right)} < 1 \right\}$$

by A_z and set

$$\alpha_{\mathfrak{k}}(z) = \sup A_z.$$

If F is extremal in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, then

$$\|F(z)\|_{\mathscr{E}}^2 \leq \left(1 - \alpha_{\mathfrak{k}}(z)\right) \mathfrak{k}_z(z)$$

for all $z \in \Omega$.

Proof. If $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right)$ consists only of the constant zero function, there is nothing to show. If there is a $u \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right)$ with $u(z_0) = 0$ and $\|M_u\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathfrak{l}^2), \mathscr{H}_{\mathfrak{k}}\right)} < 1$ which is not the zero function, then $\mathfrak{s}\colon \Omega \times \Omega \to \mathbb{C}$,

$$\mathfrak{s}(z, w) = \frac{1}{1 - u(z)\, u(w)^*}$$

defines a normalized complete Nevanlinna-Pick kernel. Then Lemma 2.1.3 shows that the map $\mathfrak{g} = \frac{\mathfrak{k}}{\mathfrak{s}}$ is positive definite. By Corollary 2.2.8 we obtain, for every extremal function $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ that

$$\|F(z)\|_{\mathscr{E}}^2 \leq \left(1 - u(z)\, u(z)^*\right) \mathfrak{k}_z(z)$$

holds for all $z \in \Omega$. Thus, the assertion follows. □

In the following let $\mathbb{B}_d \subset \mathbb{C}^d$ be the open Euclidean unit ball.

Proposition 2.2.10. *Let $\mathscr{H}_{\mathfrak{k}} \subset \mathbb{C}^{\mathbb{B}_d}$ be a functional Hilbert space given by a reproducing kernel $\mathfrak{k}\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ and let $\varphi\colon \mathbb{B}_d \to L\left(\mathbb{C}^d, \mathbb{C}\right)$ be defined by*

$$\varphi(z)(\alpha) = \sum_{i=1}^{d} z_i \alpha_i.$$

Then $\varphi \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\left(\mathbb{C}^d\right), \mathscr{H}_{\mathfrak{k}}(\mathbb{C})\right)$ with $\left\|M_\varphi\right\|_{L\left(\mathscr{H}_{\mathfrak{k}}(\mathbb{C}^d), \mathscr{H}_{\mathfrak{k}}(\mathbb{C})\right)} \le 1$ or equivalently, the tuple $M_z = (M_{z_1}, \ldots, M_{z_d}) \in L\left(\mathscr{H}_{\mathfrak{k}}\right)^d$ consisting of the multiplication operators with the coordinate functions is a well-defined row contraction on $\mathscr{H}_{\mathfrak{k}}$, if and only if the map

$$\mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, (z,w) \to \mathfrak{k}(z,w)\left(1 - \langle z, w\rangle_{\mathbb{C}^d}\right)$$

is positive definite or if and only if there is a positive definite function $\mathfrak{g}\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ with

$$\mathfrak{k}(z,w) = \mathfrak{g}(z,w)\frac{1}{1 - \langle z, w\rangle_{\mathbb{C}^d}} \quad (z, w \in \mathbb{B}_d).$$

Proof. This follows from [5, Satz 1.11 (b)]. □

Let

$$\mathfrak{s}\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ \mathfrak{s}(z,w) = \frac{1}{1 - \langle z, w\rangle_{\mathbb{C}^d}}$$

be the reproducing kernel of the Drury-Arveson space, $\mathfrak{g}\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ a positive definite map and $\mathfrak{k} = \mathfrak{s}\mathfrak{g}$, such that $\mathscr{H}_{\mathfrak{k}}$ consists of analytic functions. By Proposition 2.2.10 it follows that the tuple $M_z = (M_{z_1}, \ldots, M_{z_d}) \in L\left(\mathscr{H}_{\mathfrak{k}}\right)^d$ consisting of the multiplication operators with the coordinate functions is a well-defined row contraction on $\mathscr{H}_{\mathfrak{k}}$. By [11, Satz 2.1.7] the row contraction M_z is even pure.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \setminus \{0\}$, we write z^α for the function

$$\mathbb{B}_d \to \mathbb{C},\ z \mapsto z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

In [7] a function $\theta\colon \mathbb{B}^d \to L(\mathbb{C}, \mathscr{E})$ is called $\mathfrak{k}$-inner if $\theta 1_{\mathbb{C}} \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, $\|\theta 1_{\mathbb{C}}\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ and

$$\langle z^\alpha \theta 1_{\mathbb{C}}, \theta 1_{\mathbb{C}}\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 0$$

holds for all $\alpha \in \mathbb{N}^d \setminus \{0\}$.

Remark 2.2.11. By [6, Proposition 1.7.9] the subalgebra

$$M\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)=\left\{M_\varphi;\ \varphi\in\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)\right\}\subset L\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$$

is τ_w^*-closed. Identifying $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ with $M\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ via the map

$$\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)\to M\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right),\ \varphi\mapsto M_\varphi$$

we call the topology induced by τ_w^* on $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ the weak*-topology on $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. An application of [8, Lemma 1.16] and [6, Proposition 1.7.11] yields that the map

$$\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\right)\to\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right),\ \varphi\mapsto\varphi\operatorname{Id}_{\mathscr{E}}$$

is weak*-continuous.

By the following Remark and [7] we see that extremal functions play an essential role in the theory of wandering subspaces for shift-invariant subspaces.

Remark 2.2.12. For an extremal function $F\in\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, the map

$$F^{L(\mathbb{C},\mathscr{E})}\colon\mathbb{B}^d\to L(\mathbb{C},\mathscr{E}),\ F^{L(\mathbb{C},\mathscr{E})}(z)(\lambda)=\lambda F(z)$$

is obviously $\mathfrak{k}$-inner. Indeed, by definition $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=1$ and $\langle z^\alpha F,F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=z^\alpha(0)=0$ for all $\alpha\in\mathbb{N}^d\setminus\{0\}$

Set $\mathscr{H}^\infty(\mathfrak{k})=\overline{\mathbb{C}[z]}^{\text{weak}^*}\subset\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\right)$. With Theorem 2.1.16 we get the following equivalence for $\mathfrak{k}$-inner functions:

Proposition 2.2.13. *For $F\in\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, the following conditions are equivalent:*

(i) $F^{L(\mathbb{C},\mathscr{E})}\colon\mathbb{B}^d\to L(\mathbb{C},\mathscr{E}),\ F^{L(\mathbb{C},\mathscr{E})}(z)(\lambda)=\lambda F(z)$ *is $\mathfrak{k}$-inner,*

(ii) $F\in\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ *with* $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=\|M_F\|_{L\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)}=1$,

(iii) $V_F=1$ *and* $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=1$.

(iv) $\langle\varphi F,F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=\varphi(0)$ *for all* $\varphi\in\mathscr{H}^\infty(\mathfrak{k})$.

Proof. (i)⇒(ii) Let $F^{L(\mathbb{C},\mathscr{E})}$ be $\mathfrak{k}$-inner. By [7, Theorem 6.2], the function F is a contractive multiplier from $\mathscr{H}_{\mathfrak{s}}$ to $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. But then

$$1\geq\|M_F\|_{L\left(\mathscr{H}_{\mathfrak{s}},\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)}\geq\left\|M_F1_{\mathscr{H}_{\mathfrak{s}}}\right\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}=1.$$

(ii)⇒(iii) Since $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ we can use the uniqueness of the decomposition $F = \frac{\Phi}{1-\Psi}$ from Theorem 2.1.16 (ii) to see that $\Phi = F$ and $\Psi = 0$. Thus we deduce $V_F = 1$.

(iii)⇒(i) Let $V_F = 1$. Because of $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ the equation

$$\langle F, \mathfrak{s}_z F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1 = \mathfrak{s}_z(0)$$

holds for all $z \in \mathbb{B}_d$. Since the functions $\mathbb{B}_d \to \mathbb{C},\ z \mapsto z_i\ (i = 1, \ldots, d)$ are elements of $\mathrm{Mult}(\mathscr{H}_{\mathfrak{s}})$ the map $z^\alpha\ \left(\alpha \in \mathbb{N}^d\right)$ can be approximated by a sequence $\left(\varphi_n^{(\alpha)}\right)_{n\in\mathbb{N}}$ with

$$\varphi_n^{(\alpha)} = \sum_{k=0}^{N_n} a_k^{(n)} \mathfrak{s}_{z_{k,n}} \in \mathrm{span}\{\mathfrak{s}_z;\ z \in \mathbb{B}_d\}$$

in $\mathscr{H}_{\mathfrak{s}}$. By Theorem 2.1.16 (ii)

$$F = \Phi \in \mathrm{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right) \text{ with } \|M_F\|_{L\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \leq 1.$$

Thus, we conclude

$$\left\|\varphi_n^{(\alpha)} F - \varphi_m^{(\alpha)} F\right\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \leq \left\|\varphi_n^{(\alpha)} - \varphi_m^{(\alpha)}\right\|_{\mathscr{H}_{\mathfrak{s}}}$$

for all $n, m \in \mathbb{N}$. Hence $\left(\varphi_n^{(\alpha)} F\right)_{n\in\mathbb{N}}$ is a Cauchy sequence and

$$g = \lim_{n\to\infty} \varphi_n^{(\alpha)} F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$$

exists. Since the point evaluations on $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ are continuous, we have

$$g(z) = \lim_{n\to\infty} \varphi_n^{(\alpha)}(z) F(z) = z^\alpha F(z)$$

for all $z \in \mathbb{B}_d$ and hence

$$\begin{aligned}
\langle z^\alpha F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \left\langle \lim_{n\to\infty} \varphi_n^{(\alpha)} F, F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \lim_{n\to\infty} \left\langle \varphi_n^{(\alpha)} F, F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \lim_{n\to\infty} \left\langle \sum_{k=0}^{N_n} a_k^{(n)} \mathfrak{s}_{z_{k,n}} F, F\right\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \lim_{n\to\infty} \sum_{k=0}^{N_n} a_k^{(n)} \mathfrak{s}_{z_{n,k}}(0) \\
&= \lim_{n\to\infty} \varphi_n^{(\alpha)}(0) \\
&= 0.
\end{aligned}$$

(iv)⇒(i) By (iv) we obtain that $\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = \langle 1F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 1$ and $\langle z^\alpha F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = z^\alpha(0)$ $= 0$ for all $\alpha \in \mathbb{N}^d \setminus \{0\}$, since 1 and z^α are in $\mathscr{H}^\infty(\mathfrak{k})$. Hence the map $F^{L(\mathbb{C},\mathscr{E})} : \mathbb{B}^d \to L(\mathbb{C},\mathscr{E})$ is $\mathfrak{k}$-inner.

(i)⇒(iv) Let $\varphi \in \mathscr{H}^\infty(\mathfrak{k})$ and $(\varphi_i)_{i\in I}$ be a net in $\mathbb{C}[z]$ with $\varphi_i = \sum_{|\alpha|\le N_i} a_\alpha^{(i)} z^\alpha$ $(i \in I)$ and weak*-$\lim_{i\in I}\varphi_i = \varphi$ in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\right)$. Hence weak*-$\lim_{i\in I}\varphi_i = \varphi$ also in $\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ by Remark 2.2.11 above. Since $F^{L(\mathbb{C},\mathscr{E})}$ is $\mathfrak{k}$-inner we have $\langle z^\alpha F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} = 0$ for all $\alpha \in \mathbb{N}^d \setminus \{0\}$ and thus

$$
\begin{aligned}
\varphi(0) &= \langle \varphi, \mathfrak{k}_0\rangle_{\mathscr{H}_{\mathfrak{k}}} \\
&= \lim_{i\in I}\left\langle M_{\varphi_i} 1_{\mathscr{H}_{\mathfrak{k}}}, \mathfrak{k}_0\right\rangle_{\mathscr{H}_{\mathfrak{k}}} \\
&= \lim_{i\in I}\langle \varphi_i, \mathfrak{k}_0\rangle_{\mathscr{H}_{\mathfrak{k}}} \\
&= \lim_{i\in I}\varphi_i(0) \\
&= \lim_{i\in I}\sum_{|\alpha|\le N_i} a_\alpha^{(i)} \langle z^\alpha F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \lim_{i\in I}\langle M_{\varphi_i} F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&= \langle \varphi F, F\rangle_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}
\end{aligned}
$$

□

Remark 2.2.14. In [6, Example 3.2.3] it is shown that under the additional assumption

$$\mathfrak{k}\left(e^{it}z, e^{it}w\right) = \mathfrak{k}(z,w)$$

for all $z, w \in \mathbb{B}_d$ and $t \in \mathbb{R}$ the identity

$$\mathscr{H}^\infty(\mathfrak{k}) = \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}\right)$$

holds. In this case condition (iv) is equivalent to the assertion that F is extremal.

3 Multiplier Theorem

3.1 A multiplier criterion

Let F be an element of $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. In the following we want to describe a sufficient condition for F to be a multiplier from $\mathscr{H}_{\mathfrak{s}}$ to $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. Under suitable additional hypotheses we show that $F \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$, whenever the real part of its Sarason function is bounded.

