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Pairs of Commuting Contractions

Master's Thesis

submitted by

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advised by

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Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

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Introduction

The classical Wold decomposition for isometries ([15, Chapter I.1]) states that every isometry on a Hilbert space decomposes into a direct sum of a unilateral shift and a unitary operator.

More precisely, for every isometry $V \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ there exists a Hilbert space $\mathcal{W}$ such that V is unitarily equivalent to $(M_z \otimes \mathrm{id}_{\mathcal{W}}) \oplus U$, where $M_z \otimes \mathrm{id}_{\mathcal{W}}$ is the unilateral shift on the $\mathcal{W}$-valued Hardy space $\mathrm{H}^2(\mathbb{D}, \mathcal{W}) \cong \mathrm{H}^2(\mathbb{D}) \otimes \mathcal{W}$ and U is a unitary operator on

$$\mathcal{H}^0 = \bigcap_{n=0}^{\infty} V^n \mathcal{H}.$$

Every unilateral shift operator as above is an isometry with $\mathcal{H}^0 = \{0\}$. These isometries are called *pure*. A pair $(V_1, V_2) \in \mathcal{L}(\mathcal{H})^2$ of commuting isometries is said to be *pure* if the product $V_1 V_2$ is a pure isometry.

Pure pairs of commuting isometries have been studied by C. A. Berger, L. A. Coburn and A. Lebow in 1978 (see [5]). The following result is of particular interest for the purpose of this thesis:

Theorem 1 (Berger, Coburn, Lebow)**.** Let $V \in \mathcal{L}(\mathcal{H})$ be a pure isometry and let $V_1, V_2 \in \mathcal{L}(\mathcal{H})$ be commuting isometries. Then the following are equivalent:

(i) $V = V_1 V_2$.

(ii) There exist a Hilbert space $\mathcal{E}$, a unitary operator $U \in \mathcal{L}(\mathcal{E})$ and an orthogonal projection $P \in \mathcal{L}(\mathcal{E})$ such that the operator-valued functions $\Phi, \Psi \colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ defined by

$$\Phi(z) = \left(P + zP^{\perp}\right) U^* \qquad \text{and} \qquad \Psi(z) = U \left(P^{\perp} + zP\right) \qquad (z \in \mathbb{D})$$

induce a pure pair $(M_\Phi, M_\Psi) \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{E}))^2$ of isometric multiplication operators with

$$M_\Phi M_\Psi = M_\Psi M_\Phi = M_z \otimes \mathrm{id}_{\mathcal{E}}$$

and such that

$$(V_1, V_2, V) \cong (M_\Phi, M_\Psi, M_z \otimes \mathrm{id}_{\mathcal{E}}).$$

Pure isometries $V \in \mathcal{L}(\mathcal{H})$ are exactly those isometries that are of class $C_{\cdot 0}$, that is, for which $\lim\limits_{n\to\infty} V^{*n} = 0$ in the strong operator topology. In the following, we will consider the more general case of contractions of class $C_{\cdot 0}$. Analogously, we say that a pair $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ of commuting contractions is of class $C_{\cdot 0}$ if the contraction T_1T_2 is of class $C_{\cdot 0}$. We denote by $\mathcal{D}_S = \overline{(\mathrm{id}_{\mathcal{H}} - SS^*)\mathcal{H}}$ the defect space of a given contraction $S \in \mathcal{L}(\mathcal{H})$ and assume that $\mathcal{H}$ is separable.

In 2017, B. K. Das, J. Sarkar and S. Sarkar obtained an extension of the Berger-Coburn-Lebow result for pairs of commuting contractions (see [8]). The purpose of the present thesis is to give a complete proof of the results from [8] and to provide the necessary tools. The main result of this thesis will be the following:

Theorem 2 (Das, Sarkar, Sarkar)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction of class $C_{\cdot 0}$ and let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions. Then the following are equivalent:

(i) $T = T_1T_2$.

(ii) There exist a Hilbert space $\mathcal{E}$, a unitary operator $U \in \mathcal{L}(\mathcal{E})$ and an orthogonal projection $P \in \mathcal{L}(\mathcal{E})$ such that the operator-valued mappings $\Phi, \Psi\colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ defined by

$$\Phi(z) = (P + zP^{\perp})U^* \qquad \text{and} \qquad \Psi(z) = U(P^{\perp} + zP) \qquad (z \in \mathbb{D})$$

induce a pure pair $(M_\Phi, M_\Psi) \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{E}))^2$ of commuting isometries with

$$M_\Phi M_\Psi = M_\Psi M_\Phi = M_z \otimes \mathrm{id}_{\mathcal{E}}$$

and such that

$$(T_1, T_2, T) \cong P_{\mathcal{Q}}(M_\Phi, M_\Psi, M_z \otimes \mathrm{id}_{\mathcal{E}})|_{\mathcal{Q}}$$

with a suitable joint $(M_\Phi^*, M_\Psi^*, M_z^* \otimes \mathrm{id}_{\mathcal{E}})$-invariant subspace $\mathcal{Q} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{E})$.

(iii) There exist $\mathcal{L}(\mathcal{D}_T)$-valued polynomials φ, ψ of degree at most 1 such that

$$P_{\mathcal{Q}} M_z \otimes \mathrm{id}_{\mathcal{D}_T}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\psi\varphi}|_{\mathcal{Q}}$$

and

$$(T_1, T_2) \cong P_{\mathcal{Q}}(M_\varphi, M_\psi)|_{\mathcal{Q}}$$

for a suitable joint (M_φ^*, M_ψ^*)-invariant subspace $\mathcal{Q} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{D}_T)$.

As an application of this result, one can prove the following version of von Neumann's inequality:

Theorem 3 (Das, Sarkar, Sarkar)**.** Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions of class $C_{\cdot 0}$ with finite dimensional defect spaces $\mathcal{D}_{T_1}, \mathcal{D}_{T_2}$. Then there exists an algebraic variety $V \subseteq \mathbb{C}^2$ with $V \cap \mathbb{T}^2 \neq \mathbb{T}^2$ such that

$$\|p(T_1, T_2)\| \leq \|p\|_{V \cap \mathbb{T}^2}$$

for all polynomials $p \in \mathbb{C}[z_1, z_2]$.

Earlier in 2017, B. K. Das and J. Sarkar studied the case of commuting contractions $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ where T_1 or T_2 is of class $C_{\cdot 0}$ (see [7]). Using a similar approach as in [8], they obtained the following dilation theorem:

Theorem 4 (Das, Sarkar)**.** Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions with finite dimensional defect spaces $\mathcal{D}_{T_1}, \mathcal{D}_{T_2}$ and let T_1 be of class $C_{\cdot 0}$. Then there exists an analytic operator-valued function $\Phi \colon \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T_1})$ which induces an isometric multiplication operator $M_\Phi \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1}))$ such that

$$(T_1, T_2) \cong P_{\mathcal{Q}} \left(M_z \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_\Phi \right) \big|_{\mathcal{Q}}$$

for some joint $\left(M_z^* \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_\Phi^* \right)$-invariant subspace $\mathcal{Q} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1})$.

Using Theorem 4, one finds another improvement of von Neumann's inequality for commuting pairs of contractions which reads as follows:

Theorem 5 (Das, Sarkar)**.** Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be commuting contractions of class $C_{\cdot 0}$ with finite dimensional defect spaces $\mathcal{D}_{T_1}, \mathcal{D}_{T_2}$. Then there exists a distinguished variety $V \subseteq \mathbb{D}^2$ such that

$$\|p(T_1, T_2)\| \leq \|p\|_V$$

for all polynomials $p \in \mathbb{C}[z_1, z_2]$.

In the particular case that T_1 and T_2 are commuting contractive matrices of class $C_{\cdot 0}$ (or, equivalently, without unimodular eigenvalues) Theorem 5 was first proved by Agler and McCarthy in [2].

Besides the main result described in Theorem 2, we will explicitly present the arguments used in [7] to give a proof of Theorem 4 and Theorem 5.

The structure of the thesis is the following. In the first three chapters we provide the tools that are needed in both [8] and [7]. The fourth chapter is divided into two cases. We first consider commuting contractions $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ with finite dimensional defect spaces such that T_1 is of class $C_{\cdot 0}$ and give a proof of Theorem 4. Subsequently, we drop the assumption of finite dimensional defect spaces and consider commuting contractions $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ such that the product $T_1 T_2$ is of class $C_{\cdot 0}$ to prove Theorem 2. Finally, in the last chapter we give proofs of Theorem 3 and Theorem 5.

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1. Reproducing Kernels

Let $\mathcal{E}$ be a Hilbert space and let X be an arbitrary set. The set of all mappings from X to $\mathcal{E}$ will be denoted by $\mathcal{E}^X$.

1.1. Functional Hilbert Spaces

Firstly, we define vector-valued functional Hilbert spaces using the same approach as in [3] and [4].

Definition 1.1. A Hilbert space $\mathcal{H} \subseteq \mathcal{E}^X$ is called a *functional Hilbert space* if the point evaluations

$$\delta_\lambda \colon \mathcal{H} \longrightarrow \mathcal{E}, \qquad f \longmapsto f(\lambda)$$

are continuous for all $\lambda \in X$.

Since positive definite functions play a key role in the theory of functional Hilbert spaces, we recall the definition of positive definiteness.

Definition 1.2. A function $K \colon X \times X \to \mathcal{L}(\mathcal{E})$ satisfying

$$\sum_{i,j=1}^{n} \langle K(\lambda_i, \lambda_j)\, h_j, h_i \rangle \geq 0$$

for all finite sequences $(\lambda_i)_{i=1}^n$ in X and $(h_i)_{i=1}^n$ in $\mathcal{E}$ is called *positive definite*.

Remark 1.3. *(i) Using the identification $\mathbb{C} \cong \mathcal{L}(\mathbb{C})$, where a complex number $\alpha \in \mathbb{C}$ is considered as the multiplication operator*

$$M_\alpha \colon \mathbb{C} \longrightarrow \mathbb{C}, \qquad z \longmapsto \alpha z,$$

a function $K \colon X \times X \to \mathbb{C}$ is positive definite if and only if

$$\sum_{i,j=1}^{n} K(\lambda_i, \lambda_j)\, z_j \overline{z_i} \geq 0$$

holds for all finite sequences $(\lambda_i)_{i=1}^n$ in X and $(z_i)_{i=1}^n$ in $\mathbb{C}$.

(ii) A function $K\colon X \times X \to \mathcal{L}(\mathcal{E})$ is positive definite if and only if all finite operator matrices

$$\left(K\left(\lambda_i, \lambda_j\right)\right)_{i,j=1}^n \in M_n\left(\mathcal{E}\right) \cong \mathcal{L}(\mathcal{E}^n) \qquad \left(n \geq 1, \lambda_1, ..., \lambda_n \in X\right)$$

define positive operators on $\mathcal{E}^n$. Since positive operators are self-adjoint, positive definite functions satisfy $K\left(\lambda, \mu\right)^* = K\left(\mu, \lambda\right)$ for all $\lambda, \mu \in X$.

Proofs of the following results can be found in the first chapter of [3].

Theorem 1.4. *Let $\mathcal{H} \subseteq \mathcal{E}^X$ be a Hilbert space. The following are equivalent:*

(i) The space $\mathcal{H}$ is a functional Hilbert space.

(ii) There exists a function $K\colon X \times X \to \mathcal{L}(\mathcal{E})$ such that

$$K\left(\cdot, \mu\right) x \in \mathcal{H}$$

and

$$\left\langle f, K\left(\cdot, \mu\right) x\right\rangle = \left\langle f\left(\mu\right), x\right\rangle$$

hold for all $x \in \mathcal{E}, \mu \in X$ and $f \in \mathcal{H}$.

In this case, the function K is uniquely determined by $K\left(\lambda, \mu\right) = \delta_\lambda \delta_\mu^*$ for all $\lambda, \mu \in X$.

In the setting of Theorem 1.4 the function K is called the *reproducing kernel of the functional Hilbert space $\mathcal{H}$*.

Lemma 1.5. *Let $\mathcal{H} \subseteq \mathcal{E}^X$ be a functional Hilbert space. Then we have:*

(i) The reproducing kernel K of $\mathcal{H}$ is positive definite.

(ii) $\bigvee \left\{K\left(\cdot, \mu\right) x \mid \mu \in X, x \in \mathcal{E}\right\} = \mathcal{H}$.

Theorem 1.6 (Moore)**.** *Every positive definite function $K\colon X \times X \to \mathcal{L}(\mathcal{E})$ induces a unique functional Hilbert space $\mathcal{H}_K \subseteq \mathcal{E}^X$ with reproducing kernel K.*

Remark 1.7. *Let $\mathcal{H} \subseteq \mathcal{E}^X$ be a functional Hilbert space with reproducing kernel $K\colon X \times X \to \mathcal{L}(\mathcal{E})$ and let $Y \subseteq X$ be a subset. The positive definite function $K|_{Y \times Y}\colon Y \times Y \to \mathcal{L}(\mathcal{E})$ induces a functional Hilbert space $\mathcal{H}_Y$, which can be identified with the subspace*

$$\bigvee \left\{K\left(\cdot, \mu\right) x \mid \mu \in Y, x \in \mathcal{E}\right\}$$

of $\mathcal{H}$. More precisely, the restriction map

$$\varphi\colon \mathcal{H} \longrightarrow \mathcal{H}_Y, \qquad f \longmapsto f|_Y$$

is a well-defined contraction and its adjoint induces a unitary operator $\varphi^*\colon \mathcal{H}_Y \to \bigvee \left\{K\left(\cdot, \mu\right) x \mid \mu \in Y, x \in \mathcal{E}\right\}$.

Theorem 1.8. *If $K \colon X \times X \to \mathbb{C}$ is positive definite, then*

$$K_{\mathcal{E}} \colon X \times X \longrightarrow \mathcal{L}(\mathcal{E}), \qquad (\lambda, \mu) \longmapsto K(\lambda, \mu)\, \mathrm{id}_{\mathcal{E}}$$

is positive definite and there is a uniquely determined unitary operator $V \colon \mathcal{H}_K \otimes \mathcal{E} \to \mathcal{H}_{K_{\mathcal{E}}}$ such that

$$V(f \otimes x) = f \cdot x$$

holds for all $f \in \mathcal{H}_K$ and $x \in \mathcal{E}$, where $\mathcal{H}_K \subseteq \mathbb{C}^X$ and $\mathcal{H}_{K_{\mathcal{E}}} \subseteq \mathcal{E}^X$ denote the functional Hilbert spaces with reproducing kernels K and $K_{\mathcal{E}}$ respectively.

1.2. Multipliers of Functional Hilbert Spaces

Let $\mathcal{E}_1, \mathcal{E}_2$ be Hilbert spaces and let $\mathcal{H}_i \subseteq \mathcal{E}_i^X$ $(i = 1, 2)$ be functional Hilbert spaces with reproducing kernels $K_i \colon X \times X \to \mathcal{L}(\mathcal{E}_i)$ $(i = 1, 2)$.

Definition 1.9. The functions in

$$\mathcal{M}\left(\mathcal{H}_1, \mathcal{H}_2\right) = \{\phi \colon X \to \mathcal{L}\left(\mathcal{E}_1, \mathcal{E}_2\right) \mid \phi f \in \mathcal{H}_2 \text{ for all } f \in \mathcal{H}_1\}$$

are called *multipliers between $\mathcal{H}_1$ and $\mathcal{H}_2$*. Here $\phi f \colon X \to \mathcal{E}_2$ is defined by

$$(\phi f)(\lambda) = \phi(\lambda) f(\lambda) \qquad (\lambda \in X)$$

for every $f \colon X \to \mathcal{E}_1$.
For $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, we call

$$M_\phi \colon \mathcal{H}_1 \longrightarrow \mathcal{H}_2, \qquad f \longmapsto \phi f$$

the *multiplication operator with symbol ϕ*. Whenever $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we will write $\mathcal{M}(\mathcal{H})$ instead of $\mathcal{M}(\mathcal{H}, \mathcal{H})$.

Remark 1.10. *For $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, an elementary application of the closed graph thereom shows that $M_\phi \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. One obtains a norm*

$$\|\phi\|_{\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)} = \|M_\phi\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \qquad (\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2))$$

on $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ if

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \longrightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2), \qquad \phi \longmapsto M_\phi$$

is injective. This holds true if, for example, $\mathcal{H}_1$ contains all constant functions.

Remark 1.11. *Let $\mathcal{E}$ be a non-trivial Hilbert space. If $\mathcal{H} \subseteq \mathcal{E}^X$ is a functional Hilbert space that contains all constant functions, then the reproducing kernel $K\colon X \times X \to \mathcal{L}(\mathcal{E})$ of $\mathcal{H}$ satisfies*

$$K(\mu, \mu) \geq 0 \qquad \text{and} \qquad K(\mu, \mu) \neq 0$$

for all $\mu \in X$. This follows from the observation that

$$\|K(\cdot, \mu)\, x\|^2 = \langle K(\mu, \mu)\, x, x\rangle \leq \|K(\mu, \mu)\|\, \|x\|^2$$

for all $\mu \in X$ and $x \in \mathcal{E}$.

Corollary 1.12. *Let $\mathcal{H} \subseteq \mathbb{C}^X$ be a functional Hilbert space with reproducing kernel $K\colon X \times X \to \mathbb{C}$ such that $\mathcal{H}$ contains all constant functions. Then the inequality*

$$\sup_{\lambda \in X} \|\phi(\lambda)\| \leq \|\phi\|_{\mathcal{M}\left(\mathcal{H}_{K_{\mathcal{E}_1}}, \mathcal{H}_{K_{\mathcal{E}_2}}\right)}$$

holds for each multiplier $\phi \in \mathcal{M}\left(\mathcal{H}_{K_{\mathcal{E}_1}}, \mathcal{H}_{K_{\mathcal{E}_2}}\right)$.