In the following, where not otherwise stated, all function will spaces be $\mathbb{C}$-valued. For a measure space (X, σ, μ) and a given Banach space E we denote by $\mathfrak{L}_0(\mu, E)$ the μ-measurable functions. Here a function $f\colon X \to E$ is said to be μ-measurable if there is a sequence (f_j) of simple functions such that $f_j \to f$ μ-almost everywhere as $j \to \infty$ (for more details see [3, X.1 Measurable functions]). Let Ω be an open subset of $\mathbb{R}^d$, $\mu_1, \ldots, \mu_K$ finite positive measures on the Borel-σ-Algebra $B(\Omega)$. Suppose that $\mathscr{D} = \mathscr{H}_{\mathfrak{k}} \cap C^N(\Omega) \subset \mathscr{H}_{\mathfrak{k}}$ $(N \in \mathbb{N})$ is dense and $\operatorname{Mult}(\mathscr{H}_{\mathfrak{s}}) \subset C^N(\Omega)$. Let $\left(a_\alpha^{(l)}\right)_{|\alpha| \leq N}$ $(l = 1, \ldots, K)$ be families of μ_l-measurable functions on Ω and let $c_1, c_2 > 0$. Further suppose

$$c_1 \|f\|_{\mathscr{H}_{\mathfrak{k}}}^2 \leq \sum_{l=1}^{K} \int_\Omega \left| \sum_{|\alpha| \leq N} a_\alpha^{(l)} \partial^\alpha f(z) \right|^2 d\mu_l(z) \leq c_2 \|f\|_{\mathscr{H}_{\mathfrak{k}}}^2 \tag{3.1}$$

holds for all $f \in \mathscr{D}$. Then the operators

$$L^{(l)}\colon \mathscr{D} \to \mathfrak{L}^2(\mu_l),\ L^{(l)} f = \sum_{|\alpha| \leq N} a_\alpha^{(l)} \partial^\alpha f \quad (l = 1, \ldots, K)$$

are well defined and linear with

$$c_1 \|f\|_{\mathscr{H}_{\mathfrak{k}}}^2 \leq \sum_{l=1}^{K} \left\| L^{(l)} f \right\|_{L^2(\mu_l)}^2 \leq c_2 \|f\|_{\mathscr{H}_{\mathfrak{k}}}^2. \tag{3.2}$$

for all $f \in \mathscr{D}$. For a given Hilbert space $\mathscr{E}$ and $l = 1, \ldots, K$, let $L_{\mathscr{E}}^{(l)}\colon \mathscr{D} \otimes_{\mathrm{alg}} \mathscr{E} \to \mathfrak{L}^2(\mu_l, \mathscr{E})$ be the linear map with $L_{\mathscr{E}}^{(l)}(f \otimes x) = L^{(l)}(f)x$ for $f \in \mathscr{D}$ and $x \in \mathscr{E}$. One can show that each element $f \in \left(\mathscr{D} \otimes_{\mathrm{alg}} \mathscr{E}\right) \setminus \{0\}$ can be written as $f = \sum_{j=1}^r f_j \otimes x_j$ with $f_1, \ldots, f_r \in \mathscr{D}$ and a suitable orthonormal system $(x_j)_{j=0}^r$. For any f of this form, we use the identity

$\sum_{l=1}^{K}\sum_{j=1}^{r}\left\|L^{(l)}f_j\right\|_{L^2(\mu_l)}^2=\sum_{l=1}^{K}\left\|L_{\mathscr{E}}^{(l)}f\right\|_{L^2(\mu_l,\mathscr{E})}$ to obtain

$$\begin{aligned} c_1\|f\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^2 &= c_1\sum_{j=1}^{r}\|f_j\|_{\mathscr{H}_{\mathfrak{k}}}^2 \\ &\leq \sum_{l=1}^{K}\left\|L_{\mathscr{E}}^{(l)}f\right\|_{L^2(\mu_l,\mathscr{E})} \\ &\leq c_2\sum_{j=1}^{r}\|f_j\|_{\mathscr{H}_{\mathfrak{k}}}^2 \\ &= c_2\|f\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^2 . \end{aligned} \tag{3.3}$$

Remark 3.1.1. From the above assumptions that $\mathscr{D}=\mathscr{H}_{\mathfrak{k}}\cap C^N(\Omega)$ and $\mathrm{Mult}(\mathscr{H}_{\mathfrak{s}})\subset C^N(\Omega)$ it follows that $\mathrm{Mult}(\mathscr{H}_{\mathfrak{s}})\mathscr{D}\subset\mathscr{D}$ (see Lemma 2.1.3) and
$\mathrm{span}\{\mathfrak{s}_z;\ z\in\Omega\}\left(\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}\right)\subset\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}$.

Remark 3.1.2. For $F\in\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, $x>0$ and $t\in\mathbb{R}$, Proposition 2.1.13 shows that

$$\frac{1}{V_F+x-it}=\frac{1}{2x}\left(1-\frac{V_F-(x+it)}{V_F+(x-it)}\right)\in\mathrm{Mult}(\mathscr{H}_{\mathfrak{s}})\subset C^N(\Omega).$$

Since $\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}\cong\{\sum_{i=1}^{r}f_ix_i;\ f_i\in\mathscr{D},x_i\in\mathscr{E}\}$ and inequality (3.3) holds for all F in the dense subspace $\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}\subset\mathscr{H}_{\mathfrak{k}}\otimes\mathscr{E}\cong\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$, the operators $L_{\mathscr{E}}^{(l)}\colon\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}\to\mathfrak{L}^2(\mu_l,\mathscr{E})$ $(l=1,\dots,K)$ extend uniquely to bounded linear operators

$$\tilde{L}_{\mathscr{E}}^{(l)}\colon\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\to L^2(\mu_l,\mathscr{E})$$

with

$$\tilde{L}_{\mathscr{E}}^{(l)}\left(\sum_{i=1}^{r}f_ix_i\right)=\left[L_{\mathscr{E}}^{(l)}\left(\sum_{i=1}^{r}f_i\otimes x_i\right)\right]\quad(f_1,\dots,f_r\in\mathscr{D},x_1\dots x_r\in\mathscr{E})$$

such that the estimates

$$c_1\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^2\leq\sum_{l=1}^{K}\left\|\tilde{L}_{\mathscr{E}}^{(l)}F\right\|_{L^2(\mu_l,\mathscr{E})}^2\leq c_2\|F\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^2 \tag{3.4}$$

hold for all $F\in\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$.

In [10] Kaluza studied weighted Bergman spaces A_α^p $(\alpha\in\mathbb{R},p\in[1,\infty))$. The spaces $A_\alpha^2=\mathscr{O}(\mathbb{B}_d)\cap L^2(\mathbb{B}_d,\nu_\alpha)$ $(\alpha>-1)$ are equipped with the norm

$$\|f\|_{2,\alpha}=\left(\int_{\mathbb{B}_d}|f(z)|^2\,d\nu_\alpha(z)\right)^{\frac{1}{2}}\quad\left(f\in A_\alpha^2\right)$$

where $\nu_\alpha = c_\alpha \left(1-|z|^2\right)^\alpha \nu$, ν is the normalized Lebesgue measure on $\mathbb{B}_d$ and c_α is a normalization constant turning ν_α into a probability measure. Let

$$R\colon \mathscr{O}(\mathbb{B}_d) \to \mathscr{O}(\mathbb{B}_d),\ (Rf)(z) = \sum_{\vartheta=1}^{d} z_\vartheta (\partial_\vartheta f)(z),$$

be the operator associating with each function in $\mathscr{O}(\mathbb{B}_d)$ its radial derivative. One can show ([10, Lemma 3.10]) that, for $N \in \mathbb{N}^*$, there are coefficients $c_{N,\alpha} \in \mathbb{N}$ $(0 < |\alpha| \le N)$ such that

$$\left(R^N f\right)(z) = \sum_{0<|\alpha|\le N} c_{N,\alpha} z^\alpha (\partial^\alpha f)(z) \quad (f \in \mathscr{O}(\mathbb{B}_d), z \in \mathbb{B}_d).$$

Let $\alpha \in \mathbb{R}$ and let $N \in \mathbb{N}$ be the smallest natural number such that $\alpha + 2N > -1$. The linear subspace

$$A^2_\alpha = \left\{f \in \mathscr{O}(\mathbb{B}_d);\ R^N f \in L^2(\mathbb{B}_d, \nu_{\alpha+2N})\right\} \subset \mathscr{O}(\mathbb{B}_d)$$

equipped with the norm

$$\|f\|_{2,\alpha} = |f(0)| + \left(\int_{\mathbb{B}_d} \left|R^N f(z)\right|^2 d\nu_{\alpha+2N}(z)\right)^{\frac{1}{2}}$$

becomes a continuously embedded Banach space $A^2_\alpha \subset \mathscr{O}(\mathbb{B}_d)$ [10, Satz 5.11]. Let us fix a real number $\alpha > -(d+1)$. One can show that

$$A^2_\alpha \times A^2_\alpha \to \mathbb{C}, \left\langle \sum_{m\in\mathbb{N}^d} a_m z^m, \sum_{m\in\mathbb{N}^d} b_m z^m \right\rangle = \sum_{m\in\mathbb{N}^d} \frac{m!\Gamma(d+1+\alpha)}{\Gamma(d+1+|m|+\alpha)} a_m \overline{b_m}$$

is a well-defined scalar product which turns A^2_α into a functional Hilbert space with reproducing kernel [10, p. 127]

$$K_\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{d+\alpha+1}} \quad (z,w \in \mathbb{B}_d).$$

By a standard application of the closed graph theorem the norm $\|\cdot\|_{2,\alpha}$ is equivalent to the norm of A^2_α as a functional Hilbert space with reproducing kernel K_α. In the following we consider A^2_α as a reproducing kernel Hilbert space. For $-(d+1) < \alpha \le -d$, the spaces A^2_α are known to be complete Nevanlinna-Pick spaces normalized at $z_0 = 0$. For $\alpha \ge -d$, the tuple $M_z \in L\left(A^2_\alpha\right)^d$ is a well-defined row contraction and hence $K_\alpha = K_{-d}\mathscr{g}$ with the normalized complete Nevanlinna-Pick kernel K_{-d} and a suitable positive definite kernel $\mathscr{g}$. The space A^2_{-d} is the Drury-Arveson space, the space $A^2_{-1} = H^2(\mathbb{B}_d)$ is the Hardy space on $\mathbb{B}_d$ and $A^2_0 = L^2_a(\mathbb{B}_d)$ is the unweighted Bergman space. It is elementary to check that the functional Hilbert spaces A^2_α with $\alpha > -(d+1)$ satisfy condition (3.1) with $K = 2$, $\mu_1 = \delta_0$, $\mu_2 = \nu_{\alpha+2N}$, $\mathscr{D} = A^2_\alpha$ and $L^{(1)}\colon A^2_\alpha \hookrightarrow L^2(\delta_0),\ f \mapsto f$, $L^{(2)} = R^N\colon A^2_\alpha \to L^2(\mathbb{B}_d, \nu_{\alpha+2N})$.

Proposition 3.1.3. *Let $a, b \in \mathbb{C}$ with $\operatorname{Re} a, \operatorname{Re} b > 0$, then*

$$\int_{-\infty}^{\infty} \left(\frac{1}{a - iy} + \frac{1}{b + iy} \right) dy = 2\pi.$$

Proof. We have

$$\frac{1}{a - iy} + \frac{1}{b + iy} = \frac{a + b}{(y + ia)(y - ib)}.$$

for all $y \in \mathbb{R}$. Since $\operatorname{Re} a, \operatorname{Re} b > 0$ the function

$$f \colon \{z \in \mathbb{C};\ \operatorname{Im} z > 0\} \to \mathbb{C},\ f(z) = \frac{1}{(z + ia)(z - ib)}$$

has only the simple pole $c = ib$. Thus, by a standard result from function theory we conclude

$$\operatorname{res}(f, c) = \lim_{z \to c} (z - c) f(z) = \lim_{z \to ib} \frac{1}{z + ia} = \frac{1}{i(a + b)}$$

Hence, by a corollary of the residue theorem we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{1}{a - iy} + \frac{1}{b + iy} \right) dy &= (a + b) \int_{-\infty}^{\infty} \frac{1}{(y + ia)(y - ib)} dy \\ &= (a + b)(2\pi i \operatorname{res}(f, c)) \\ &= 2\pi. \end{aligned}$$

□

In the following we denote by $H = \{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$ the right half plane in $\mathbb{C}$ and by

$$P \colon H \times \mathbb{R} \to \mathbb{R},\ P(z, t) = \frac{\operatorname{Re} z}{\pi \left((\operatorname{Re} z)^2 + (\operatorname{Im} z - t)^2 \right)}$$

the Poisson kernel on the right half plane. Let $v \colon \overline{H} \to \mathbb{R}$ be continuous, bounded and subharmonic on H. Since the map $\mathbb{C} \to \mathbb{C},\ z \mapsto -iz$ is holomorphic, the function

$$\tilde{v} \colon \overline{\{z \in \mathbb{C};\ \operatorname{Im} z > 0\}} \to \mathbb{R},\ \tilde{v}(z) = v\left(\overline{-iz}\right)$$

is continuous, bounded and subharmonic on $\{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$ by Proposition 4.0.7. Therefore

$$u_{\tilde{v}}(z) = \begin{cases} \int_{-\infty}^{\infty} \frac{\operatorname{Im} z}{\pi\left((\operatorname{Im} z)^2 + (\operatorname{Re} z - t)^2\right)} \tilde{v}(t)\, dt & \text{for } z \in \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}, \\ \tilde{v}(z) & \text{for } z \in \partial \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}. \end{cases}$$

is continuous, bounded with $\|\tilde{v}\|_\infty$ and harmonic on $\{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$ by [4, 7.3].

Theorem 3.1.4. *If $v\colon \overline{H} \to \mathbb{R}$ is continuous, bounded and subharmonic on H, then we have*

$$v(z) \leq \int_{-\infty}^{\infty} P(z,t)\, v(it)\, dt < \infty$$

for all $z \in H$.

Proof. Set $u\colon \overline{H} \to \mathbb{R}$,

$$u(z) = \begin{cases} \int_{-\infty}^{\infty} P(z,t)\, v(it)\, dt & \text{for } z \in H, \\ v(z) & \text{for } z \in \partial H. \end{cases}$$

Then an easy calculation shows that u is continuous, bounded by $\|v\|_\infty$ and harmonic on H, since $u(z) = u_{\tilde{v}}\left(\overline{-iz}\right)$ for all $z \in \overline{H}$. For $k \in \mathbb{N}_{\geq 1}$, we define

$$h_k\colon \overline{H} \to \mathbb{R},\; h_k(z) = v(z) - u(z) - \frac{1}{k}\log|z+1|\,.$$

Since $\log|z+1|$ and u are harmonic on H and v is subharmonic on H we deduce that h_k is subharmonic on H as sum of subharmonic functions. Next let $z_0 \in H$ be arbitrary. Set $A_k = e^{2\|v\|_\infty k}$ and

$$G = \{z \in \mathbb{C};\; 0 < \operatorname{Re} z < \max(2\operatorname{Re} z_0, A_k)\,, |\operatorname{Im} z| < \max(2|\operatorname{Im} z_0|\,, A_k)\}\,.$$

Then $z_0 \in G \subset H$ and G is simply connected. Since $v(z) - u(z) \leq 2\|v\|_\infty$ for all $z \in \overline{H}$ and

$$\frac{1}{k}\log|z+1| \geq \frac{1}{k}\log A_k = 2\|v\|_\infty$$

for all $z \in \partial G$ with $\operatorname{Re} z > 0$, we have $h_k(z) \leq 0$ for all $z \in \partial G$. Since h_k is continuous on $\overline{G}$ and subharmonic on G, we have $h_k(z) \leq 0$ for all $z \in G$ by [4, 11.3] and in particular $h_k(z_0) \leq 0$. Thus, we conclude

$$v(z_0) - u(z_0) = \lim_{k\to\infty} h_k(z_0) \leq 0.$$

Since $z_0 \in H$ was chosen arbitrary, the claim holds. □

Proposition 3.1.5. *Let $U \subset \mathbb{C}$ be open, $\mathscr{E}$ a Hilbert space, $f\colon U \to \mathscr{E}$ holomorphic and $p \in [1,\infty)$. Then, the function*

$$U \to \mathbb{R},\; z \to \|f(z)\|_{\mathscr{E}}^p$$

is continuous, bounded and subharmonic on U.