Proof. Let $\phi \in \mathcal{M}\left(\mathcal{H}_{K_{\mathcal{E}_1}}, \mathcal{H}_{K_{\mathcal{E}_2}}\right)$ be given. An elementary exercise shows that

$$M_\phi^* K(\cdot, \mu)x = K(\cdot, \mu)\phi(\mu)^* x$$

for $\mu \in X$ and $x \in \mathcal{E}_2$. Using the identifications explained in Theorem 1.8 one easily obtains that

$$\begin{aligned} K(\mu, \mu)\, \|\phi(\mu)^* x\|^2 &= \|K(\cdot, \mu)\, \phi(\mu)^* x\|^2 \\ &\leq \left\|M_\phi^*\right\|^2 \|K(\cdot, \mu)\, x\|^2 \\ &= \|M_\phi\|^2\, K(\mu, \mu)\, \|x\|^2 \end{aligned}$$

for $\mu \in X$ and $x \in \mathcal{E}_2$. Since $K(\mu, \mu) > 0$ by Remark 1.11, we may conclude that

$$\|\phi(\mu)\|^2 = \sup_{\|x\| \leq 1} \|\phi(\mu)^* x\|^2 \leq \|M_\phi\|^2$$

for all $\mu \in X$. □

1.3. Vector-valued Hardy Spaces

In this section, we introduce the functional Hilbert space over the unit disk $\mathbb{D}$ known as the Hardy space. This functional Hilbert space will play a fundamental role throughout the remainder of this thesis.

Definition 1.13. The space

$$\mathrm{H}^2(\mathbb{D}) = \left\{ f = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{O}(\mathbb{D}) \;\middle|\; \|f\|^2_{\mathrm{H}^2(\mathbb{D})} = \sum_{n=0}^{\infty} |f_n|^2 < \infty \right\}$$

is, together with the inner product

$$\left\langle \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} g_n z^n \right\rangle_{\mathrm{H}^2(\mathbb{D})} = \sum_{n=0}^{\infty} f_n \overline{g_n} \qquad \left(\sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} g_n z^n \in \mathcal{O}(\mathbb{D}) \right),$$

a functional Hilbert space with reproducing kernel

$$K \colon \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}, \qquad (z, w) \longmapsto \frac{1}{1 - z\overline{w}}.$$

The Hilbert space $\mathrm{H}^2(\mathbb{D})$ is called the *Hardy space over* $\mathbb{D}$.

The $\mathcal{E}$-valued Hardy space on $\mathbb{D}$ is defined by

$$\mathrm{H}^2(\mathbb{D}, \mathcal{E}) = \left\{ f = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \;\middle|\; \|f\|^2_{\mathrm{H}^2(\mathbb{D},\mathcal{E})} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{E}} < \infty \right\}.$$

This space, equipped with the inner product

$$\left\langle \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} g_n z^n \right\rangle_{\mathrm{H}^2(\mathbb{D},\mathcal{E})} = \sum_{n=0}^{\infty} \langle f_n, g_n \rangle_{\mathcal{E}} \qquad \left(\sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} g_n z^n \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \right),$$

is a functional Hilbert space with reproducing kernel $K_{\mathcal{E}} = K \cdot \mathrm{id}_{\mathcal{E}}$ (see e.g. [13, Section 1.15]).

As explained in Theorem 1.8, one can identify the Hilbert space tensor product $\mathrm{H}^2(\mathbb{D}) \otimes \mathcal{E}$ with the $\mathcal{E}$-valued Hardy space $\mathrm{H}^2(\mathbb{D}, \mathcal{E})$. We shall use this identification throughout the whole paper.

Corollary 1.14. *The Hilbert space tensor product* $\mathrm{H}^2(\mathbb{D}) \otimes \mathcal{E}$ *is isometrically isomorphic to* $\mathrm{H}^2(\mathbb{D}, \mathcal{E})$ *via the unitary operator*

$$V \colon \mathrm{H}^2(\mathbb{D}) \otimes \mathcal{E} \longrightarrow \mathrm{H}^2(\mathbb{D}, \mathcal{E})$$

uniquely determined by

$$V(f \otimes x) = f \cdot x$$

for all $f \in \mathrm{H}^2(\mathbb{D})$ *and* $x \in \mathcal{E}$.

One can show that

$$\sum_{n=0}^{\infty}\|f_n\|^2 = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi}\|f(re^{it})\|^2 dt$$

for $f = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{O}(\mathbb{D},\mathcal{E})$ (cf. [13, Section 1.15]). We are now able to identify the multipliers of vector-valued Hardy spaces over $\mathbb{D}$.

Proposition 1.15. *The identity*

$$\mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right) = \left\{\phi \in \mathcal{O}(\mathbb{D},\mathcal{L}(\mathcal{E})) \;\middle|\; \|\phi\|_{\mathbb{D}} = \sup_{z\in\mathbb{D}}\|\phi(z)\| < \infty\right\}$$

holds and the multiplier norm is given by

$$\|\phi\|_{\mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right)} = \|\phi\|_{\mathbb{D}}$$

for all $\phi \in \mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right)$.

Proof. Let $\phi\colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ be a bounded holomorphic function. For $f \in \mathrm{H}^2(\mathbb{D},\mathcal{E})$, the function $\phi f\colon \mathbb{D} \to \mathcal{E}$ is analytic with

$$\frac{1}{2\pi}\int_0^{2\pi}\left\|(\phi f)\left(re^{it}\right)\right\|^2 dt \leq \|\phi\|_{\mathbb{D}}^2\|f\|_{\mathrm{H}^2(\mathbb{D},\mathcal{E})}^2$$

for $0 < r < 1$. Thus $\phi \in \mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right)$ with $\|\phi\|_{\mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right)} \leq \|\phi\|_{\mathbb{D}}$.

Conversely, let $\phi \in \mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right)$ be given. Then

$$\mathbb{D} \longrightarrow \mathcal{E}, \qquad z \longmapsto \phi(z)x$$

belongs to $\mathrm{H}^2(\mathbb{D},\mathcal{E})$ and hence is analytic for every $x \in \mathcal{E}$. A well known application of the uniform boundedness principle implies that the operator-valued map $\phi\colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ is analytic. By Corollary 1.12 we know that

$$\|\phi\|_{\mathbb{D}} \leq \|\phi\|_{\mathcal{M}\left(\mathrm{H}^2(\mathbb{D},\mathcal{E})\right)}.$$

This observation completes the proof. □

2. $C_{\cdot 0}$-Contractions and Inner Functions

The purpose of this chapter is to provide basic definitions and preliminaries that are needed to prove the main results of the thesis. Our starting point is the Wold decomposition for isometries. For the rest of this thesis, let $\mathcal{H}$ be a separable Hilbert space.

2.1. Wold Decomposition

We begin with the definition of reducing subspaces.

Definition 2.1. Let $T \in \mathcal{L}(\mathcal{H})$.

(i) A closed linear subspace $\mathcal{M} \subseteq \mathcal{H}$ is called *invariant for* T if $T\mathcal{M} \subseteq \mathcal{M}$.

(ii) A closed linear subspace $\mathcal{M} \subseteq \mathcal{H}$ is called *reducing for* T if $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are invariant for T.

Definition 2.2. Two operators $T \in \mathcal{L}(\mathcal{H}_1)$ and $S \in \mathcal{L}(\mathcal{H}_2)$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be *unitarily equivalent* if there exists a unitary operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$UT = SU.$$

In this case, we write $T \cong S$.

In the setting of Definition 2.2, it is easy to see that $T \cong S$ if and only if $T^* \cong S^*$.

The following theorem describes the Wold decomposition for isometries. Its proof can be found in [15, Chapter I.1].

Theorem 2.3 (Wold decomposition)**.** *Let $V \in \mathcal{L}(\mathcal{H})$ be an isometry. Then $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ decomposes into the orthogonal sum of the closed subspaces*

$$\mathcal{H}^0 = \bigcap_{n=0}^{\infty} V^n \mathcal{H} \qquad \text{and} \qquad \mathcal{H}^1 = \bigvee_{n=0}^{\infty} V^n \mathcal{W},$$

where $\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}$. Furthermore, the spaces $\mathcal{H}^0$ and $\mathcal{H}^1$ are reducing for V, the operator $V|_{\mathcal{H}^0}$ is unitary and the operator $V|_{\mathcal{H}^1}$ is unitarily equivalent to the unilateral shift

$$M_z \otimes \mathrm{id}_{\mathcal{W}}\colon \mathrm{H}^2(\mathbb{D}, \mathcal{W}) \longrightarrow \mathrm{H}^2(\mathbb{D}, \mathcal{W}), \qquad f \longmapsto zf$$

via the unitary operator

$$U\colon \mathcal{H}^1 \longrightarrow \mathrm{H}^2(\mathbb{D},\mathcal{W}), \qquad \sum_{n=0}^{\infty} V^n x_n \longmapsto \sum_{n=0}^{\infty} x_n z^n.$$

Remark 2.4. *Let $V \in \mathcal{L}(\mathcal{H})$ be an isometry and let $\mathcal{M} \subseteq \mathcal{H}$ be a closed invariant subspace for V such that $V|_{\mathcal{M}}$ is unitary. Then we obtain*

$$\mathcal{M} = \bigcap_{n=0}^{\infty} V^n \mathcal{M} \subseteq \bigcap_{n=0}^{\infty} V^n \mathcal{H}.$$

Thus, the space $\mathcal{H}^0$ in Theorem 2.3 is the largest reducing subspace for V such that $V|_{\mathcal{H}^0}$ is unitary.