Proof. Let $w \in \mathscr{E}$ and $\overline{D_r(z_0)} \subset U$. Then we have by Cauchy's integral formula:

$$\begin{aligned}\langle f(z_0), w\rangle_{\mathscr{E}} &= \frac{1}{2\pi i}\int_{D_r(z_0)} \frac{\langle f(z,w)_{\mathscr{E}}\rangle}{z-z_0}dz \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\langle f\left(z_0 + re^{it}\right), w\right\rangle_{\mathscr{E}} dt.\end{aligned}$$

Thus, we conclude

$$\begin{aligned}|\langle f(z_0), w\rangle_{\mathscr{E}}| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\left\langle f\left(z_0 + re^{it}\right), w\right\rangle_{\mathscr{E}}\right| dt \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|f\left(z_0 + re^{it}\right)\right\|_{\mathscr{E}} dt\, \|w\|_{\mathscr{E}}\end{aligned}$$

and in particular

$$\|f(z_0)\|_{\mathscr{E}}^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|f\left(z_0 + re^{it}\right)\right\|_{\mathscr{E}} dt\, \|f(z_0)\|_{\mathscr{E}}.$$

Therefore

$$\|f(z_0)\|_{\mathscr{E}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|f\left(z_0 + re^{it}\right)\right\|_{\mathscr{E}} dt$$

and

$$U \to \mathbb{R},\ z \to \|f(z)\|_{\mathscr{E}}$$

is subharmonic. One can easily show that $\varphi\colon \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} x^p & \text{for } x > 0, \\ 0 & \text{for } x \leq 0 \end{cases}$$

is monotonically increasing and convex for $1 \leq p < \infty$. Hence

$$U \to \mathbb{R},\ z \to \|f(z)\|_{\mathscr{E}}^p = \|f(z)\|_{\mathscr{E}}^p \quad (1 \leq p < \infty)$$

is subharmonic. □

Proposition 3.1.6. *Let $g \in C^N(\Omega)$ be a function with $0 \notin g(\Omega)$. Then for $\beta \in \mathbb{N}^d$ with $|\beta| = j \in \{1,\ldots,N\}$ the partial derivative $\partial^\beta\left(\frac{1}{g}\right)$ is a linear combination of functions of the form*

$$\frac{\prod_{i=1}^r (\partial^{\alpha_i} g)}{g^{k+1}},$$

where $k \in \{1,\ldots,j\}$, $r \in \{1,\ldots,j\}$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{N}^d \setminus \{0\}$ are multindices with

$$|\alpha_1| + \cdots + |\alpha_r| = j.$$

Proof. This assertion follows by a finite induction on $j = 1, \dots, N$. $\square$

Remark 3.1.7. For $F \in \mathscr{H}_k(\mathscr{E})$ and $w \in \mathbb{C}$ with $\operatorname{Re} w > 0$ we have $\frac{1}{V_F + w} \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}}) \subset C^N(\Omega)$ by Remark 3.1.2 and thus for $\beta \in \mathbb{N}^d$ with $|\beta| = j \in \{1, \dots, N\}$, it follows that

$$\partial^\beta \left(\frac{1}{V_F + w} \right) = \sum_{k=1}^{j} \frac{1}{(V_F + w)^{k+1}} b_{\beta,k}$$

with suitable functions $b_{\beta,k} = b_{\beta,k}(F) \in C(\Omega)$ not depending on w.

Remark 3.1.8. For $F \in \mathscr{H}_k(\mathscr{E})$, $w \in \mathbb{C}$ with $\operatorname{Re} w > 0$ and $f = \sum_{i=1}^r f_i \otimes x_i \in \mathscr{D} \otimes_{\text{alg}} \mathscr{E}$, we find that

$$\begin{aligned}
&L_{\mathscr{E}}^{(l)} \left(\frac{f}{V_F + w} \right) \\
&= \sum_{i=1}^r L^{(l)} \left(\frac{f_i}{V_F + w} \right) x_i \\
&= \sum_{i=1}^r \sum_{|\alpha| \le N} a_\alpha^{(l)} \partial^\alpha \left(\frac{f_i}{V_F + w} \right) x_i \\
&= \sum_{i=1}^r \sum_{|\alpha| \le N} \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} a_\alpha^{(l)} \partial^\beta \left(\frac{1}{V_F + w} \right) \left(\partial^{\alpha-\beta} f_i \right) x_i \\
&= \frac{1}{V_F + w} L_{\mathscr{E}}^{(l)}(f) + \sum_{|\alpha| \le N} \sum_{0 \ne \beta \le \alpha} \binom{\alpha}{\beta} a_\alpha^{(l)} \partial^\beta \left(\frac{1}{V_F + w} \right) \left(\partial^{\alpha-\beta} f \right) \\
&= \frac{1}{V_F + w} L_{\mathscr{E}}^{(l)}(f) + \sum_{j=1}^N \sum_{|\beta| = j} \partial^\beta \left(\frac{1}{V_F + w} \right) \sum_{|\alpha| \le N, \beta \le \alpha} \binom{\alpha}{\beta} a_\alpha^{(l)} \left(\partial^{\alpha-\beta} f \right) \\
&= \frac{1}{V_F + w} L_{\mathscr{E}}^{(l)}(f) + \sum_{k=1}^N \frac{1}{(V_F + w)^{k+1}} L_{\mathscr{E}}^{(l,k)}(f),
\end{aligned}$$

with linear maps $L_{\mathscr{E}}^{(l,k)} \colon \mathscr{D} \otimes_{\text{alg}} \mathscr{E} \to \mathfrak{L}_0(\mu_l, \mathscr{E})$ acting as

$$L_{\mathscr{E}}^{(l,k)}(f) = \sum_{j=k}^N \sum_{|\beta| = j} b_{\beta,k} \sum_{|\alpha| \le N, \beta \le \alpha} \binom{\alpha}{\beta} a_\alpha^{(l)} \left(\partial^{\alpha-\beta} f \right)$$

Note that the operators $L_{\mathscr{E}}^{(l,k)} \colon \mathscr{D} \otimes_{\text{alg}} \mathscr{E} \to \mathfrak{L}_0(\mu_l, \mathscr{E})$ are partial differential operators acting as

$$L_{\mathscr{E}}^{(l,k)}(f) = \sum_{|\alpha| \le N-1} c_\alpha^{(l,k)} \partial^\alpha (f)$$

with suitably defined functions $c_\alpha^{(l,k)} = c_\alpha^{(l,k)}(F) \in \mathfrak{L}_0(\mu_l)$. Defining

$$L_{\mathscr{E}}^{(l,0)} = L_{\mathscr{E}}^{(l)} \colon \mathscr{D} \otimes_{\text{alg}} \mathscr{E} \to \mathfrak{L}^2(\mu_l, \mathscr{E}),$$

the above result can be written as

$$L_{\mathscr{E}}^{(l)}\left(\frac{f}{V_F+w}\right)=\sum_{k=0}^{N}\frac{1}{(V_F+w)^{k+1}}L_{\mathscr{E}}^{(l,k)}(f)\quad \left(l=1\ldots,K,f\in\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}\right).$$

Lemma 3.1.9. *Let $\Omega,\mu_l,L_{\mathscr{E}}^{(l)}$ for $l=1,\ldots,K$, $\mathscr{H}_{\mathfrak{k}}$ and $\mathscr{D}$ be as above. Let $b_0,\cdots,b_m\in\mathbb{C}$, $z_0,\cdots,z_m\in\Omega$, and $h=\sum_{i=0}^{m}b_is_{z_i}$. Let $F=\sum_{i=0}^{r}f_i\otimes x_i\in\mathscr{D}\otimes_{alg}\mathscr{E}$ with $f_0,\ldots,f_r\in\mathscr{D}$ and an orthonormal system $(x_i)_{i=0}^{r}$. By V_F we denote the Sarason function of F. Then we have*

$$\int_{\Omega}\frac{\left\|L_{\mathscr{E}}^{(l)}(hF)(z)\right\|_{\mathscr{E}}^{2}}{(\mathrm{Re}V_F(z)+3)^{2N+1}}d\mu_l(z)\le C_N\|h\|_{\mathscr{H}_{\mathfrak{s}}}^{2}$$

for some $C_N>0$ and all $l=1,\ldots,K$.

Proof. Fix $l\in\{l=1,\ldots,K\}$. Since $\mathrm{Re}V_F\ge 0$ and V_F is in $\mathrm{Mult}(\mathscr{H}_{\mathfrak{s}})\subset C^N(\Omega)$, the measurable function $\frac{1}{(\mathrm{Re}V_F+3)^{2N+1}}$ is bounded. By Remark 3.1.1, hF is in $\mathscr{D}\otimes_{\mathrm{alg}}\mathscr{E}$ and thus since $L_{\mathscr{E}}^{(l)}hF$ is square integrable the function $\frac{\left\|L_{\mathscr{E}}^{(l)}hF\right\|_{\mathscr{E}}^{2}}{(\mathrm{Re}V_F+3)^{2N+1}}$ is integrable. Note that $\frac{1}{V_F+\overline{a}}h\in\mathscr{H}_{\mathfrak{s}}$ by Remark 3.1.2. By Proposition 2.1.13 and an easy calculation we conclude for $a\in\mathbb{C}$ with $\mathrm{Re}\,a>0$

$$\begin{aligned}
&4\,\mathrm{Re}\,a\left\|\frac{1}{V_F+\overline{a}}hF\right\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^{2}\\
&\le\|h\|_{\mathscr{H}_{\mathfrak{s}}}^{2}-\left\|\frac{V_F-a}{V_F+\overline{a}}h\right\|_{\mathscr{H}_{\mathfrak{s}}}^{2}\\
&=4\,\mathrm{Re}\,a\,\mathrm{Re}\left\langle\frac{1}{V_F+\overline{a}}h,h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}-4(\mathrm{Re}\,a)^2\left\|\frac{1}{V_F+\overline{a}}h\right\|_{\mathscr{H}_{\mathfrak{s}}}^{2}\\
&\le 4\,\mathrm{Re}\,a\,\mathrm{Re}\left\langle\frac{1}{V_F+\overline{a}}h,h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}.
\end{aligned}$$

Hence, inequality (3.3) yields

$$\begin{aligned}
0&\le\int_{\Omega}\left\|L_{\mathscr{E}}^{(l)}\left(\frac{1}{V_F+\overline{a}}hF\right)(z)\right\|_{\mathscr{E}}^{2}d\mu_l(z)\\
&\le c_2\left\|\frac{1}{V_F+\overline{a}}hF\right\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^{2}\\
&\le c_2\,\mathrm{Re}\left\langle\frac{1}{V_F+\overline{a}}h,h\right\rangle_{\mathscr{H}_{\mathfrak{s}}}.
\end{aligned}$$

Next let $a = x + iy$ with $x > 0$ be arbitrary. By Proposition 3.2.7 we have

$$\int_{-\infty}^{\infty} \left(\frac{1}{V_F(z) + x - iy} + \frac{1}{\overline{V_F(w)} + x + iy} \right) dy = 2\pi.$$

for all $z, w \in \Omega$. We now want to show that $\mathbb{R} \times \Omega \to \mathbb{R}$,

$$(t,z) \mapsto \left\| L_{\mathscr{E}}^{(l)} \left(\frac{G}{V_F + x - it} \right)(z) \right\|_{\mathscr{E}}^2$$

is $(\lambda \times \mu_l)$-measurable where $G = \sum_{i=1}^{s} g_i \otimes y_i \in \mathscr{D} \otimes_{\text{alg}} \mathscr{E}$ is a fixed element with $g_1, \ldots, g_s \in \mathscr{D}$ and an orthonormal system $(y_i)_{i=1}^{s}$ in $\mathscr{E}$. Then $\frac{G}{V_F + x - it} = \sum_{i=1}^{s} \frac{g_i}{V_F + x - it} y_i \in \mathscr{D} \otimes_{\text{alg}} \mathscr{E}$ and

$$\left\| L_{\mathscr{E}}^{(l)} \left(\frac{G}{V_F + x - it} \right)(z) \right\|_{\mathscr{E}}^2 = \sum_{i=1}^{s} \left| L^{(l)} \left(\frac{g_i}{V_F + x - it} \right)(z) \right|^2.$$

To show that $\mathbb{R} \times \Omega \to \mathbb{R}$,

$$(t,z) \mapsto \left\| L_{\mathscr{E}}^{(l)} \left(\frac{G}{V_F + x - it} \right)(z) \right\|_{\mathscr{E}}^2$$

is $(\lambda \times \mu_l)$-measurable, it hence suffices to show that, for $g \in \mathscr{D}$, the function $\mathbb{R} \times \Omega \to \mathbb{C}$,

$$(t,z) \mapsto L^{(l)} \left(\frac{g}{V_F + x - it} \right)(z) = \sum_{|\alpha| \le N} a_{\alpha}^{(l)}(z)\, \partial^{\alpha} \left(\frac{g}{V_F + x - it} \right)(z)$$

is $(\lambda \times \mu_l)$-measurable. Because of

$$\partial^{\alpha} \left(\frac{g}{V_F + x - it} \right)(z) = \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} \partial^{\beta} \left(\frac{1}{V_F + x - it} \right)(z) \left(\partial^{\alpha - \beta} g \right)(z)$$

it suffices to show the $(\lambda \times \mu_l)$-measurability of $\mathbb{R} \times \Omega \to \mathbb{C}$,

$$(t,z) \mapsto \partial^{\beta} \left(\frac{1}{V_F + x - it} \right)(z).$$

But since

$$\frac{1}{V_F + x} \in \operatorname{Mult}(\mathscr{H}_{\mathfrak{s}}) \subset C^N(\Omega)$$

by Remark 3.1.2 the function $\mathbb{R} \times \Omega \to \mathbb{C}$,

$$(t,z) \mapsto \frac{1}{V_F + x - it}(z).$$

even belongs to $C^N(\mathbb{R}\times\Omega)$. Using that, $h=\sum_{i=0}^m b_i s_{z_i}$ and by Fubini's theorem we have

$$\begin{aligned}
&\int_\Omega\int_\mathbb{R}\left\|L_\mathscr{E}^{(l)}\left(\frac{1}{V_F+x-iy}hF\right)(z)\right\|_\mathscr{E}^2(y)\,d\lambda\,d\mu_l(z)\\
&=\int_\mathbb{R}\left(\int_\Omega\left\|L_\mathscr{E}^{(l)}\left(\frac{1}{V_F+x-iy}hF\right)(z)\right\|_\mathscr{E}^2 d\mu_l(z)\right)d\lambda(y)\\
&\le c_2\int_{-\infty}^{\infty}\operatorname{Re}\left\langle\frac{1}{V_F+x-iy}h,h\right\rangle_{\mathscr{H}_\mathbf{s}}dy\\
&=\frac{c_2}{2}\sum_{i,j=0}^m b_j\overline{b_i}s_{z_j}(z_i)\int_{-\infty}^{\infty}\left(\frac{1}{V_F(z_i)+x-iy}+\frac{1}{\overline{V_F(z_j)}+x+iy}\right)dy\\
&=c_2\pi\sum_{i,j=0}^m b_j\overline{b_i}s_{z_j}(z_i)\\
&=c_2\pi\|h\|_{\mathscr{H}_\mathbf{s}}^2.
\end{aligned}$$