Definition 2.5. A contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be *completely non-unitary* if there is no reducing subspace $\{0\} \neq \mathcal{M} \subseteq \mathcal{H}$ for T such that $T|_{\mathcal{M}}$ is unitary.

More generally, in [15, Chapter I.3] is proven that a similar decomposition exists for contractions on Hilbert spaces.

Theorem 2.6. *Every contraction $T \in \mathcal{L}(\mathcal{H})$ admits a unique orthogonal decomposition of $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ into reducing subspaces $\mathcal{H}^0$ and $\mathcal{H}^1$ for $T = T_0 \oplus T_1 \in \mathcal{L}(\mathcal{H}^0 \oplus \mathcal{H}^1)$ such that $T_0 \in \mathcal{L}(\mathcal{H}^0)$ is unitary and $T_1 \in \mathcal{L}(\mathcal{H}^1)$ is completely non-unitary. The space $\mathcal{H}^0$ is given by*

$$\mathcal{H}^0 = \{h \in \mathcal{H} \mid \|T^n h\| = \|h\| = \|T^{*n} h\| \text{ for all } n \in \mathbb{N}\}.$$

For isometries, this decomposition coincides with the Wold decomposition.

We are particularly interested in isometries with trivial unitary part $\mathcal{H}^0 = \{0\}$. This leads to the following definition.

Definition 2.7. An isometry $V \in \mathcal{L}(\mathcal{H})$ is called *pure* if $\mathcal{H}^0 = \{0\}$. Moreover, we call a commuting pair of isometries $(V_1, V_2) \in \mathcal{L}(\mathcal{H})^2$ *pure* if the isometry $V_1 V_2$ is pure.

Remark 2.8. *By the Wold decomposition theorem, pure isometries act, up to unitary equivalence, as vector-valued unilateral shifts*

$$M_z \otimes \mathrm{id}_{\mathcal{E}}\colon \mathrm{H}^2(\mathbb{D},\mathcal{E}) \longrightarrow \mathrm{H}^2(\mathbb{D},\mathcal{E}), \qquad f \longmapsto zf$$

with a suitable Hilbert space $\mathcal{E}$.

Another class of operators we are interested in is introduced in the following definition.

Definition 2.9. We say that a given contraction $T \in \mathcal{L}(\mathcal{H})$ is of *class $C_{\cdot 0}$* if

$$\lim_{n\to\infty} T^{*n} h = 0$$

for all $h \in \mathcal{H}$. In this case, we call T a *$C_{\cdot 0}$-contraction.*

Example 2.10. *Let $\mathcal{E}$ be a Hilbert space. The unilateral shift*

$$M_z \otimes \mathrm{id}_{\mathcal{E}} \colon \mathrm{H}^2(\mathbb{D}, \mathcal{E}) \longrightarrow \mathrm{H}^2(\mathbb{D}, \mathcal{E}), \qquad f \longmapsto zf$$

is of class $C_{\cdot 0}$.

Proof. Let $w \in \mathbb{D}$ and $x \in \mathcal{E}$. Since the sequence $(M_z^{*n})_{n\in\mathbb{N}}$ is norm-bounded, the assertion follows from Lemma 1.5 and the observation that

$$\lim_{n\to\infty} M_z^{*n} K(\cdot, w) \otimes x = \lim_{n\to\infty} \overline{w}^n K(\cdot, w)x = 0. \qquad \square$$

The properties described in the Definitions 2.5, 2.7 and 2.9 coincide for isometries.

Proposition 2.11. *For an isometry $V \in \mathcal{L}(\mathcal{H})$, the following statements are equivalent:*

(i) V is completely non-unitary.

(ii) V is pure.

(iii) V is of class $C_{\cdot 0}$.

Proof. Both $\mathcal{H}^0$ and $\mathcal{H}^1$ are invariant for V. Thus, if *(i)* is true, then by Theorem 2.3, the space $\mathcal{H}^0$ is trivial and hence *(ii)* follows.

Let V be pure. Then Remark 2.8 yields $V \cong M_z \otimes \mathrm{id}_{\mathcal{E}} \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{E}))$ for a suitable Hilbert space $\mathcal{E}$. Since $M_z \otimes \mathrm{id}_{\mathcal{E}}$ is of class $C_{\cdot 0}$ by Example 2.10, so is V.
Now suppose that *(iii)* holds and let $h \in \mathcal{H}^0$ be arbitrary. Then, for all $n \in \mathbb{N}$, there exists $g_n \in \mathcal{H}$ with $h = V^n g_n$. Since

$$\|h\| = \|V^n g_n\| = \|g_n\| = \|V^{*n} V^n g_n\| = \|V^{*n} h\| \xrightarrow{(n\to\infty)} 0,$$

we conclude that $h = 0$. By Remark 2.4, the isometry V is completely non-unitary. $\square$

2.2. Transfer Functions

The goal of this section is to specify a class of operator-valued functions ϕ on $\mathbb{D}$, which induce isometric multiplication operators. For this purpose, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.

Definition 2.12. For a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

we call

$$\tau_U \colon \mathbb{D} \longrightarrow \mathcal{L}(\mathcal{H}_1), \qquad z \longmapsto A + zB\left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C$$

the *transfer function of U*.

Remark 2.13. *In the setting of Definition 2.12, we have* $\|A\|, \|B\|, \|C\|, \|D\| \leq 1$. *Hence* $\tau_U\colon \mathbb{D} \to \mathcal{L}(\mathcal{H}_1)$ *is a well-defined analytic function.*

The following lemma is based on a standard computation for transfer functions. A proof can also be found in [1, Theorem 6.5].

Lemma 2.14. *For* U *and* τ_U *as in Definition 2.12, it follows that*

$$\mathrm{id}_{\mathcal{H}_1} - \tau_U(z)^* \tau_U(z) = \left(1 - |z|^2\right) C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C$$

for all $z \in \mathbb{D}$.

Proof. Since $U^*U = \mathrm{id}_{\mathcal{H}_1 \oplus \mathcal{H}_2}$, we conclude that

$$\begin{pmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\mathcal{H}_1} & 0 \\ 0 & \mathrm{id}_{\mathcal{H}_2} \end{pmatrix}.$$

Thus, it follows

$$\begin{aligned}
\mathrm{id}_{\mathcal{H}_1} - \tau_U(z)^* \tau_U(z) &= \mathrm{id}_{\mathcal{H}_1} - \left(A^* + \overline{z} C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} B^*\right) \left(A + zB \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C\right) \\
&= \mathrm{id}_{\mathcal{H}_1} - A^*A - zA^*B \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C - \overline{z} C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} B^*A \\
&\qquad - \overline{z} z C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} B^*B \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C \\
&= C^*C + zC^*D \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C + \overline{z} C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} D^*C \\
&\qquad - |z|^2 C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} \left(\mathrm{id}_{\mathcal{H}_2} - D^*D\right) \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C \\
&= C^* \big[\mathrm{id}_{\mathcal{H}_2} + zD \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} + \overline{z} \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} D^* \\
&\qquad - |z|^2 \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} \left(\mathrm{id}_{\mathcal{H}_2} - D^*D\right) \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} \big] C \\
&= C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} \big[\left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right) \left(\mathrm{id}_{\mathcal{H}_2} - zD\right) + z \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right) D \\
&\qquad + \overline{z} D^* \left(\mathrm{id}_{\mathcal{H}_2} - zD\right) - |z|^2 \left(\mathrm{id}_{\mathcal{H}_2} - D^*D\right) \big] \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C \\
&= C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} \big[\mathrm{id}_{\mathcal{H}_2} - zD - \overline{z} D^* + |z|^2 D^*D + zD - |z|^2 D^*D \\
&\qquad + \overline{z} D^* - |z|^2 D^*D - |z|^2 \mathrm{id}_{\mathcal{H}_2} + |z|^2 D^*D \big] \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C \\
&= C^* \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} \left(1 - |z|^2\right) \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} C
\end{aligned}$$

for all $z \in \mathbb{D}$. □

Remark 2.15. *Let* τ_U *as in Definition 2.12. Then Lemma 2.14 yields* $\|\tau_U(z)\| \leq 1$ *for all* $z \in \mathbb{D}$.

Remark 2.16. *Let* $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *be unitary. Lemma 2.14 applied to the unitary operator* $U^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *yields that*

$$\begin{aligned}
\mathrm{id}_{\mathcal{H}_1} - \tau_U(z) \tau_U(z)^* &= \mathrm{id}_{\mathcal{H}_1} - \tau_{U^*}(\overline{z})^* \tau_{U^*}(\overline{z}) \\
&= \left(1 - |z|^2\right) B \left(\mathrm{id}_{\mathcal{H}_2} - zD\right)^{-1} \left(\mathrm{id}_{\mathcal{H}_2} - \overline{z} D^*\right)^{-1} B^*
\end{aligned}$$

for all $z \in \mathbb{D}$.