Note that by Remark 3.1.8

$$L_\mathscr{E}^{(l)}\left(\frac{hF}{V_F+x-iy}\right)=\sum_{k=0}^N\frac{1}{(V_F+x-iy)^{k+1}}L_\mathscr{E}^{(l,k)}(hF).$$

Further we have for $z\in\Omega$, that the map

$$\begin{aligned}
&U_z\colon\overline{\{w\in\mathbb{C};\ \operatorname{Re}w>0\}}^\mathbb{C}\to\mathscr{E},\\
&U_z(w)=\sum_{k=0}^N\frac{1}{(V_F(z)+x+\overline{w})^k}L_\mathscr{E}^{(l,k)}(hF)(z)
\end{aligned}$$

is continuous. Due to $\operatorname{Re}V_F(z),\operatorname{Re}(\overline{w})\ge 0$ and $x>0$ it is also bounded by

$$\sum_{k=0}^N\frac{1}{x^k}\left\|L_\mathscr{E}^{(l,k)}(hF)(z)\right\|_\mathscr{E}.$$

The map

$$\{w\in\mathbb{C};\ \operatorname{Re}w>0\}\to\mathscr{E},\ w\mapsto U_z(\overline{w})$$

is analytic. Hence

$$\{w\in\mathbb{C};\ \operatorname{Re}w>0\}\to\mathbb{R}_{\ge0},\ w\mapsto\|U_z(\overline{w})\|_\mathscr{E}^2$$

subharmonic on $\{w\in\mathbb{C};\ \operatorname{Re}w>0\}$ by Proposition 3.1.5. Therefore

$$\|U_z\|_\mathscr{E}^2:\overline{\{w\in\mathbb{C};\ \operatorname{Re}w>0\}}^\mathbb{C}\to\mathbb{R}_{\ge0},\ \|U_z\|_\mathscr{E}^2(w)=\|U_z(w)\|_\mathscr{E}^2$$

is continuous, bounded and subharmonic on $\{w \in \mathbb{C};\ \operatorname{Re} w > 0\}$ by Proposition 4.0.7. Note that

$$\begin{aligned} P\colon \{w \in \mathbb{C};\ \operatorname{Re} w > 0\} \times \mathbb{R} \to \mathbb{R},\ P(w,y) &= \frac{\operatorname{Re} w}{\pi \left((\operatorname{Re} w)^2 + (\operatorname{Im} w - y)^2 \right)} \\ &= \frac{\operatorname{Re} w}{\pi |w - iy|^2} \end{aligned}$$

is the Poisson kernel for the right-half plane. Thus we have by Theorem 3.1.4

$$\|U_z(w)\|_{\mathscr{E}}^2 \leq \int_{-\infty}^{\infty} \frac{\operatorname{Re} w}{\pi |w - iy|^2} \|U_z(iy)\|_{\mathscr{E}}^2\, dy$$

for all $w \in \mathbb{C}$ with $\operatorname{Re} w > 0$. For $t, x \in \mathbb{R}$ with $2|t| \leq x$, $z \in \Omega$, we can use $\operatorname{Re} V_F(z) \geq 0$ to show that

$$\begin{aligned} &2|V_F(z) + x + it - iy|^2 - |V_F(z) + x - iy|^2 \\ &= \operatorname{Re} V_F(z)^2 + 2 \operatorname{Re} V_F(z) x + \left(x^2 - 2t^2\right) + (\operatorname{Im} V_F(z) - y + 2t)^2 \\ &\geq \operatorname{Re} V_F(z)^2 + 4 \operatorname{Re} V_F(z) |t| + 2|t|^2 + (\operatorname{Im} V_F(z) - y + 2t)^2 \\ &\geq 0. \end{aligned}$$

and hence

$$\frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x + it - iy|^2} \leq 2 \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x - iy|^2}.$$

Thus, we conclude for $x \in [1,2]$, $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and μ_l-almost every $z \in \Omega$:

$$\begin{aligned} &\int_{\mathbb{R}} \left\| L_{\mathscr{E}}^{(l)} \left(\frac{1}{V_F + x - iy} hF \right)(z) \right\|_{\mathscr{E}}^2 d\lambda(y) \\ &= \frac{\pi}{\operatorname{Re} V_F(z) + x} \int_{-\infty}^{\infty} \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x - iy|^2} \|U_z(iy)\|_{\mathscr{E}}^2\, dy \\ &\geq \frac{\pi}{2(\operatorname{Re} V_F(z) + x)} \int_{-\infty}^{\infty} \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x + it - iy|^2} \|U_z(iy)\|_{\mathscr{E}}^2\, dy \\ &\geq \frac{\pi}{2(\operatorname{Re} V_F(z) + x)} \|U_z(V_F(z) + x + it)\|_{\mathscr{E}}^2 \\ &= \frac{\pi}{2(\operatorname{Re} V_F(z) + x)} \left\| \sum_{k=0}^{N} (2 \operatorname{Re} V_F(z) + 2x - it)^{-k} L_{\mathscr{E}}^{(l,k)}(hF)(z) \right\|_{\mathscr{E}}^2 \\ &= \frac{\pi \left\| \sum_{k=0}^{N} (2 \operatorname{Re} V_F(z) + 2x - it)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)(z) \right\|_{\mathscr{E}}^2}{|2 \operatorname{Re} V_F(z) + 2x - it|^{2N}\, 2(\operatorname{Re} V_F(z) + x)} \\ &\geq \frac{\pi \left\| \sum_{k=0}^{N} (2 \operatorname{Re} V_F(z) + 2x - it)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)(z) \right\|_{\mathscr{E}}^2}{2^{2N+1} (\operatorname{Re} V_F(z) + 3)^{2N+1}}. \end{aligned}$$

Set $d\nu^{(l)} = \frac{d\mu_l}{(\operatorname{Re} V_F(z)+3)^{2N+1}}$. Since $\sum_{k=0}^{N}(2\operatorname{Re} V_F + 2x - it)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)$ is μ_l-measurable, we conclude that $\sum_{k=0}^{N}(2\operatorname{Re} V_F + 2x - it)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)$ is in $\mathfrak{L}^2\left(\nu^{(l)}, \mathscr{E}\right)$ and the estimate from above yields

$$\begin{aligned}
&\left\| \sum_{k=0}^{N} (2\operatorname{Re} V_F + 2x - it)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF) \right\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)}^2 \\
&= \int_\Omega \frac{\left\| \sum_{k=0}^{N} (2\operatorname{Re} V_F(z) + 2x - it)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)(z) \right\|_{\mathscr{E}}^2}{(\operatorname{Re} V_F(z) + 3)^{2N+1}} d\mu_l(z) \\
&\leq \frac{2^{2N+1}}{\pi} \int_\Omega \int_{\mathbb{R}} \left\| L_{\mathscr{E}}^{(l)} \left(\frac{1}{V_F + x - iy} hF \right)(z) \right\|_{\mathscr{E}}^2 d\lambda(y)\, d\mu_l(z) \\
&\leq c_2 2^{2N+1} \|h\|_{\mathscr{H}_{\mathfrak{s}}}^2 \\
&= A_N \|h\|_{\mathscr{H}_{\mathfrak{s}}}^2
\end{aligned}$$

for $(x,t) \in [1,2] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ with $A_N = c_2 2^{2N+1}$. For $u \in \mathfrak{L}^2\left(\nu^{(l)}, \mathscr{E}\right)$ we define

$$\begin{aligned}
&p_u \colon \Omega \times \left\{ w \in \mathbb{C}; \operatorname{Re} w \in [2,4], \operatorname{Im} w \in \left[-\frac{1}{2}, \frac{1}{2}\right] \right\} \to \mathbb{C}, \\
&p_u(z,w) = \left\langle \sum_{k=0}^{N} (2\operatorname{Re} V_F(z) + w)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)(z), u(z) \right\rangle_{\mathscr{E}}
\end{aligned}$$

This is a polynomial in w of degree $\leq N$ with $\left\langle L_{\mathscr{E}}^{(l)}(hF)(z), u(z) \right\rangle_{\mathscr{E}}$ as coefficient of w^N. For all $z \in \Omega$, we have by the Cauchy-Schwarz and the Hölder inequality

$$\begin{aligned}
&\int_\Omega |p_u(z,w)|\, d\nu^{(l)}(z) \\
&= \int_\Omega \left| \left\langle \sum_{k=0}^{N} (2\operatorname{Re} V_F(z) + w)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)(z), u(z) \right\rangle_{\mathscr{E}} \right| d\nu^{(l)}(z) \\
&\leq \int_\Omega \left\| \sum_{k=0}^{N} (2\operatorname{Re} V_F(z) + w)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF)(z) \right\|_{\mathscr{E}} \|u(z)\|_{\mathscr{E}}\, d\nu^{(l)}(z) \\
&\leq \left\| \sum_{k=0}^{N} (2\operatorname{Re} V_F + w)^{N-k} L_{\mathscr{E}}^{(l,k)}(hF) \right\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \|u\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \\
&\leq A_N^{\frac{1}{2}} \|u\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \|h\|_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

for all $w \in \mathbb{C}$ with $\operatorname{Re} w \in [2,4]$ and $\operatorname{Im} w \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Since $|p_u|$ is $\nu_l \times \lambda$-measurable we can apply Fubini's theorem and the Cauchy integral formula to conclude that:

$$\begin{aligned}
&\left|\int_\Omega \left\langle L^{(l)}_{\mathscr{E}}(hF)(z), u(z)\right\rangle_{\mathscr{E}} d\nu^{(l)}(z)\right| \\
&= \left|\int_\Omega \frac{p_u^{(N)}(z,3)}{N!} d\nu^{(l)}(z)\right| \\
&= \left|\int_\Omega \frac{1}{2\pi i}\int_{\partial D_{\frac{1}{2}}(3)} \frac{p_u(z,\xi)}{(\xi-3)^{N+1}} d\xi d\nu^{(l)}(z)\right| \\
&= \frac{2^{N-1}}{\pi}\left|\int_\Omega\int_{-\pi}^{\pi} p_u\left(z, 3+\frac{1}{2}e^{it}\right) e^{-iNt} dt d\nu^{(l)}(z)\right| \\
&\leq \frac{2^{N-1}}{\pi}\int_\Omega\int_{-\pi}^{\pi}\left| p_u\left(z, 3+\frac{1}{2}e^{it}\right)\right| dt d\nu^{(l)}(z) \\
&= \frac{2^{N-1}}{\pi}\int_{-\pi}^{\pi}\int_\Omega\left| p_u\left(z, 3+\frac{1}{2}e^{it}\right)\right| d\nu^{(l)}(z) dt \\
&\leq \frac{2^{N-1}}{\pi}\int_{-\pi}^{\pi} A_N^{\frac{1}{2}} \|u\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \|h\|_{\mathscr{H}_{\mathfrak{s}}} dt \\
&= 2^N A_N^{\frac{1}{2}} \|u\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \|h\|_{\mathscr{H}_{\mathfrak{s}}} \\
&= B_N \|u\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \|h\|_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

with $B_N = 2^N A_N^{\frac{1}{2}}$. From $L^{(l)}_{\mathscr{E}}(hF) \in \mathfrak{L}^2\left(\nu^{(l)},\mathscr{E}\right)$ and

$$\left\langle L^{(l)}_{\mathscr{E}}(hF), u\right\rangle_{\mathfrak{L}^2\left(\nu^{(l)},\mathscr{E}\right)} \leq B_N \|u\|_{L^2\left(\nu^{(l)},\mathscr{E}\right)} \|h\|_{\mathscr{H}_{\mathfrak{s}}}$$

for all $u \in \mathfrak{L}^2\left(\nu^{(l)},\mathscr{E}\right)$ it follows that

$$\left\|L^{(l)}_{\mathscr{E}}(hF)\right\|_{\mathfrak{L}^2\left(\nu^{(l)},\mathscr{E}\right)} \leq B_N \|h\|_{\mathscr{H}_{\mathfrak{s}}}$$

and therefore

$$\int_\Omega \frac{\left\|L^{(l)}_{\mathscr{E}}(hF)(z)\right\|^2_{\mathscr{E}}}{(\operatorname{Re} V_F(z)+3)^{2N+1}} d\mu_l(z) \leq B_N^2 \|h\|^2_{\mathscr{H}_{\mathfrak{s}}}$$

□

Theorem 3.1.10. *Let $\Omega, \mu_l, \tilde{L}_{\mathscr{E}}^{(l)}$, $\mathscr{H}_{\mathfrak{k}}$ and $\mathscr{D}$ be as above and assume that*

$$c_1 \|G\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \leq \sum_{l=1}^{K} \left\| \tilde{L}_{\mathscr{E}}^{(l)} G \right\|^2_{L^2(\mu_l, \mathscr{E})} \leq c_2 \|G\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}.$$

holds for all $G \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$. If $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and $\operatorname{Re} V_F$ is bounded on Ω, then $F \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$, and there exists a constant $c_N > 0$ depending only on N such that

$$\|F\|_{\operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)} \leq c_N \left(\|\operatorname{Re} V_F\|_\infty + 3\right)^{N+\frac{1}{2}}.$$