We continue with a very simple but useful result about contractions that will be needed in the following.

Lemma 2.17. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a contraction and let* $h \in \mathcal{H}$ *be arbitrary. Then* $Th = h$ *if and only if* $T^*h = h$.

Proof. Suppose that $Th = h$. Then $\langle h, T^*h \rangle = \langle Th, h \rangle = \langle h, h \rangle = \|h\|^2$ and hence

$$\|h - T^*h\|^2 = \|h\|^2 - 2\,\mathrm{Re}\,\langle h, T^*h \rangle + \|T^*h\|^2 = \|h\|^2 - 2\,\|h\|^2 + \|T^*h\|^2 \leq 0,$$

because $\|T^*h\| \leq \|h\|$. Thus $T^*h = h$. The converse assertion follows by symmetry. □

Remark 2.18. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a contraction, let* $\lambda \in \mathbb{T}$ *and* $h \in \mathcal{H}$ *be given. By applying Lemma 2.17 to the contraction* $\frac{1}{\lambda}T$, *we obtain that* $Th = \lambda h$ *if and only if* $T^*h = \overline{\lambda}h$. *In particular, it follows that a completely non-unitary contraction cannot possess any unimodular eigenvalues. This is due to the fact that any unimodular eigenvalue of* T *provides an eigenspace* $\mathcal{M}$ *such that* $T|_{\mathcal{M}}$ *is unitary.*

Proposition 2.19. *Let* $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *be unitary. If* $A \in \mathcal{L}(\mathcal{H}_1)$ *is completely non-unitary, then, for all* $z \in \mathbb{D}$, *the operator* $\tau_U(z)$ *does not have any unimodular eigenvalues.*

Proof. Let $z \in \mathbb{D}$ and assume that $\tau_U(z)$ has a unimodular eigenvalue, that is

$$\tau_U(z)v = \lambda v$$

for some $v \in \mathcal{H}_1 \backslash \{0\}$ and $\lambda \in \mathbb{T}$. Since $\tau_U(z)$ is a contraction by Remark 2.15, Remark 2.18 yields that

$$\tau_U(z)^*v = \overline{\lambda}v,$$

and hence that

$$(\mathrm{id}_{\mathcal{H}_1} - \tau_U(z)^*\tau_U(z))\,v = 0.$$

Thus, Lemma 2.14 yields $Cv = 0$ and the definition of τ_U implies

$$Av = \tau_U(z)v = \lambda v.$$

Then A has a non-trivial unitary part and is therefore not completely non-unitary. □

Proposition 2.20. *Let* $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *be a unitary operator. Let* $A = A_0 \oplus A_1 \in \mathcal{L}(\mathcal{H}_1^0 \oplus \mathcal{H}_1^1)$ *be the orthogonal decomposition of* $A \in \mathcal{L}(\mathcal{H}_1)$ *into its* *unitary part* A_0 *on* $\mathcal{H}_1^0$ *and its completely non-unitary part* A_1 *on* $\mathcal{H}_1^1$ *as in Theorem 2.6. Then*

$$U_1 = \begin{pmatrix} A_1 & B \\ C|_{\mathcal{H}_1^1} & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1^1 \oplus \mathcal{H}_2)$$

is unitary and the transfer function τ_U *of* U *admits the decomposition*

$$\tau_U(z) = \begin{pmatrix} A_0 & 0 \\ 0 & \tau_{U_1}(z) \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1^0 \oplus \mathcal{H}_1^1)$$

for all $z \in \mathbb{D}$.

Proof. Because U is unitary, we have

$$A^*A + C^*C = \mathrm{id}_{\mathcal{H}_1} \qquad \text{and} \qquad AA^* + BB^* = \mathrm{id}_{\mathcal{H}_1}.$$

Since $A^*A|_{\mathcal{H}_1^0} = AA^*|_{\mathcal{H}_1^0} = \mathrm{id}_{\mathcal{H}_1^0}$, we conclude that

$$\mathcal{H}_1^0 \subseteq \ker C \qquad \text{and} \qquad \mathcal{H}_1^0 \subseteq \ker B^*,$$

or, equivalently,

$$\mathcal{H}_1^0 \subseteq \ker C \qquad \text{and} \qquad \operatorname{Im} B \subseteq \mathcal{H}_1^1.$$

Thus, one easily checks that

$$U_1 = \begin{pmatrix} A_1 & B \\ C|_{\mathcal{H}_1^1} & D \end{pmatrix} \in \mathcal{L}\left(\mathcal{H}_1^1 \oplus \mathcal{H}_2\right)$$

is isometric and surjective again and hence unitary.

The matrix representation of $\tau_U(z) = A + zB(\mathrm{id}_{\mathcal{H}_2} - zD)^{-1}C \in \mathcal{L}(\mathcal{H}_1)$ $(z \in \mathbb{D})$ with respect to the decomposition $\mathcal{H}_1 = \mathcal{H}_1^0 \oplus \mathcal{H}_1^1$ is given by

$$\tau_U(z) = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 + zB(\mathrm{id}_{\mathcal{H}_2} - zD)^{-1}C|_{\mathcal{H}_1^1} \end{pmatrix}$$

for all $z \in \mathbb{D}$. Hence $\tau_U(z) = A_0 \oplus \tau_{U_1}(z) \in \mathcal{L}\left(\mathcal{H}_1^0 \oplus \mathcal{H}_1^1\right)$ $(z \in \mathbb{D})$, where

$$\tau_{U_1} \colon \mathbb{D} \longrightarrow \mathcal{L}\left(\mathcal{H}_1^1\right), \qquad z \longmapsto A_1 + zB(\mathrm{id}_{\mathcal{H}_2} - zD)^{-1}C|_{\mathcal{H}_1^1}$$

is the transfer function of U_1. $\square$

Let m be the normalized, one-dimensional Lebesgue-measure on $\mathbb{T}$.

Definition 2.21. A bounded analytic function $\phi\colon \mathbb{D} \to \mathcal{L}(\mathcal{H})$ is called *inner* if the limit

$$\text{SOT-}\lim_{r \uparrow 1} \phi(rz)$$

exists in the strong operator topology and is an isometry on $\mathcal{H}$ for m-almost every $z \in \mathbb{T}$.

One can show that the limit in Definition 2.21 exists for every bounded analytic function $\phi\colon \mathbb{D} \to \mathcal{L}(\mathcal{H})$ for m-almost every $z \in \mathbb{T}$. For a proof see [15, Chapter V.2].

For $f \in \mathrm{H}^2(\mathbb{D}, \mathcal{H})$ one can show that there is an m-zero set $N \subseteq \mathbb{T}$ such that the limit

$$f^*(z) = \lim_{r \uparrow 1} f(rz) \in \mathcal{H}$$

exists in the norm-topology of $\mathcal{H}$ for every $z \in \mathbb{T} \setminus N$. The function

$$f^*\colon \mathbb{T} \longrightarrow \mathcal{H}, \qquad z \longmapsto \begin{cases} \lim_{r\uparrow 1} f(rz), & z \notin N \\ 0, & z \in N \end{cases}$$

defines an element in $\mathrm{L}^2(\mathbb{T}, \mathcal{H})$ which is independent of the choice of N and satisfies the identity

$$\|f\|^2_{\mathrm{H}^2(\mathbb{D},\mathcal{H})} = \int_{\mathbb{T}} \|f^*(z)\|^2 dm(z).$$

For a bounded analytic function $\phi\colon \mathbb{D} \to \mathcal{L}(\mathcal{H})$, there is a zero set $N \subseteq \mathbb{T}$ such that the limit

$$\phi^*(z) = \text{SOT-}\lim_{r\uparrow 1} \phi(rz) \in \mathcal{L}(\mathcal{H})$$

exists in the strong operator topology of $\mathcal{L}(\mathcal{H})$ for every $z \in \mathbb{T} \setminus N$. The function

$$\phi^*\colon \mathbb{T} \longrightarrow \mathcal{L}(\mathcal{H}), \qquad z \longmapsto \begin{cases} \text{SOT-}\lim_{r\uparrow 1} \phi(rz), & z \in \mathbb{T} \setminus N \\ 0, & z \in N \end{cases}$$

defines an element in $\mathrm{L}^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ which is independent of the choice of N and satisfies the identity

$$\|\phi\|_{\mathbb{D}} = \|\phi^*\|_{\mathrm{L}^\infty(\mathbb{T},\mathcal{L}(\mathcal{H}))}.$$

Proofs of these results can be found in [13, Sections 4.5-4.7].

Let $\phi\colon \mathbb{D} \to \mathcal{L}(\mathcal{H})$ be a bounded analytic function and let $\phi^*\colon \mathbb{T} \to \mathcal{L}(\mathcal{H})$ be the radial limit of ϕ defined as above. Denote by

$$\rho\colon\ \mathrm{H}^2(\mathbb{D}, \mathcal{H}) \longrightarrow \mathrm{L}^2(\mathbb{T}, \mathcal{H}), \qquad f \longmapsto f^*$$

the isometric embedding associating with each function $f \in \mathrm{H}^2(\mathbb{D}, \mathcal{H})$ its radial limit $f^* \in \mathrm{L}^2(\mathbb{T}, \mathcal{H})$. For $f \in \mathrm{H}^2(\mathbb{D}, \mathcal{H})$, there is an m-zero set $N \subseteq \mathbb{T}$ such that

$$\phi^*(z) = \text{SOT-}\lim_{r\uparrow 1} \phi(rz) \quad \text{and} \quad f^*(z) = \lim_{r\uparrow 1} f(rz)$$

hold for all $z \in \mathbb{T} \setminus N$. Since

$$\begin{aligned}\|\phi(rz)f(rz) - \phi^*(z)f^*(z)\| &= \|\phi(rz)(f(rz) - f^*(z)) + (\phi(rz) - \phi^*(z))f^*(z)\| \\ &\leq \|\phi\|_{\mathbb{D}} \|f(rz) - f^*(z)\| + \|(\phi(rz) - \phi^*(z))f^*(z)\| \xrightarrow{(r\uparrow 1)} 0\end{aligned}$$

for every $z \in \mathbb{T} \setminus N$, it follows that the multiplication operators

$$M_\phi \colon \mathrm{H}^2(\mathbb{D}, \mathcal{H}) \longrightarrow \mathrm{H}^2(\mathbb{D}, \mathcal{H}) \qquad \text{and} \qquad M_{\phi^*} \colon \mathrm{L}^2(\mathbb{T}, \mathcal{H}) \longrightarrow \mathrm{L}^2(\mathbb{T}, \mathcal{H})$$

satisfy the intertwining relation

$$\rho M_\phi = M_{\phi^*} \rho.$$

Lemma 2.22. *For an inner multiplier $\phi \in \mathcal{M}\left(\mathrm{H}^2(\mathbb{D}, \mathcal{H})\right)$, the multiplication operator $M_\phi \colon \mathrm{H}^2(\mathbb{D}, \mathcal{H}) \to \mathrm{H}^2(\mathbb{D}, \mathcal{H})$ is an isometry.*

Proof. By definition, there exists an m-zero set $N \subseteq \mathbb{T}$ such that $\phi^*(z) \in \mathcal{L}(\mathcal{H})$ is an isometry for every $z \in \mathbb{T} \setminus N$. But then, for $f \in \mathrm{H}^2(\mathbb{D}, \mathcal{H})$,

$$\int_{\mathbb{T}} \|\phi^*(z)f(z)\|^2 \, dm(z) = \int_{\mathbb{T}} \|f(z)\|^2 \, dm(z)$$

and hence

$$\|M_\phi f\|_{\mathrm{H}^2(\mathbb{D},\mathcal{H})} = \|(\rho M_\phi)\, f\|_{\mathrm{L}^2(\mathbb{T},\mathcal{H})} = \|(M_{\phi^*}\rho)\, f\|_{\mathrm{L}^2(\mathbb{T},\mathcal{H})} = \|f\|_{\mathrm{H}^2(\mathbb{D},\mathcal{H})}. \qquad \square$$

Proposition 2.23. *Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ be a unitary operator and let $D = D_0 \oplus D_1 \in \mathcal{L}\left(\mathcal{H}_2^0 \oplus \mathcal{H}_2^1\right)$ be the orthogonal decomposition of $D \in \mathcal{L}(\mathcal{H}_2)$ from Theorem 2.6. If $\sigma(D_1) \subseteq \mathbb{D}$, then the transfer function τ_U of U extends to a holomorphic map*

$$\tau_U \colon D_R(0) \longrightarrow \mathcal{L}(\mathcal{H}_1)$$

on a disc with radius $R > 1$ and $\tau_U(z) \in \mathcal{L}(\mathcal{H}_1)$ is a unitary operator for every $z \in \mathbb{T}$. The condition $\sigma(D_1) \subseteq \mathbb{D}$ is satisfied, for instance, if $\dim \mathcal{H}_2 < \infty$ or if $D = 0$. In particular, the transfer function τ_U is inner.

Proof. Since

$$CC^* + DD^* = \mathrm{id}_{\mathcal{H}_2} \qquad \text{and} \qquad DD^*|_{\mathcal{H}_2^0} = \mathrm{id}_{\mathcal{H}_2^0},$$

we find that $\mathcal{H}_2^0 \subseteq \ker C^*$, or, equivalently,

$$\operatorname{Im} C \subseteq \mathcal{H}_2^1.$$

But then the hypothesis that $\sigma(D_1) \subseteq \mathbb{D}$ implies that the transfer function τ_U of U has the form

$$\tau_U(z) = A + zB(\mathrm{id}_{\mathcal{H}_2^1} - zD_1)^{-1}C \qquad (z \in \mathbb{D})$$

and hence extends to a holomorphic function $\tau_U \colon D_R(0) \to \mathcal{L}(\mathcal{H}_1)$ on an open disc $D_R(0)$ with $R > 1$. In the same way one finds that the function

$$(\mathrm{id}_{\mathcal{H}_2} - \overline{z}D^*)^{-1}(\mathrm{id}_{\mathcal{H}_2} - zD)^{-1}C = (\mathrm{id}_{\mathcal{H}_2^1} - \overline{z}D_1^*)^{-1}(\mathrm{id}_{\mathcal{H}_2^1} - zD_1)^{-1}C \colon \mathbb{D} \longrightarrow \mathcal{L}(\mathcal{H}_2)$$

admits a continuous extension to an open neighborhood of $\overline{\mathbb{D}}$. But then Lemma 2.14 implies that

$$\tau_U(z)^*\tau_U(z) = \mathrm{id}_{\mathcal{H}_1}$$

for all $z \in \mathbb{T}$. Since

$$D^* = \left(D^*|_{\mathcal{H}_2^0}\right) \oplus \left(D^*|_{\mathcal{H}_2^1}\right)$$

is the decomposition of D^* into its unitary part $D^*|_{\mathcal{H}_2^0}$ and completely non-unitary part $D^*|_{\mathcal{H}_2^1}$ and since $\sigma\left(D^*|_{\mathcal{H}_2^1}\right) \subseteq \mathbb{D}$, it follows in exactly the same way from Remark 2.16 that also

$$\tau_U(z)\tau_U(z)^* = \mathrm{id}_{\mathcal{H}_1}$$

for all $z \in \mathbb{T}$. The remaining assertion in Proposition 2.23 obviously holds. $\square$

3. Sz.-Nagy's and Foias' Dilation Theory

The aim of this chapter is to prepare the construction of dilations for two commuting contractions. We begin by proving some elementary results on defect operators.

Definition 3.1. For a contraction $S \in \mathcal{L}(\mathcal{H})$, we call $D_S = (\mathrm{id}_{\mathcal{H}} - SS^*)^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H})$ the *defect operator of* S. Furthermore, we define the *defect space of* S by $\mathcal{D}_S = \overline{D_S \mathcal{H}} \subseteq \mathcal{H}$.

Lemma 3.2. *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be commuting contractions. Define $T = T_1 T_2$. For all $h \in \mathcal{H}$, we have:*

(i) $\|D_{T_1} T_2^* h\|^2 + \|D_{T_2} h\|^2 = \|D_{T_1} h\|^2 + \|D_{T_2} T_1^* h\|^2$,

(ii) $\|D_T h\|^2 = \|D_{T_1} h\|^2 + \|D_{T_2} T_1^* h\|^2$.

Proof. By definition

$$\begin{aligned} T_2 D_{T_1}^2 T_2^* + D_{T_2}^2 &= T_2 T_2^* - T_2 T_1 T_1^* T_2^* + (\mathrm{id}_{\mathcal{H}} - T_2 T_2^*) \\ &= (\mathrm{id}_{\mathcal{H}} - T_1 T_1^*) + T_1 T_1^* - T_1 T_2 T_2^* T_1^* \\ &= D_{T_1}^2 + T_1 D_{T_2}^2 T_1^* \end{aligned}$$

and therefore

$$\begin{aligned} \|D_{T_1} T_2^* h\|^2 + \|D_{T_2} h\|^2 &= \left\langle T_2 D_{T_1}^2 T_2^* h, h \right\rangle + \left\langle D_{T_2}^2 h, h \right\rangle \\ &= \left\langle \left(T_2 D_{T_1}^2 T_2^* + D_{T_2}^2 \right) h, h \right\rangle \\ &= \left\langle \left(D_{T_1}^2 + T_1 D_{T_2}^2 T_1^* \right) h, h \right\rangle \\ &= \left\langle D_{T_1}^2 h, h \right\rangle + \left\langle T_1 D_{T_2}^2 T_1^* h, h \right\rangle \\ &= \|D_{T_1} h\|^2 + \|D_{T_2} T_1^* h\|^2 \end{aligned}$$

as well as

$$\begin{aligned} \|D_T h\|^2 &= \left\langle D_T^2 h, h \right\rangle = \left\langle \left(\mathrm{id}_{\mathcal{H}} - T_1 T_2 (T_1 T_2)^* \right) h, h \right\rangle \\ &= \left\langle \left(\mathrm{id}_{\mathcal{H}} - T_1 T_1^* \right) h, h \right\rangle + \left\langle \left(T_1 T_1^* - T_1 T_2 T_2^* T_1^* \right) h, h \right\rangle \\ &= \left\langle D_{T_1}^2 h, h \right\rangle + \left\langle T_1 D_{T_2}^2 T_1^* h, h \right\rangle \\ &= \|D_{T_1} h\|^2 + \|D_{T_2} T_1^* h\|^2 \end{aligned}$$

for all $h \in \mathcal{H}$. $\square$

Remark 3.3. *For commuting contractions $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and $T = T_1T_2$, Lemma 3.2 (ii) implies that there is a unique isometry $V\colon \mathcal{D}_T \longrightarrow \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$ satisfying*

$$V\left(D_T h\right) = \left(D_{T_1}h, D_{T_2}T_1^*h\right)$$

for all $h \in \mathcal{H}$.