Proof. Let $F \in \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ be a function such that $\operatorname{Re} V_F$ is bounded on Ω. Let $h \in \mathscr{H}_{\mathfrak{s}}$. Since

$$\left\{ \sum_{i=1}^{r} f_i x_i;\ f_i \in \mathscr{D}, x_i \in \mathscr{E} \right\} \cong \mathscr{D} \otimes_{\text{alg}} \mathscr{E} \subset \mathscr{H}_{\mathfrak{k}} \otimes \mathscr{E} \cong \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$$

and $\operatorname{span}\{s_z;\ z \in \Omega\} \subset \mathscr{H}_{\mathfrak{s}}$ are dense there are sequences $(F_n)_{n\in\mathbb{N}}$ in $\{\sum_{i=1}^r f_i x_i;\ f_i \in \mathscr{D}, x_i \in \mathscr{E}\}$ and $(h_n)_{n\in\mathbb{N}}$ in $\operatorname{span}\{s_z;\ z \in \Omega\}$ with $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\text{-}\lim_{n\to\infty} F_n = F$ and $\mathscr{H}_{\mathfrak{s}}\text{-}\lim_{n\to\infty} h_n = h$. In this case $\operatorname{Re} V_{F_n}$ converges pointwise to $\operatorname{Re} V_F$. For $n \in \mathbb{N}$ and $l = 1, \ldots, K$ we define $d\nu_n^{(l)} = \frac{d\mu_l}{(\operatorname{Re} V_{F_n} + 3)^{2N+1}}$. Since the functions $\frac{1}{(\operatorname{Re} V_{F_n} + 3)^{2N+1}}$ are bounded μ_l-measurable functions the inclusion mappings $i_n \colon L^2\left(\mu^{(l)}, \mathscr{E}\right) \to L^2\left(\nu_n^{(l)}, \mathscr{E}\right)$, $[f] \mapsto [f]$ are continuous linear with $\|i_n\| \leq \left\| \frac{1}{(\operatorname{Re} V_{F_n} + 3)^{2N+1}} \right\|_{\infty, \mu_l}^{\frac{1}{2}} \leq 1$. Now let $\tilde{h} \in \operatorname{span}\{s_z;\ z \in \Omega\}$. Then $\tilde{h} \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$. Since the operators $M_{\tilde{h}} \colon \mathscr{H}_{\mathfrak{k}}(\mathscr{E}) \to \mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and $\tilde{L}_{\mathscr{E}}^{(l)} \colon \mathscr{H}_{\mathfrak{k}} \to L^2(\mu_l, \mathscr{E})$ are continuous we have:

$$\begin{aligned}
&\left| \left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F \right\|^2_{L^2\left(\nu_n^{(l)}, \mathscr{E}\right)} - \left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F_n \right\|^2_{L^2\left(\nu_n^{(l)}, \mathscr{E}\right)} \right| \\
&\leq \left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F - \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F_n \right\|_{L^2\left(\nu_n^{(l)}, \mathscr{E}\right)} \left(\left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F \right\|_{L^2\left(\nu_n^{(l)}, \mathscr{E}\right)} + \left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F_n \right\|_{L^2\left(\nu_n^{(l)}, \mathscr{E}\right)} \right) \\
&\leq \left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F - \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F_n \right\|_{L^2(\mu_l, \mathscr{E})} \left(\left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F \right\|_{L^2(\mu_l, \mathscr{E})} + \left\| \tilde{L}_{\mathscr{E}}^{(l)} \tilde{h} F_n \right\|_{L^2(\mu_l, \mathscr{E})} \right) \\
&\stackrel{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

Using the Lemma of Fatou we conclude:

$$
\begin{aligned}
&\limsup_{n\to\infty}\left|\int_\Omega\left(\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_F+3)^{2N+1}}-\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F_n\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_{F_n}+3)^{2N+1}}\right)d\mu_l\right|\\
&\le\limsup_{n\to\infty}\left|\int_\Omega\left(\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_F+3)^{2N+1}}-\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_{F_n}+3)^{2N+1}}\right)d\mu_l\right|\\
&\quad+\limsup_{n\to\infty}\left|\int_\Omega\left(\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_{F_n}+3)^{2N+1}}-\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F_n\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_{F_n}+3)^{2N+1}}\right)d\mu_l\right|\\
&\le\int_\Omega\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F\right)\right\|_{\mathscr{E}}^2\limsup_{n\to\infty}\left|\frac{1}{(\operatorname{Re}V_F+3)^{2N+1}}-\frac{1}{(\operatorname{Re}V_{F_n}+3)^{2N+1}}\right|d\mu_l\\
&\quad+\limsup_{n\to\infty}\left|\left\|\tilde{L}_{\mathscr{E}}^{(l)}\tilde{h}F\right\|_{L^2\left(\nu_n^{(l)},\Omega\right)}^2-\left\|\tilde{L}_{\mathscr{E}}^{(l)}\tilde{h}F_n\right\|_{L^2\left(\nu_n^{(l)},\Omega\right)}^2\right|\\
&=0
\end{aligned}
$$

Using Lemma 3.1.9, we see in particular that

$$
\begin{aligned}
&\int_\Omega\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_F+3)^{2N+1}}d\mu_l\\
&=\lim_{n\to\infty}\int_\Omega\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left(\tilde{h}F_n\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_{F_n}+3)^{2N+1}}d\mu_l\\
&\le C_N\left\|\tilde{h}\right\|_{\mathscr{H}_{\mathfrak{s}}}^2.
\end{aligned}
$$

Since $\tilde{h}\in\operatorname{span}\{s_z;\,z\in\Omega\}$ was arbitrary we have

$$
\int_\Omega\frac{\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left((h_l-h_k)F\right)\right\|_{\mathscr{E}}^2}{(\operatorname{Re}V_F+3)^{2N+1}}d\mu_l\le C_N\left\|h_l-h_k\right\|_{\mathscr{H}_{\mathfrak{s}}}^2
$$

for all $k,l\in\mathbb{N}$ and therefore by the inequality from the assumption

$$
\begin{aligned}
\left\|h_lF-h_kF\right\|_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})}^2&\le\frac{1}{c_1}\sum_{l=1}^{K}\int_\Omega\left\|\tilde{L}_{\mathscr{E}}^{(l)}\left((h_l-h_k)F\right)\right\|_{\mathscr{E}}^2d\mu_l\\
&\le\frac{C_NK}{c_1}\left(\left\|\operatorname{Re}V_F\right\|_\infty+3\right)^{2N+1}\left\|h_l-h_k\right\|_{\mathscr{H}_{\mathfrak{s}}}^2
\end{aligned}
$$

for all $k,l\in\mathbb{N}$. Hence $(h_lF)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ and

$$
g=\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\text{-}\lim_{l\to\infty}h_lF
$$

exists. Since the point evaluations on $\mathscr{H}_{\mathfrak{s}}$ and $\mathscr{H}_{\mathfrak{k}}(\mathscr{E})$ are continuous we have

$$g(z) = \mathscr{E}\text{-}\lim_{l\to\infty}(h_l(z)F(z)) = h(z)F(z)$$

for all $z \in \Omega$ and hence $hF = g = \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\text{-}\lim_{l\to\infty} h_l F$. We conclude $F \in \operatorname{Mult}\left(\mathscr{H}_{\mathfrak{s}}, \mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right)$ and

$$\begin{aligned}
\|hF\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} &= \lim_{l\to\infty}\|h_l F\|^2_{\mathscr{H}_{\mathfrak{k}}(\mathscr{E})} \\
&\le \lim_{l\to\infty} c_N\left(\|\operatorname{Re} V_F\|_\infty + 3\right)^{2N+1}\|h_l\|^2_{\mathscr{H}_{\mathfrak{s}}} \\
&= c_N\left(\|\operatorname{Re} V_F\|_\infty + 3\right)^{2N+1}\|h\|^2_{\mathscr{H}_{\mathfrak{s}}}
\end{aligned}$$

with $c_N = \frac{C_N K}{c_1}$. $\square$

3.2 Counterexample

We now want to give a counterexample for the reverse implication in Theorem 3.1.10. We show that there exist functions in certain weighted Dirichlet spaces, which are multipliers such that the real part of their Sarason function is unbounded.

In the following, we write $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ for the unit disk and $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ for the unit circle in $\mathbb{C}$. Further,

$$\mathscr{O}(\mathbb{D}) = \{f\colon \mathbb{D} \to \mathbb{C};\ f \text{ holomorphic}\}$$

should be the set of all holomorphic functions on $\mathbb{D}$ and dm the normalized arc length measure on the unit circle $\mathbb{T}$. Furthermore we denote by $\hat{f}(n)$ $(n \in \mathbb{Z})$ the Fourier coefficients of a function $f \in L^2(\mathbb{T})$, by

$$H^2(\mathbb{T}) = \left\{f \in L^2(\mathbb{T});\ \hat{f}(n) = 0 \text{ for } n < 0\right\}$$

the Hardy space on the unit circle, by

$$H^2(\mathbb{D}) = \left\{f \in \mathscr{O}(\mathbb{D});\ \sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^2\, dm(z) < \infty\right\}$$

the Hardy space on the unit disc and by $H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions on $\mathbb{D}$. Finally, we write dA for the normalized area measure on the unit disk $\mathbb{D}$.

Definition 3.2.1. Let $0 < \alpha < 1$. The vector spaces

$$D_\alpha = \left\{f \in H^2(\mathbb{D});\ \|f\|_\alpha^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1 - |z|^2\right)^\alpha dA(z) < \infty\right\}$$

equipped with the norm $\|\cdot\|_\alpha$ are called weighted Dirichlet spaces.

Let $d\mu_\alpha(z) = -\left(1 - |z|^2\right) \Delta \left(1 - |z|^2\right)^\alpha dA(z)$ where Δ denotes the Laplace operator. Then an easy calculation shows that

$$d\mu_\alpha = 4\alpha \left(1 - |z|^2\right)^\alpha + 4|z|^2 \alpha(1-\alpha) \left(1 - |z|^2\right)^{\alpha-1} dA(z)$$

is a finite positive Borel-measure on $\overline{\mathbb{D}}$. Let $H^2(\mathbb{D}) \to H^2(\mathbb{T}), f \mapsto [f^*]$ be the canonical isometric isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$. For $f \in H^2(\mathbb{D})$ and $\zeta \in \overline{\mathbb{D}}$, we define the local Dirichlet integral by

$$D_\zeta(f) = \begin{cases} \int_{\mathbb{T}} \left|\frac{f^*(z) - f(\zeta)}{z - \zeta}\right|^2 dm(z) & \text{if } \zeta \in \mathbb{D}, \\ \int_{\mathbb{T}} \left|\frac{f^*(z) - f^*(\zeta)}{z - \zeta}\right|^2 dm(z) & \text{if } \zeta \in \mathbb{T} \end{cases}.$$

Proposition 3.2.2. *For every $f \in D_\alpha$, we have*

$$\|f\|_\alpha^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{\overline{\mathbb{D}}} D_\zeta(f)\, d\mu_\alpha(\zeta).$$

Proof. For a proof see [18, Introduction] and note that $D_\alpha = D(\mu_\alpha)$ in [18, Introduction]. $\square$

Proposition 3.2.3. *For $0<\alpha<1$ the weighted Dirichlet space D_α is a functional Hilbert space $\mathscr{H}_{s^\alpha}$ given by a reproducing kernel of the form*

$$s^\alpha\colon \mathbb{D}\times\mathbb{D}\to\mathbb{C},\ s^\alpha(z,w) = 1+\sum_{n=0}^{\infty} c_n^\alpha (z\overline{w})^n$$

with coefficients $c_0^\alpha = 1$, $c_n^\alpha \in [0,\infty)$ for $n\geq 1$ such that the limit $\lim_{n\to\infty} \frac{c_n^\alpha}{(n+1)^{\alpha-1}} \in (0,\infty)$ exists. The induced norm coincides with $\|\cdot\|_\alpha$.

Notation 3.2.4. We denote by s_1^α the analytic function

$$\mathbb{D}\to\mathbb{C},\ z\mapsto 1+\sum_{n=1}^{\infty} c_n^\alpha z^n.$$

Note that $\lim_{w\to 1} s_w^\alpha(z) = \lim_{w\to 1} s_1^\alpha(\overline{w}z) = s_1^\alpha(z)$ holds for all $z\in\mathbb{D}$.

Proposition 3.2.5. *The map s^α is a normalized complete Nevanlinna-Pick kernel.*

Proof. For a proof note that $D_\alpha = D(\mu_\alpha)$ as it is shown in [18, Introduction] and see [19]. $\square$

Proposition 3.2.6. *For every $f\in D_\alpha$, we have*

$$\begin{aligned}\operatorname{Re} V_f(z) &= \int_{\mathbb{T}} \frac{\left(1-|z|^2\right)|f(\zeta)|^2}{\left|1-\overline{\zeta}z\right|^2} dm(\zeta) \\ &\quad + \int_{\overline{\mathbb{D}}} \left(2\operatorname{Re} s_z^\alpha(\zeta)-1\right) D_\zeta(f)\, d\mu_\alpha(\zeta) \\ &\geq 2\int_{\overline{\mathbb{D}}} \operatorname{Re}\left(s_z^\alpha\right)(\zeta) D_\zeta(f)\, d\mu_\alpha(\zeta) - \|f\|_\alpha^2 .\end{aligned}$$

Proof. For a proof see [18, Proposition 4 and Corollary 3]. $\square$

Proposition 3.2.7. *For all $z,w\in\mathbb{D}$ we have $\operatorname{Re} \mathbf{s}_w^\alpha(z) \geq \frac{1}{2}$.*

Proof. For a proof see [18, Theorem 2]. $\square$

Lemma 3.2.8. *(i) We have $\left|1-\frac{1}{\mathbf{s}_1^\alpha(z)}\right| < 1$ for all $z\in\mathbb{D}$ and $\left(1-\frac{1}{\mathbf{s}_1^\alpha}\right)\in \operatorname{Mult}(D_\alpha)$ with $\left\|M_{\left(1-\frac{1}{\mathbf{s}_1^\alpha}\right)}\right\|_{L(D_\alpha)} \leq 1$.*

(ii) The maps $\mathbf{s}_1^\alpha$ and $\left(\mathbf{s}_1^\alpha\right)'$ are non-negative on $[0,1)$ and

$$\lim_{r\to 1}\frac{s_1^{\alpha}(r)}{(1-r)^{-\alpha}}\equiv const,\quad \lim_{r\to 1}\frac{\left(s_1^{\alpha}\right)'(r)}{(1-r)^{-\alpha-1}}\equiv const,$$

where the constants involved are positive real numbers and only depend on α.

(iii) There exist $\varepsilon_\alpha\in(0,1)$ *and* $\delta_\alpha>0$ *such that*

$$\operatorname{Re} s_1^{\alpha}(z)\geq \delta_\alpha(1-|z|)^{-\alpha}$$

whenever z *belongs to the set* $S=\{z\in\mathbb{D};\ |z-|z||<\varepsilon_\alpha(1-|z|)\}$.

Proof. (i) Since s^{α} is a normalized complete Nevanlinna-Pick kernel, there are functions $u_n\colon\mathbb{D}\to\mathbb{C}$ $(n\in\mathbb{N})$ with

$$s^{\alpha}(z,w)=\frac{1}{1-\sum_{n=0}^{\infty}u_n(z)\overline{u_n(w)}}\qquad(z,w\in\mathbb{D}).$$

and $\left(\sum_{n=0}^{\infty}|u_n(z)|^2\right)^{\frac{1}{2}}<1$ for all $z\in\mathbb{D}$. By Cauchy-Schwarz we have

$$\begin{aligned}\left|\sum_{n=0}^{\infty}u_n(z)\overline{u_n(w)}\right| &\leq\left(\sum_{n=0}^{\infty}|u_n(z)|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}|u_n(w)|^2\right)^{\frac{1}{2}}\\ &\leq\left(\sum_{n=0}^{\infty}|u_n(z)|^2\right)^{\frac{1}{2}}\end{aligned}$$

for all $z,w\in\mathbb{D}$. Hence

$$\begin{aligned}\left|1-\frac{1}{s_1^{\alpha}(z)}\right| &=\lim_{w\to 1}\left|1-\frac{1}{s_1^{\alpha}(z\overline{w})}\right|\\ &=\lim_{w\to 1}\left|\sum_{n=0}^{\infty}u_n(z)\overline{u_n(w)}\right|\\ &\leq\left(\sum_{n=0}^{\infty}|u_n(z)|^2\right)^{\frac{1}{2}}\\ &<1.\end{aligned}$$

for all $z\in\mathbb{D}$. Now, let $h\in D_\alpha$. Using the notations from Lemma 2.1.1 we then have for all $z,w\in\mathbb{D}$:

$$\begin{aligned}\left(1-\frac{1}{s_w^{\alpha}(z)}\right)h(z) &=\sum_{n=0}^{\infty}u_n(z)\overline{u_n(w)}h(z)\\ &=\mathfrak{U}(z)\left(\left(\overline{u_n(w)}h(z)\right)_{n\in\mathbb{N}}\right)\\ &=M_{\mathfrak{U}}\left(\left(\overline{u_n(w)}h\right)_{n\in\mathbb{N}}\right)(z).\end{aligned}$$

Hence

$$\left(1-\frac{1}{\boldsymbol{s}_w^{\alpha}}\right)h = M_{\mathfrak{U}}\left(\left(\overline{u_n(w)}h\right)_{n\in\mathbb{N}}\right)\in D_\alpha,$$

that is $\left(1-\frac{1}{\boldsymbol{s}_w^{\alpha}}\right)\in \operatorname{Mult}(D_\alpha)$ for all $w\in\mathbb{D}$. Since $\|M_{\mathfrak{U}}\|_{L\left(\mathscr{H}_{\boldsymbol{s}^\alpha}(l^2),\mathscr{H}_{\boldsymbol{s}^\alpha}\right)}\le 1$ we further conclude

$$\begin{aligned}
\left\|\left(1-\frac{1}{s_w^{\alpha}}\right)h\right\|^2_{\mathscr{H}_{\boldsymbol{s}^\alpha}} &\le \left\|\left(\overline{u_n(w)}h\right)_{n\in\mathbb{N}}\right\|^2_{\mathscr{H}_{\boldsymbol{s}^\alpha}(l^2)}\\
&= \sum_{n=0}^{\infty}|u_n(w)|^2\,\|h\|^2_{\mathscr{H}_{\boldsymbol{s}^\alpha}}\\
&\le \|h\|^2_{\mathscr{H}_{\boldsymbol{s}^\alpha}}
\end{aligned}$$

for all $w\in\mathbb{D}$ and $h\in\mathscr{H}_{\boldsymbol{s}^\alpha}=D_\alpha$. Hence we have $\left\|M_{\left(1-\frac{1}{\boldsymbol{s}_w^{\alpha}}\right)}\right\|_{L(D_\alpha)}\le 1$ for all $w\in\mathbb{D}$. Because

$$\lim_{w\to 1}\left(1-\frac{1}{s_w^{\alpha}}\right)(z)=\left(1-\frac{1}{s_1^{\alpha}}\right)(z)$$

holds for all $z\in\mathbb{D}$ and $\left\|1-\frac{1}{s_w^{\alpha}}\right\|_{L(D_\alpha)}\le 1$ holds for all $w\in\mathbb{D}$, we can use Corollary 4.0.2 to find that $\left(1-\frac{1}{\boldsymbol{s}_1^{\alpha}}\right)\in\operatorname{Mult}(D_\alpha)$ with $\left\|M_{\left(1-\frac{1}{\boldsymbol{s}_1^{\alpha}}\right)}\right\|_{L(D_\alpha)}\le 1.$

(ii) The maps $\boldsymbol{s}_1^{\alpha}$ and $\left(\boldsymbol{s}_1^{\alpha}\right)'$ are positive on $[0,1)$ by definition. By L'Hôpital's rule, $\lim_{r\to 1}\frac{-\log(r)}{1-r}=\lim_{r\to 1}\frac{1}{r}$ holds. For $s<1$ we have by [20, Eq. (9.3)] that

$$\begin{aligned}
\lim_{r\to 1}\frac{\sum_{n=1}^{\infty}n^{-s}r^n}{(1-r)^{s-1}} &= \lim_{r\to 1}\frac{Li_s(r)}{(1-r)^{s-1}}\\
&= \lim_{r\to 1}\frac{Li_s\left(e^{\log(r)}\right)(-\log(r))^{s-1}}{(-\log(r))^{s-1}(1-r)^{s-1}}\\
&= \lim_{r\to 1}\frac{Li_s\left(e^{\log(r)}\right)}{(-\log(r))^{s-1}r^{s-1}}\\
&= \Gamma(1-s)
\end{aligned}$$

where Γ denotes the Gamma function and $Li_s(r)=\sum_{n=1}^{\infty}n^{-s}r^n$ the polylogarithm. For

$\gamma \in (0,2)$ we conclude for $s = 1-\gamma < 1$

$$\begin{aligned}
\lim_{r\to 1} \frac{1+\sum_{n=1}^{\infty}(n+1)^{\gamma-1} r^n}{(1-r)^{-\gamma}} &= \lim_{r\to 1} \frac{1+\sum_{n=1}^{\infty}(n+1)^{-s} r^n}{(1-r)^{s-1}} \\
&= \lim_{r\to 1} \frac{\sum_{n=1}^{\infty} n^{-s} r^n}{r(1-r)^{s-1}} \\
&= \Gamma(1-s) \\
&= c_{1\gamma}
\end{aligned}$$

with $c_{1\gamma} = \Gamma(\gamma)$. We have

$$s_1^{\alpha}(r) = \sum_{n=0}^{\infty} c_n^{\alpha} r^n \text{ and } (s_1^{\alpha})'(r) = \sum_{n=0}^{\infty} c_{n+1}^{\alpha}(n+1) r^n.$$

for all $r \in [0,1)$. Setting $c = \lim_{n\to\infty} \frac{c_n^{\alpha}}{(n+1)^{\alpha-1}} \in \mathbb{R}_{>0}$, we have

$$\lim_{n\to\infty} \frac{(n+1)c_{n+1}^{\alpha}}{(n+1)^{\alpha}} = \lim_{n\to\infty} \frac{c_{n+1}^{\alpha}}{(n+2)^{\alpha-1}} \left(\frac{n+2}{n+1}\right)^{\alpha-1} = c.$$

Define $c_n^{\alpha+1} = (n+1)c_{n+1}^{\alpha}$ for $n \geq 0$. We finish the proof by establishing that

$$\frac{\sum_{n=0}^{\infty} c_n^{\gamma} r^n}{(1-r)^{-\gamma}} \xrightarrow{r\to 1} c_{1\gamma} c.$$

For this, let $\varepsilon > 0$ be arbitrary. Then for $\gamma \in \{\alpha, \alpha+1\}$, there exists an index $N_{\varepsilon} > 0$ such that

$$\left| \frac{c_n^{\gamma}}{(n+1)^{\gamma-1}} - c \right| < \frac{\varepsilon}{c_{1\gamma}}$$

holds for all $n \geq N_\varepsilon + 1$. Hence we have

$$
\begin{aligned}
&\limsup_{r\to 1}\left|\frac{\sum_{n=0}^{\infty} c_n^\gamma r^n}{(1-r)^{-\gamma}} - c_{1\gamma} c\right| \\
&\leq \limsup_{r\to 1}\left|\frac{c_0^\gamma - c}{(1-r)^{-\gamma}}\right| + \limsup_{r\to 1}\left|\frac{\sum_{n=1}^{\infty}\left(\frac{c_n^\gamma}{(n+1)^{\gamma-1}} - c\right)(n+1)^{\gamma-1} r^n}{(1-r)^{-\gamma}}\right| \\
&\quad + \limsup_{r\to 1}\left|\frac{c\left(1+\sum_{n=1}^{\infty}(n+1)^{\gamma-1} r^n\right)}{(1-r)^{-\gamma}} - c_{1\gamma} c\right| \\
&\leq \limsup_{r\to 1}\frac{\sum_{n=1}^{N_\varepsilon}\left|\frac{c_n^\gamma}{(n+1)^{\gamma-1}} - c\right|(n+1)^{\gamma-1} r^n}{(1-r)^{-\gamma}} \\
&\quad + \limsup_{r\to 1}\frac{\sum_{n=N_\varepsilon+1}^{\infty}\left|\frac{c_n^\gamma}{(n+1)^{\gamma-1}} - c\right|(n+1)^{\gamma-1} r^n}{(1-r)^{-\gamma}} \\
&\leq \limsup_{r\to 1}\frac{\frac{\varepsilon}{c_{1\gamma}}\left(1+\sum_{n=1}^{\infty}(n+1)^{\gamma-1} r^n\right)}{(1-r)^{-\gamma}} \\
&= \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was chosen arbitrary, the claim follows.
(iii) By (ii) the function

$$[0,1) \to \mathbb{R},\ r \mapsto \frac{\left(s_1^\alpha\right)'(r)}{(1-r)^{-\alpha-1}}$$

has a continuous extension to the compact intervall $[0,1]$. Hence there exists a constant $M_\alpha > 0$ such that

$$\frac{\left(s_1^\alpha\right)'(|z|)}{(1-|z|)^{-\alpha-1}} \leq M_\alpha$$

holds for all $z \in \mathbb{D}$. Therefore we have

$$\left|\left(s_1^\alpha\right)'(z)\right| \leq \left(s_1^\alpha\right)'(|z|) \leq M_\alpha (1-|z|)^{-\alpha-1}$$

for all $z \in \mathbb{D}$. If $0 < \varepsilon < 1$ and $|z - |z|| < \varepsilon(1-|z|)$, it follows that

$$
\begin{aligned}
|\operatorname{Re} s_1^\alpha(z) - \operatorname{Re} s_1^\alpha(|z|)| &\leq |s_1^\alpha(z) - s_1^\alpha(|z|)| \\
&= \left|\int_{[|z|,z]} \left(s_1^\alpha\right)'(\xi)\, d\xi\right| \\
&\leq |z - |z|| \sup_{t\in[0,1]} \left|\left(s_1^\alpha\right)'(t|z| + (1-t)z)\right| \\
&\leq M_\alpha \varepsilon (1-|z|)^{-\alpha}.
\end{aligned}
$$

Here we used the obvious estimate

$$1-|z| \leq 1-|t\,|z|+(1-t)z| \quad (z \in \mathbb{D}, t \in [0,1]).$$

As above we conclude with (ii) that the function

$$[0,1) \to \mathbb{R},\; r \mapsto \frac{s_1^{\alpha}(r)}{(1-r)^{-\alpha}}$$

has a continuous extension to the compact intervall $[0,1]$. Hence there exists a constant $C_{\alpha} > 0$ such that

$$\frac{s_1^{\alpha}(|z|)}{(1-|z|)^{-\alpha}} \geq C_{\alpha}$$

for all $z \in \mathbb{D}$. Using $\operatorname{Re} s_1^{\alpha}(|z|) = s_1^{\alpha}(|z|) \geq 1$ and setting $\varepsilon_{\alpha} = \min\left(\frac{C_{\alpha}}{2(M_{\alpha}+1)}, \frac{1}{2}\right)$ we have

$$\varepsilon_{\alpha} M_{\alpha}(1-|z|)^{-\alpha} \leq \frac{1}{2} \operatorname{Re} s_1^{\alpha}(|z|).$$

for all $z \in \mathbb{D}$. Therefore we conclude

$$\begin{aligned} 2\varepsilon_{\alpha} M_{\alpha}(1-|z|)^{-\alpha} - \operatorname{Re} s_1^{\alpha}(z) &\leq |\operatorname{Re} s_1^{\alpha}(z) - \operatorname{Re} s_1^{\alpha}(|z|)| \\ &\leq M_{\alpha}\varepsilon_{\alpha}(1-|z|)^{-\alpha}, \end{aligned}$$

and consequently

$$\operatorname{Re} s_1^{\alpha}(z) \geq M_{\alpha}\varepsilon_{\alpha}(1-|z|)^{-\alpha},$$

for all $z \in \mathbb{D}$ with $|z-|z|| < \varepsilon_{\alpha}(1-|z|)$. □

The following inequality is essential for the proof that certain multipliers of the weighted Dirichlet spaces D_{α} with $0 < \alpha < 1$ have a Sarason function with unbounded real part.

Lemma 3.2.9. *Let S be the set from Lemma 3.2.8 (iii). There exists a real number $c_{\alpha} > 0$, such that for all $f \in D_{\alpha}$,*

$$\|f\|_{\alpha}^2 + \sup_{z\in\mathbb{D}} \operatorname{Re} V_f(z) \geq c_{\alpha} \int_S |f'(\zeta)|^2 \, dA(\zeta).$$

Proof. Let $f \in D_{\alpha}$. By Proposition 3.2.6 we have

$$\|f\|_{\alpha}^2 + \operatorname{Re} V_f(z) \geq \int_{\mathbb{D}} 2 \operatorname{Re} s_z^{\alpha}(\zeta) D_{\zeta}(f) \, d\mu_{\alpha}(\zeta).$$

for all $z \in \mathbb{D}$. Since $\operatorname{Re} s_z^\alpha \geq \frac{1}{2}$ for all $z \in \mathbb{D}$ by Proposition 3.2.7 we can use Fatou's lemma to conclude

$$\begin{aligned}\|f\|_\alpha^2 + \sup_{z \in \mathbb{D}} \operatorname{Re} V_f(z) &\geq \liminf_{z \to 1} \int_{\mathbb{D}} 2 \operatorname{Re} s_z^\alpha(\zeta) D_\zeta(f)\, d\mu_\alpha(\zeta) \\ &\geq \int_{\mathbb{D}} 2 \operatorname{Re} s_1^\alpha(\zeta) D_\zeta(f)\, d\mu_\alpha(\zeta).\end{aligned}$$

Let $\zeta \in \mathbb{D}$ and set

$$\tilde{g}_\zeta : \mathbb{D} \setminus \{\zeta\} \to \mathbb{C},\ \tilde{g}_\zeta(z) = \frac{f(z) - f(\zeta)}{z - \zeta}.$$

By Riemann's theorem on removable singularities the function $\tilde{g}_\zeta$, has a holomorphic extension g_ζ on $\mathbb{D}$. Since g_ζ is continuous we have $g_\zeta(\zeta) = f'(\zeta)$ and since $D_\zeta(f) < \infty$ we get $g_\zeta \in H^2(\mathbb{D})$. Note that $\frac{1-|z|^2}{|z-\zeta|} \leq 2$ for all $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$. Thus, by the Cauchy integral formula, we have

$$\begin{aligned}\frac{1}{2}\left(1 - |z|^2\right)|g(rz)|^2 &= \frac{1}{4\pi}\left|\int_{-\pi}^{\pi} \frac{1-|z|^2}{e^{it} - z} g\left(re^{it}\right)^2 ie^{it}\, dt\right| \\ &\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1-|z|^2}{|e^{it} - z|}\left|g\left(re^{it}\right)\right|^2 dt \\ &\leq \|g\|_{H^2(\mathbb{D})}^2\end{aligned}$$

for all $g \in H^2(\mathbb{D})$, $r \in [0,1)$ and $z \in \mathbb{D}$. For $r \to 1$ and $g = g_\zeta$ from above we obtain that

$$\begin{aligned}\frac{1}{2}\left(1 - |\zeta|^2\right)\left|f'(\zeta)\right|^2 &= \frac{1}{2}\left(1 - |\zeta|^2\right)\left|g_\zeta(\zeta)\right|^2 \\ &\leq \left\|g_\zeta\right\|_{H^2(\mathbb{D})}^2 \\ &= D_\zeta(f)\end{aligned}$$

for all $\zeta \in \mathbb{D}$. Hence we conclude

$$\|f\|_\alpha^2 + \sup_{z \in \mathbb{D}} \operatorname{Re} V_f(z) \geq \int_{\mathbb{D}} \operatorname{Re} s_1^\alpha(\zeta)\left(1 - |\zeta|^2\right)\left|f'(\zeta)\right|^2 d\mu_\alpha(\zeta).$$