In the following proposition we construct a unitary operator on a certain Hilbert space, which will be crucial in the following chapter. In particular, one should pay attention to the construction of the Hilbert space $\mathcal{D}$.

Proposition 3.4. *For commuting contractions $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, there exist a Hilbert space $\mathcal{D}$ and a unitary operator $U\colon (\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2} \to (\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}$ satisfying*

$$U\left(\begin{pmatrix}\begin{pmatrix}0\\ D_{T_1}T_2^*h\end{pmatrix}\\ D_{T_2}h\end{pmatrix}\right) = \begin{pmatrix}\begin{pmatrix}0\\ D_{T_1}h\end{pmatrix}\\ D_{T_2}T_1^*h\end{pmatrix} \qquad (h \in \mathcal{H}). \tag{3.1}$$

Proof. Let $\mathcal{D}$ be a Hilbert space such that

$$\dim \mathcal{D} = \begin{cases} 0, & \text{if } \dim \mathcal{D}_{T_1}, \dim \mathcal{D}_{T_2} < \infty \\ \dim \mathcal{D}_{T_1} + \dim \mathcal{D}_{T_2}, & \text{else.} \end{cases}$$

If we set

$$\begin{aligned} M_1 &= \{0_{\mathcal{D}}\} \oplus \{D_{T_1}T_2^*h \oplus D_{T_2}h \mid h \in \mathcal{H}\},\\ M_2 &= \{0_{\mathcal{D}}\} \oplus \{D_{T_1}h \oplus D_{T_2}T_1^*h \mid h \in \mathcal{H}\}, \end{aligned}$$

we obtain that

$$(\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2} = \overline{M_i} \oplus \overline{M_i}^{\perp} \qquad (i = 1, 2),$$

where

$$\dim \overline{M_1} = \dim \overline{M_2}$$

and

$$\overline{M_i}^{\perp} = ((\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}) \ominus \overline{M_i} \qquad (i = 1, 2).$$

By Lemma 3.2 (i), $\tilde{U}\colon M_1 \to M_2$ defined by Equation 3.1 is a surjective isometry. Thus, $\tilde{U}$ extends to a unitary operator $U_1\colon \overline{M_1} \to \overline{M_2}$.

If $\dim \mathcal{D}_{T_1}, \dim \mathcal{D}_{T_2} < \infty$, then $\dim(\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2} < \infty$ and we conclude that $\dim \overline{M_1}^{\perp} = \dim \overline{M_2}^{\perp}$. On the other hand, if $\dim \mathcal{D}_{T_1} = \infty$ or $\dim \mathcal{D}_{T_2} = \infty$, we consider the set

$$\{(h, 0, 0) \mid h \in \mathcal{D}\} \subseteq \overline{M_i}^{\perp} \qquad (i = 1, 2).$$

Since $\dim\{(h,0,0) \mid h \in \mathcal{D}\} = \dim \mathcal{D}$, we find that

$$\dim \mathcal{D} \leq \dim \overline{M_i}^{\perp} \leq \dim((\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}) = \dim \mathcal{D} \qquad (i = 1, 2),$$

which yields $\dim \overline{M_1}^{\perp} = \dim \overline{M_2}^{\perp}$ (see Theorem A.5).

Hence, there is a unitary operator $U_2 \colon \overline{M_1}^{\perp} \to \overline{M_2}^{\perp}$ and, in particular, $U = U_1 \oplus U_2$ is a unitary operator satisfying Equation 3.1. $\square$

We continue with a generalization of Definition 2.2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.

Definition 3.5. We say that $(T_1, ..., T_n) \in \mathcal{L}(\mathcal{H}_1)^n$ and $(S_1, ..., S_n) \in \mathcal{L}(\mathcal{H}_2)^n$ are *unitarily equivalent* if there exists a unitary operator $U \colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$UT_i = S_i U \qquad (i = 1, ..., n).$$

In this case, we write $(T_1, ..., T_n) \cong (S_1, ..., S_n)$.

In the setting of Definition 3.5, it can be easily seen that $(T_1, ..., T_n)$ and $(S_1, ..., S_n)$ are unitarily equivalent if and only if $(T_1^*, ..., T_n^*)$ and $(S_1^*, ..., S_n^*)$ are unitarily equivalent.

Definition 3.6. Let $T = (T_1, ..., T_n) \in \mathcal{L}(\mathcal{H}_1)^n$ and $S = (S_1, ..., S_n) \in \mathcal{L}(\mathcal{H}_2)^n$ be commuting tuples of bounded linear operators. We call S a *coextension* of T if there exists an isometry $\Pi \colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\Pi T_i^* = S_i^* \Pi \qquad (i = 1, ..., n).$$

In the setting of Definition 3.6, the closed subspace $\mathcal{Q} = \Pi\mathcal{H}_1 \subseteq \mathcal{H}_2$ is invariant for $(S_1^*, ..., S_n^*)$ and

$$(T_1^*, ..., T_n^*) \cong (S_1^*, ..., S_n^*)|_{\mathcal{Q}}$$

as well as

$$(T_1, ..., T_n) \cong P_{\mathcal{Q}}(S_1, ..., S_n)|_{\mathcal{Q}},$$

where the restrictions and compressions are formed componentwise.

Moreover, since $\mathcal{Q}$ is a joint $(S_1^*, ..., S_n^*)$-invariant subspace of $\mathcal{H}_2$, we find that

$$S_1^*|_{\mathcal{Q}} \cdot ... \cdot S_n^*|_{\mathcal{Q}} = (S_1^* \cdot ... \cdot S_n^*)|_{\mathcal{Q}}$$

and hence

$$P_{\mathcal{Q}} S_1|_{\mathcal{Q}} \cdot ... \cdot P_{\mathcal{Q}} S_n|_{\mathcal{Q}} = P_{\mathcal{Q}}(S_1 \cdot ... \cdot S_n)|_{\mathcal{Q}}.$$

Thus, we conclude that

$$p(T_1, ..., T_n) \cong P_{\mathcal{Q}} p(S_1, ..., S_n)|_{\mathcal{Q}}$$

for every polynomial $p \in \mathbb{C}[z_1, ..., z_n]$.

Theorem 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $C_{\cdot 0}$-contraction. Then*

$$\Pi_T \colon \mathcal{H} \longrightarrow \mathrm{H}^2(\mathbb{D}, \mathcal{D}_T), \qquad h \longmapsto \sum_{k=0}^{\infty} \left(D_T T^{*k} h \right) z^k$$

is a well-defined isometry satisfying

$$\Pi_T T^* = (M_z^* \otimes \mathrm{id}_{\mathcal{D}_T}) \, \Pi_T.$$

Furthermore, if $V \in \mathcal{L}(\mathcal{D}_T, \mathcal{E})$ is an isometry with values in a Hilbert space $\mathcal{E}$, then

$$\Pi_T^V \colon \mathcal{H} \longrightarrow \mathrm{H}^2(\mathbb{D}, \mathcal{E}), \qquad h \longmapsto \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V \right) \Pi_T h$$

is an isometry satisfying

$$\Pi_T^V T^* = (M_z^* \otimes \mathrm{id}_{\mathcal{E}}) \, \Pi_T^V.$$

Proof. For all $h \in \mathcal{H}$ and $N \in \mathbb{N}$, one finds that

$$\begin{aligned}
\sum_{k=0}^{N} \left\| D_T T^{*k} h \right\|^2 &= \sum_{k=0}^{N} \left\langle D_T T^{*k} h, D_T T^{*k} h \right\rangle \\
&= \left\langle \sum_{k=0}^{N} T^k D_T^2 T^{*k} h, h \right\rangle = \left\langle \sum_{k=0}^{N} T^k \left(\mathrm{id}_{\mathcal{H}} - TT^* \right) T^{*k} h, h \right\rangle \\
&= \left\langle \sum_{k=0}^{N} \left(T^k T^{*k} - T^{k+1} T^{*(k+1)} \right) h, h \right\rangle = \left\langle \left(\mathrm{id}_{\mathcal{H}} - T^{N+1} T^{*(N+1)} \right) h, h \right\rangle \\
&= \|h\|^2 - \left\| T^{*(N+1)} h \right\|^2.
\end{aligned}$$

Thus, by passing to the limit as $N \to \infty$, we obtain that

$$\sum_{k=0}^{\infty} \left\| D_T T^{*k} h \right\|^2 = \|h\|^2$$

for all $h \in \mathcal{H}$, since T is a $C_{\cdot 0}$-contraction. Hence Π_T is a well-defined isometry. Furthermore, it holds

$$\begin{aligned}
\Pi_T T^* h &= \sum_{k=0}^{\infty} \left(D_T T^{*k} T^* h \right) z^k = \sum_{k=0}^{\infty} \left(D_T T^{*(k+1)} h \right) M_z^* (z^{k+1}) \\
&= (M_z^* \otimes \mathrm{id}_{\mathcal{D}_T}) \sum_{k=0}^{\infty} \left(D_T T^{*k} h \right) z^k = (M_z^* \otimes \mathrm{id}_{\mathcal{D}_T}) \, \Pi_T h
\end{aligned}$$

for all $h \in \mathcal{H}$.