Since

$$d\mu_\alpha(z) = 4\alpha\left(1 - |z|^2\right)^\alpha + 4|z|^2\alpha(1-\alpha)\left(1 - |z|^2\right)^{\alpha - 1} dA(z)$$

we conclude, using Lemma 3.2.8 (iii) and $Res_1^\alpha \geq 0$:

$$\begin{aligned}
&\int_{\mathbb{D}} \operatorname{Re} s_1^\alpha(\zeta)\left(1-|\zeta|^2\right)|f'(\zeta)|^2 d\mu_\alpha(\zeta)\\
&= \int_{\mathbb{D}} \operatorname{Re} s_1^\alpha(\zeta)\left(1-|\zeta|^2\right)|f'(\zeta)|^2\\
&\quad \left[4\alpha\left(1-|\zeta|^2\right)^{\alpha}+4|\zeta|^2\alpha(1-\alpha)\left(1-|\zeta|^2\right)^{\alpha-1}\right] dA(\zeta)\\
&\geq \int_S \operatorname{Re} s_1^\alpha(\zeta)|f'(\zeta)|^2\\
&\quad \left[4\alpha\left(1-|\zeta|^2\right)^{\alpha+1}+4|\zeta|^2\alpha(1-\alpha)\left(1-|\zeta|^2\right)^{\alpha}\right] dA(\zeta)\\
&\geq \delta_\alpha \int_S |f'(\zeta)|^2(1-|\zeta|)^{-\alpha}\\
&\quad \left[4\alpha\left(1-|\zeta|^2\right)^{\alpha+1}+4|\zeta|^2\alpha(1-\alpha)\left(1-|\zeta|^2\right)^{\alpha}\right] dA(\zeta)\\
&= \delta_\alpha \int_S |f'(\zeta)|^2\\
&\quad \left[4\alpha\left(1-|\zeta|^2\right)(1+|\zeta|)^{\alpha}+4|\zeta|^2\alpha(1-\alpha)(1+|\zeta|)^{\alpha}\right] dA(\zeta)\\
&\geq \delta_\alpha \int_S |f'(\zeta)|^2\left[4\alpha(1-\alpha)\left(1-|\zeta|^2\right)+4|\zeta|^2\alpha(1-\alpha)\right] dA(\zeta)\\
&= 4\alpha(1-\alpha)\delta_\alpha \int_S |f'(\zeta)|^2 dA(\zeta).
\end{aligned}$$

With $c_\alpha = 4\alpha(1-\alpha)\delta_\alpha$ the claim follows. □

Remark 3.2.10. Let $0<\alpha<1$ and $h \in \operatorname{Mult}(D_\alpha)$ with $M_h^* \xrightarrow{SOT} 0$. Then we can show as in [14, Lemma 2.26] that there exists a w^*-continuous algebra homomorphism

$$\Phi\colon H^\infty(\mathbb{D}) \to L(D_\alpha)$$

with $\|\Phi\|=1$, $\Phi(1)=1$, and $\Phi(f)=M_{f\circ h}$ for all $f \in H^\infty(\mathbb{D})$.

Lemma 3.2.11. *Let B be an infinite interpolating Blaschke product (cf.* [9, Chapter 7]*) with zero set $\{z_n;\, n\in\mathbb{N}\} \subset [0,1)$, that is, B is a function $\mathbb{D}\to\mathbb{C}$ with*

$$B(z) = \prod_{n\in\mathbb{N}} \frac{z-z_n}{1-z_n z} \quad (z\in\mathbb{D})$$

and for the functions

$$B_k\colon \mathbb{D}\to\mathbb{C},\; B_k(z) = \prod_{n\in\mathbb{N}, n\neq k} \frac{z-z_n}{1-z_n z} \quad (k\in\mathbb{N})$$

there exists $\delta>0$ such that $\inf_{k\in\mathbb{N}}|B_k(z_k)| \geq \delta$. *Then we have*

$$|B'(z_k)| \geq \frac{\delta}{2(1-z_k)}$$

for all $k\in\mathbb{N}$.

Proof. Set $b_k\colon \mathbb{D} \to \mathbb{C},\ b_k(z) = \frac{z-z_k}{1-z_k z}$ for $k \in \mathbb{N}$. Then we conclude that

$$B'(z) = b_k'(z) B_k(z) + b_k(z) B_k'(z).$$

for all $z \in \mathbb{D}$. Let $k \in \mathbb{N}$. Since $b_k(z_k) = 0$, $|B_k(z_k)| \geq \delta$ and $z_k \in [0,1)$ we have

$$\begin{aligned} \left|B'(z_k)\right| &= \left|b_k'(z_k)\right| |B_k(z_k)| \\ &\geq \delta \left|b_k'(z_k)\right| \\ &= \frac{\delta}{1-z_k^2} \\ &\geq \frac{\delta}{2(1-z_k)}. \end{aligned}$$

□

Proposition 3.2.12. *For* $0 < \alpha < 1$*, there exists* $u \in \operatorname{Mult}(D_\alpha)$ *such that* $\operatorname{Re} V_u$ *is unbounded in* $\mathbb{D}$.

Proof. Let $f = \left(1 - \frac{1}{s_1^\alpha}\right)$. Then Lemma 3.2.8 (i) yields $f \in \operatorname{Mult}(D_\alpha)$ with $\left\|M_f\right\|_{L(D_\alpha)} \leq 1$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$. Hence we conclude

$$D_\alpha\text{-}\lim_{n\to\infty} \left(M_f^*\right)^n \mathbf{s}_z^\alpha = D_\alpha\text{-}\lim_{n\to\infty} \overline{f(z)}^n \mathbf{s}_z^\alpha = 0.$$

Therefore the inequality $\left\|\left(M_f^*\right)^n\right\|_{L(D_\alpha)} \leq 1$ for all $n \in \mathbb{N}$ and the fact that $D_\alpha = \bigvee\left(s_z^\alpha;\ z \in \mathbb{D}\right)$ yield $\tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty}\left(M_f^*\right)^n = 0$. Using the map $\Phi\colon H^\infty(\mathbb{D}) \to L(\mathscr{H}_{\mathbf{s}})$ induced by f from Remark 3.2.10 we have $M_{g\circ f} = \Phi(g) \in L(D_\alpha)$ and hence $g \circ f \in \operatorname{Mult}(D_\alpha)$ with $\left\|M_{g\circ f}\right\|_{L(D_\alpha)} \leq \|\Phi\| \|g\|_\infty = \|g\|_\infty$ for all $g \in H^\infty(\mathbb{D})$ $(*)$. We claim that for an infinite interpolating Blaschke product B with zero set $\{z_n;\ n \in \mathbb{N}\} \subset [0,1)$ the function $\operatorname{Re} V_{B\circ f}$ is unbounded in $\mathbb{D}$. From the Blaschke condition $\sum_{n=0}^\infty (1-|z_n|) < \infty$ we get $\lim_{n\to\infty} z_n = 1$. By Lemma 3.2.8 (ii) we have

$$\lim_{r\to 1}\left(1 - \frac{1}{s_1^\alpha(r)}\right) = \lim_{r\to 1}\left(1 - \frac{(1-r)^{-\alpha}(1-r)^\alpha}{s_1^\alpha(r)}\right) = 1.$$

Since $f(0) = 0$ and $(f(r))_{0<r<1}$ is increasing, it follows that $f([0,1)) = [0,1)$ by the intermediate value theorem. Thus, we can choose a sequence $(w_n)_{n\in\mathbb{N}}$ in $[0,1)$ with $\lim_{n\to\infty} w_n = 1$ and $f(w_n) = z_n$ for all $n \in \mathbb{N}$. With the notations from Lemma 3.2.8 (iii) we choose a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ with

$$w_{n_{k+1}} > \frac{2\varepsilon_\alpha}{3} + w_{n_k}\left(1 - \frac{2\varepsilon_\alpha}{3}\right)$$

for all $k \in \mathbb{N}$. Then we can conclude that the disks

$$\Delta_k = \left\{z \in \mathbb{D};\ |z - w_{n_k}| < \frac{\varepsilon_\alpha}{3}(1 - w_{n_k})\right\} \quad (k \in \mathbb{N})$$

are disjoint since $w_{n_{k+1}} - w_{n_k} > 2\left(\frac{\varepsilon_\alpha}{3}(1-w_{n_k})\right)$. For $z \in \Delta_k$, we have

$$1-|z| \geq 1-|z-w_{n_k}|-|w_{n_k}| > \left(1-\frac{\varepsilon_\alpha}{3}\right)(1-w_{n_k})$$

and thus

$$|z-|z|| \leq 2|z-w_{n_k}| < \frac{2\varepsilon_\alpha}{3}(1-w_{n_k}) < \varepsilon_\alpha(1-|z|).$$

Hence we conclude $\Delta_k \subset S$ for all $k \in \mathbb{N}$. Since $\|B\|_\infty = 1$, Lemma 3.2.9 in combinition with $(*)$ yields

$$1+\sup_{z\in\mathbb{D}} \operatorname{Re} V_{B\circ f}(z) \geq c_\alpha \sum_{k=0}^{\infty} \int_{\Delta_k} \left|(B\circ f)'(\zeta)\right|^2 dA(\zeta)$$

and since $\left|(B\circ f)'\right|^2$ is subharmonic in $\mathbb{D}$ we get that

$$1+\sup_{z\in\mathbb{D}} \operatorname{Re} V_{B\circ f}(z) \geq \frac{\pi\varepsilon_\alpha^2 c_\alpha}{9} \sum_{k=0}^{\infty} (1-w_{n_k})^2 \left|(B\circ f)'(w_{n_k})\right|^2.$$

Hence it suffices to show that $\left((1-w_{n_k})\left|(B\circ f)'(w_{n_k})\right|\right)_{k\in\mathbb{N}}$ is not a zero sequence. By Lemma 3.2.11 there exists a $\delta > 0$ such, that

$$\left|B'(z_k)\right| \geq \frac{\delta}{(1-z_k)}$$

for all $k \in \mathbb{N}$. We have $z_n = f(w_n)$ and thus $1-z_n = \frac{1}{s_1^\alpha(w_n)}$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
(1-w_{n_k})\left|(B\circ f)'(w_{n_k})\right| &= (1-w_{n_k})\left|B'(z_{n_k})\right| \frac{\left(s_1^\alpha\right)'(w_{n_k})}{\left(s_1^\alpha(w_{n_k})\right)^2} \\
&\geq \delta(1-w_{n_k}) \frac{\left(s_1^\alpha\right)'(w_{n_k})}{s_1^\alpha(w_{n_k})} \\
&= \delta \frac{\left(s_1^\alpha\right)'(w_{n_k})(1-w_{n_k})^{-\alpha}}{(1-w_{n_k})^{-\alpha-1} s_1^\alpha(w_{n_k})}
\end{aligned}$$

for all $k \in \mathbb{N}$ and the result follows by Lemma 3.2.8(ii). □

4 Appendix

In the following let $\mathscr{E}$ be a complex Hilbert space and Ω an arbitrary set.

Theorem 4.0.1. *Let $\mathscr{H}_{\mathfrak{k}} \subset \mathscr{E}^{\Omega}$ be a functional Hilbert space with reproducing kernel $\mathfrak{k}\colon \Omega \times \Omega \to L(\mathscr{E})$. Let $(h_\alpha)_{\alpha \in A}$ be a net in $\mathscr{H}_{\mathfrak{k}}$ and $c \geq 0$ with $\|h_\alpha\|_{\mathscr{H}_{\mathfrak{k}}} \leq c$ for all $\alpha \in A$. If $h \in \mathscr{E}^{\Omega}$ with $\mathscr{E}\text{-}\lim_\alpha h_\alpha(z) = h(z)$ for all $z \in \Omega$, then $h \in \mathscr{H}_{\mathfrak{k}}$ with $\|h\|_{\mathscr{H}_{\mathfrak{k}}} \leq c$ and $\tau_w^{\mathscr{H}_{\mathfrak{k}}}\text{-}\lim_\alpha h_\alpha = h$.*

Proof. By Alaoglu-Bourbaki the net $(h_\alpha)_{\alpha \in A}$ has a weakly convergent subnet $(h_{\alpha_i}) \xrightarrow{i} \tilde{h} \in \mathscr{H}_{\mathfrak{k}}$. Then $\|\tilde{h}\|_{\mathscr{H}_{\mathfrak{k}}} \leq c$ and since weak convergence in $\mathscr{H}_{\mathfrak{k}}$ implies pointwise convergence, we find that $h = \tilde{h}$. The last assertion follows from the fact that the net $(h_\alpha)_{\alpha \in A}$ is normbounded and $\left(\langle \cdot, h_\alpha \rangle_{\mathscr{H}_{\mathfrak{k}}}\right) \xrightarrow{\alpha} \langle \cdot, h \rangle_{\mathscr{H}_{\mathfrak{k}}}$ pointwise on the total subset $\left\{\langle \cdot, \mathfrak{k}_z \rangle;\, z \in \Omega\right\} \subset \mathscr{H}_{\mathfrak{k}}$. □

Corollary 4.0.2. *Let $\mathscr{H}_i \subset \mathscr{E}_i^{\Omega}$ $(i = 1, 2)$ be functional Hilbert spaces and let $(\varphi_k)_{k \in \mathbb{N}}$ be a sequence in $\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ such that $(\varphi_k)_{k \in \mathbb{N}}$ converges pointwise on Ω to a function $\varphi\colon \Omega \to L(\mathscr{E}_1, \mathscr{E}_2)$ and such that $\|M_{\varphi_k}\|_{L(\mathscr{H}_1, \mathscr{H}_2)} \leq c$ for all $k \in \mathbb{N}$. Then $\varphi \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ with $\|M_\varphi\|_{L(\mathscr{H}_1, \mathscr{H}_2)} \leq c$ and $\varphi f = \tau_w^{\mathscr{H}_{\mathfrak{k}}}\text{-}\lim_k \varphi_k f$ for all $f \in \mathscr{H}_1$.*

Proof. Let $f \in \mathscr{H}_1$. Then we have $\lim_{k \to \infty} \varphi_k(z) f(z) = \varphi(z) f(z)$ for all $z \in \Omega$ and

$$\|\varphi_k f\|_{\mathscr{H}_2} \leq \|M_{\varphi_k}\|_{L(\mathscr{H}_1, \mathscr{H}_2)} \|f\|_{\mathscr{H}_1} \leq c \|f\|_{\mathscr{H}_1}$$

for all $k \in \mathbb{N}$. By Theorem 4.0.1 it follows that $\varphi f \in \mathscr{H}_2$ with $\|\varphi f\|_{\mathscr{H}_2} \leq c \|f\|_{\mathscr{H}_1}$ and $\tau_w^{\mathscr{H}_2}\text{-}\lim_k \varphi_k f = \varphi f$ holds. Hence we conclude φ is in $\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ with $\|M_\varphi\|_{L(\mathscr{H}_1, \mathscr{H}_2)} \leq c$. □

Proposition 4.0.3. *Let H be a Hilbert space and $(x_n)_{n \in \mathbb{N}}$ a sequence in H, which converges weakly to a $x \in H$. Then we have $\|x\|_H \leq \liminf_{n \to \infty} \|x_n\|_H$.*

Proof. Otherwise, one could choose a real number r with

$$\|x\|_H > r > \liminf_{n \to \infty} \|x_n\|_H.$$

But then there would be a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ with $\|x_{n_k}\|_H < r$ for all $k \in \mathbb{N}$. Since $x = \tau_w\text{-}\lim_{k \to \infty} x_{n_k}$ and since $\{x \in H;\, \|x\|_H \leq r\} \subset H$ is τ_w-closed, we would obtain the contradiction that $\|x\|_H \leq r < \|x\|_H$. □

Proposition 4.0.4. *Let (X,σ,μ) be a measure space. Then there exists a unique unitary operator $V\colon L^2(\mu)\otimes\mathscr{E}\to L^2(\mu,\mathscr{E})$ with $V([f]\otimes v)=[fv]$ for all $[f]\in L^2(\mu)$ and $v\in\mathscr{E}$.*

Proof. The map

$$B\colon L^2(\mu)\times\mathscr{E}\to L^2(\mu,\mathscr{E}),B([f],v)=[fv]$$

is well defined due to

$$\int_\Omega \|f(z)v\|_{\mathscr{E}}^2\,d\mu(z)=\left(\int_\Omega |f(z)|^2\,d\mu(z)\right)\|v\|_{\mathscr{E}}$$

for all $[f]\in L^2(\mu)$ and $v\in\mathscr{E}$. Furthermore B is bilinear. Because of the universal property of the algebraic tensor product $\otimes_a$ there exists a unique linear map $V_a\colon L^2(\mu)\otimes_a\mathscr{E}\to L^2(\mu,\mathscr{E})$ with $V([f]\otimes_a v)=[fv]$ for all $[f]\in L^2(\mu)$ and $v\in\mathscr{E}$. By a standard result from functional analysis we have that V_a has a unique extension V to the Hilbert space $L^2(\mu)\otimes\mathscr{E}$. To check that the map V is an isometry let $k\in\mathbb{N}$ and $\left[f^{(i)}\right]\in L^2(\mu)$, $v^{(i)}\in\mathscr{E}$ for $i=1\cdots,k$. Then we deduce

$$\begin{aligned}
&\left\|V\sum_{i=0}^{k}\left[f^{(i)}\right]\otimes v^{(i)}\right\|_{L^2(\mu,\mathscr{E})}^2\\
&=\left\|\sum_{i=0}^{k}\left[f^{(i)}v^{(i)}\right]\right\|_{L^2(\mu,\mathscr{E})}^2\\
&=\sum_{i,j=0}^{k}\int_\Omega\left\langle f^{(i)}(z)v^{(i)},f^{(j)}(z)v^{(j)}\right\rangle_{\mathscr{E}}d\mu(z)\\
&=\sum_{i,j=0}^{k}\int_\Omega f^{(i)}(z)\overline{f^{(j)}(z)}d\mu(z)\left\langle v^{(i)},v^{(j)}\right\rangle_{\mathscr{E}}\\
&=\sum_{i,j=0}^{k}\left\langle\left[f^{(i)}\right]\otimes v^{(i)},\left[f^{(j)}\right]\otimes v^{(j)}\right\rangle_{L^2(\mu)\otimes\mathscr{E}}\\
&=\left\|\sum_{i=0}^{k}\left[f^{(i)}\right]\otimes v^{(i)}\right\|_{L^2(\mu)\otimes\mathscr{E}}^2
\end{aligned}$$

and due to

$$L^2(\mu)\otimes\mathscr{E}=\overline{\operatorname{span}\{[f]\otimes v;\ [f]\in L^2(\mu),v\in\mathscr{E}\}}.$$

V is an isometry. We next want to show that V is surjective. Thus let $[f]\in L^2(\mu,\mathscr{E})$ be arbitrary. Then by [3, Prop. 4.8] there is a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty}\|f-f_n\|_{\mathfrak{L}^2(\mu,\mathscr{E})}=0.$$

For $n \in \mathbb{N}$ write

$$f_n = \sum_{k=0}^{N} a_k^{(n)} \mathbb{1}_{A_k^{(n)}}$$

where $a_k^{(n)} \in \mathscr{E}$ and $A_k^{(n)} \in \sigma$ $(n = 0, \ldots, N)$, such that $\mu\left(A_k^{(n)}\right) < \infty$ and $\bigcap_{k=0}^{N} A_k^{(n)} = \emptyset$. Set

$$g_n = \sum_{k=0}^{N} \left[\mathbb{1}_{A_k^{(n)}}\right] \otimes a_k^{(n)} \in L^2(\mu) \otimes \mathscr{E} \quad (n \in \mathbb{N})$$

Then we deduce $Vg_n = [f_n]$ for all $n \in \mathbb{N}$ and we conclude that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu) \otimes \mathscr{E}$ since V is an isometry. Therefore

$$g = \lim_{n\to\infty} g_n$$

exists in $L^2(\mu) \otimes \mathscr{E}$ and we have

$$\begin{aligned}
\|Vg - [f]\|_{L^2(\mu,\mathscr{E})} &= \lim_{n\to\infty} \|Vg_n - [f]\|_{L^2(\mu,\mathscr{E})} \\
&= \lim_{n\to\infty} \|[f_n] - [f]\|_{L^2(\mu,\mathscr{E})} \\
&= \lim_{n\to\infty} \|f - f_n\|_{\mathfrak{L}^2(\mu,\mathscr{E})} \\
&= 0.
\end{aligned}$$

Hence $Vg = [f]$ and V is indeed surjective. □

Set $X = \mathbb{N}$, $\sigma = \mathscr{P}(\mathbb{N})$ and $\mu = \sum_{n\in\mathbb{N}} \delta_n$ where δ_n $(n \in \mathbb{N})$ denote the Dirac measures. Then we have $L^2(\mu) = \mathfrak{l}^2$ and $L^2(\mu, \mathscr{E}) = \mathfrak{l}^2(\mathscr{E})$ and we obtain the following corollary.

Corollary 4.0.5. *There exists a unique unitary operator $V : \mathfrak{l}^2 \otimes \mathscr{E} \to \mathfrak{l}^2(\mathscr{E})$ with $V\left((x_n)_{n\in\mathbb{N}} \otimes v\right) = (x_n v)_{n\in\mathbb{N}}$ for all $(x_n)_{n\in\mathbb{N}} \in \mathfrak{l}^2$ and $v \in \mathscr{E}$.*

Corollary 4.0.6. *Let $\mathscr{H}_{\mathfrak{k}}$ be a functional Hilbert space with reproducing kernel $\mathfrak{k} \colon \Omega \times \Omega \to \mathbb{C}$ .Then there exist unitary operators*

$$\begin{aligned}
&\mathscr{H}_{\mathfrak{k}} \otimes \mathfrak{l}^2 \otimes \mathscr{E} \to \mathscr{H}_{\mathfrak{k}}(\mathscr{E}) \otimes \mathfrak{l}^2 \textit{ with } h \otimes (x_n)_{n\in\mathbb{N}} \otimes v \longmapsto hv \otimes (x_n)_{n\in\mathbb{N}}, \\
&\mathscr{H}_{\mathfrak{k}} \otimes \mathfrak{l}^2 \otimes \mathscr{E} \to \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2\right) \otimes \mathscr{E} \textit{ with } h \otimes (x_n)_{n\in\mathbb{N}} \otimes v \longmapsto (x_n h)_{n\in\mathbb{N}} \otimes v, \\
&\mathscr{H}_{\mathfrak{k}} \otimes \mathfrak{l}^2 \otimes \mathscr{E} \to \mathscr{H}_{\mathfrak{k}}\left(\mathfrak{l}^2(\mathscr{E})\right) \textit{ with } h \otimes (x_n)_{n\in\mathbb{N}} \otimes v \longmapsto (x_n hv)_{n\in\mathbb{N}}, \\
&\mathscr{H}_{\mathfrak{k}} \otimes \mathfrak{l}^2 \otimes \mathscr{E} \to \mathfrak{l}^2\left(\mathscr{H}_{\mathfrak{k}}(\mathscr{E})\right) \textit{ with } h \otimes (x_n)_{n\in\mathbb{N}} \otimes v \longmapsto (x_n hv)_{n\in\mathbb{N}}
\end{aligned}$$

for all $h \in \mathscr{H}_{\mathfrak{k}}$, $(x_n)_{n\in\mathbb{N}} \in \mathfrak{l}^2$ and $v \in \mathscr{E}$.

Proof. Use that the tensor product is associative and commutative up to an unitary operator and apply [5, Satz 1.15] as well as Corollary 4.0.5. □

Proposition 4.0.7. *If* $u\colon \Omega \to [-\infty, \infty)$ *is a subharmonic function on an open set* $\Omega \subset \mathbb{C}$ *then also the function* $\tilde{u}\colon \tilde{\Omega} = \{\bar{z};\ z \in \Omega\} \to [-\infty, \infty), \tilde{u}(z) = u(\bar{z})$ *is subharmonic.*

Proof. Let $\overline{B_r(z_0)} \subset \tilde{\Omega}$, $h\colon \overline{B_r(z_0)} \to \mathbb{R}$ be continuous and harmonic on $B_r(z_0)$ such that, $h(z) \geq \tilde{u}(z)$ holds for every $z \in \partial B_r(z_0)$. Set $\tilde{h}\colon \overline{B_r(\overline{z_0})} \to \mathbb{R}, \tilde{h}(z) = h(\bar{z})$. We next want to show that $\tilde{h}$ is harmonic on $B_r(\overline{z_0})$. Thus, let $z_1 \in B_r(\overline{z_0})$. Since h is harmonic on $B_r(z_0)$, there exists a $s \in (0, \infty)$ with $B_s(\overline{z_1}) \subset B_r(z_0)$ and a holomorphic function $f\colon B_s(\overline{z_1}) \to \mathbb{C}$, such that $\operatorname{Re} f(z) = h(z)$ for every $z \in B_s(\overline{z_1})$. Now, set $\tilde{f}\colon B_s(z_1) \to \mathbb{C}, \tilde{f}(z) = \overline{f(\bar{z})}$. Then $\tilde{f}$ is holomorphic and we have

$$\tilde{h}(z) = h(\bar{z}) = \operatorname{Re} f(\bar{z}) = \operatorname{Re} \overline{f(\bar{z})} = \operatorname{Re} \tilde{f}(z)$$

for all $z \in B_s(z_1)$. Hence, $\tilde{h}$ is harmonic on $B_r(\overline{z_0})$. Further, we have

$$\tilde{h}(z) = h(\bar{z}) \geq \tilde{u}(\bar{z}) = u(z)$$

for all $z \in \partial B_r(\overline{z_0})$. Since $\tilde{h}$ is harmonic on $B_r(\overline{z_0})$ and u is subharmonic on Ω we deduce $u(z) \leq \tilde{h}(z)$ for all $z \in B_r(\overline{z_0})$ and hence

$$\tilde{u}(z) = u(\bar{z}) \leq \tilde{h}(\bar{z}) = h(z)$$

for all $z \in B_r(z_0)$. $\square$

Bibliography

[1] J. Agler and J.E. McCarthy, *Complete Nevanlinna-Pick kernels*, J. Funct. Anal. **175** (2000), 111–124.

[2] A. Aleman, M. Hartz, J.E. MCarthy, and S. Richter, *Factorizations induced by complete Nevanlinna–Pick factors*, Advances in Mathematics **335** (2018), 372 –404.

[3] H. Amann and J. Escher, *Analysis. III*, Birkhäuser Verlag, Basel, 2009. Translated from the 2001 German original by Silvio Levy and Matthew Cargo.

[4] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, Second, Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001.

[5] C. Barbian, *Positivitätsbedingungen funktionaler Hilberträume und Anwendungen in der mehrdimensionalen Operatorentheorie.*, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 2001.

[6] ______, *Beurling-type representation of invariant subspaces in reproducing kernel Hilbert spaces.*, Dissertation, Universität des Saarlandes, Saarbrücken, 2007.

[7] M. Bhattacharjee, J. Eschmeier, Dinesh K. Keshari, and Jaydeb Sarkar, *Dilations, wandering subspaces, and inner functions*, Linear Algebra Appl. **523** (2017), 263–280.

[8] K. Everad, *Über sphärische Isometrien und exakte Sequenzen von Toeplitzalgebren.*, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 2008.

[9] J. B. Garnett, *Bounded analytic functions*, first, Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.

[10] M. Kaluza, *Bergmanräume über der Einheitskugel in C^n*, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 2009.

[11] V. Klauk, *Dilatationssätze für m-Hyperkontraktionen*, Masterarbeit, Universität des Saarlandes, Saarbrücken, 2016.

[12] G. Köthe, *Topological vector spaces. I*, Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag New York Inc., New York, 1969.

[13] ______, *Topological vector spaces. II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 237, Springer-Verlag, New York-Berlin, 1979.

[14] E. Kreuzer, *Toeplitz Extensions and Berezin transforms*, Masterthesis, Universität des Saarlandes, Saarbrücken, 2014.

[15] W. Rudin, *Functional analysis*, Second, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[16] D. Sarason, *Exposed points in H^1. I*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), 1989, pp. 485–496.

[17] ______, *Exposed points in H^1. II*, Topics in operator theory: Ernst D. Hellinger memorial volume, 1990, pp. 333–347.

[18] S. Shimorin, *Reproducing kernels and extremal functions in Dirichlet-type spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **255** (1998), no. Issled. po Lineĭn. Oper. i Teor. Funkts. 26, 198–220, 254.

[19] ———, *Complete Nevanlinna-Pick property of Dirichlet-type spaces*, J. Funct. Anal. **191** (2002), 276–296.

[20] D. Wood, *The Computation of Polylogarithms*, Technical Report 15-92*, University of Kent, Computing Laboratory, University of Kent, Canterbury, UK, 1992.