Now, let $V \in \mathcal{L}(\mathcal{D}_T, \mathcal{E})$ be an isometry. Then, $\Pi_T^V = \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \Pi_T$ is an isometry satisfying

$$\begin{aligned}
\Pi_T^V T^* h &= \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \Pi_T T^* h = \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) (M_z^* \otimes \mathrm{id}_{\mathcal{D}_T}) \Pi_T h \\
&= (M_z^* \otimes \mathrm{id}_{\mathcal{E}}) \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \Pi_T h \\
&= (M_z^* \otimes \mathrm{id}_{\mathcal{E}}) \Pi_T^V h
\end{aligned}$$

for all $h \in \mathcal{H}$. □

Corollary 3.8. *In the setting of Theorem 3.7, we have*

$$T \cong P_{\mathcal{Q}} (M_z \otimes \mathrm{id}_{\mathcal{D}_T})|_{\mathcal{Q}}$$

for the $(M_z^* \otimes \mathrm{id}_{\mathcal{D}_T})$-invariant subspace $\mathcal{Q} = \Pi_T \mathcal{H}$ of $\mathrm{H}^2(\mathbb{D}, \mathcal{D}_T)$ and

$$T \cong P_{\tilde{\mathcal{Q}}} (M_z \otimes \mathrm{id}_{\mathcal{E}})|_{\tilde{\mathcal{Q}}}$$

for the $(M_z^* \otimes \mathrm{id}_{\mathcal{E}})$-invariant subspace $\tilde{\mathcal{Q}} = \Pi_T^V \mathcal{H}$ of $\mathrm{H}^2(\mathbb{D}, \mathcal{E})$.

4. Pairs of Commuting Contractions

We are now able to prove the main results of this thesis described in the introduction.

4.1. Contractions with Finite Dimensional Defect Spaces

In this section, we only consider pairs of commuting contractions with finite dimensional defect spaces. Note that in these cases the Hilbert space $\mathcal{D}$ constructed in the proof of Proposition 3.4 vanishes.

Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions such that $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$. Proposition 3.4 provides a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \longrightarrow \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D_{T_1} T_2^* h \\ D_{T_2} h \end{pmatrix} = \begin{pmatrix} D_{T_1} h \\ D_{T_2} T_1^* h \end{pmatrix}$$

for all $h \in \mathcal{H}$. Then, the unitary operator

$$U^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} : \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \longrightarrow \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$$

satisfies

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} D_{T_1} h \\ D_{T_2} T_1^* h \end{pmatrix} = \begin{pmatrix} D_{T_1} T_2^* h \\ D_{T_2} h \end{pmatrix}$$

for all $h \in \mathcal{H}$. Since $\dim \mathcal{D}_{T_2} < \infty$, the transfer functions $\tau_U, \tau_{U^*} : \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T_2})$ of U and U^* are inner by Proposition 2.23.

Note that

$$\tilde{U} = \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} : \mathcal{D}_{T_2} \oplus \mathcal{D}_{T_1} \longrightarrow \mathcal{D}_{T_2} \oplus \mathcal{D}_{T_1}$$

is a unitary operator such that

$$\begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \begin{pmatrix} D_{T_2} T_1^* h \\ D_{T_1} h \end{pmatrix} = \begin{pmatrix} D_{T_2} h \\ D_{T_1} T_2^* h \end{pmatrix}$$

for all $h \in \mathcal{H}$. Thus, the unitary operator

$$\tilde{U}^* = \begin{pmatrix} D & C \\ B & A \end{pmatrix} : \mathcal{D}_{T_2} \oplus \mathcal{D}_{T_1} \longrightarrow \mathcal{D}_{T_2} \oplus \mathcal{D}_{T_1}$$

satisfies

$$\begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} D_{T_2} h \\ D_{T_1} T_2^* h \end{pmatrix} = \begin{pmatrix} D_{T_2} T_1^* h \\ D_{T_1} h \end{pmatrix}$$

for all $h \in \mathcal{H}$. Again, since $\dim \mathcal{D}_{T_1} < \infty$, the transfer functions $\tau_{\tilde{U}}, \tau_{\tilde{U}^*} : \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T_1})$ of $\tilde{U}$ and $\tilde{U}^*$ are inner by Proposition 2.23.

In the following, we consider those pairs of commuting contractions $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ where T_1 or T_2 is of class $C_{\cdot 0}$. We will see that the results in these two cases are symmetric.

Lemma 4.1. *Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions and let T_1 be of class $C_{\cdot 0}$. Then, it holds*

$$D_{T_1} T_2^* = A^* D_{T_1} + \sum_{i=0}^{\infty} C^* D^{*i} B^* D_{T_1} T_1^{*(i+1)},$$

where the series converges in the strong operator topology.

Proof. Let $h \in \mathcal{H}$. Using the operator U^*, we have

$$D_{T_1} T_2^* h = A^* D_{T_1} h + C^* D_{T_2} T_1^* h \tag{4.1}$$

and

$$D_{T_2} h = B^* D_{T_1} h + D^* D_{T_2} T_1^* h. \tag{4.2}$$

Replacing h by $T_1^* h$ in Equation 4.2, we have

$$D_{T_2} T_1^* h = B^* D_{T_1} T_1^* h + D^* D_{T_2} T_1^{*2} h. \tag{4.3}$$

Inserting Equation 4.3 in Equation 4.1 yields

$$\begin{aligned} D_{T_1} T_2^* h &= A^* D_{T_1} h + C^* \left(B^* D_{T_1} T_1^* h + D^* D_{T_2} T_1^{*2} h \right) \\ &= A^* D_{T_1} h + C^* B^* D_{T_1} T_1^* h + C^* D^* D_{T_2} T_1^{*2} h. \end{aligned} \tag{4.4}$$

We now repeat this step by replacing h by $T_1^* h$ in Equation 4.3, that is

$$D_{T_2} T_1^{*2} h = B^* D_{T_1} T_1^{*2} h + D^* D_{T_2} T_1^{*3} h,$$

and insert this in Equation 4.4 to observe

$$\begin{aligned} D_{T_1}T_2^*h &= A^*D_{T_1}h + C^*B^*D_{T_1}T_1^*h + C^*D^*B^*D_{T_1}T_1^{*2}h + C^*D^{*2}D_{T_2}T_1^{*3}h \\ &= A^*D_{T_1}h + \sum_{i=0}^{1} C^*D^{*i}B^*D_{T_1}T_1^{*(i+1)}h + C^*D^{*2}D_{T_2}T_1^{*3}h. \end{aligned}$$

Successively, we obtain

$$D_{T_1}T_2^*h = A^*D_{T_1}h + \sum_{i=0}^{N} C^*D^{*i}B^*D_{T_1}T_1^{*(i+1)}h + C^*D^{*(N+1)}D_{T_2}T_1^{*(N+2)}h$$

for all $h \in \mathcal{H}$ and $N \in \mathbb{N}$. Since $\|D\| \le 1$ and T_1 is of class $C_{\cdot 0}$, it follows that

$$\lim_{N\to\infty} \left\| C^*D^{*(N+1)}D_{T_2}T_1^{*(N+2)}h \right\| = 0$$

for all $h \in \mathcal{H}$. Finally, for all $h \in \mathcal{H}$, we conclude

$$\begin{aligned} \left\| D_{T_1}T_2^*h - A^*D_{T_1}h - \sum_{i=0}^{N} C^*D^{*i}B^*D_{T_1}T_1^{*(i+1)}h \right\| &= \left\| C^*D^{*(N+1)}D_{T_2}T_1^{*(N+2)}h \right\| \\ &\le \left\| T_1^{*(N+2)}h \right\| \xrightarrow{(N\to\infty)} 0. \end{aligned}$$

□

A similar proof using the operator $\tilde{U}^*$ instead of U^* yields the following remark.

Remark 4.2. *Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions and let T_2 be of class $C_{\cdot 0}$. Then we have*

$$D_{T_2}T_1^* = DD_{T_2} + \sum_{i=0}^{\infty} CA^iBD_{T_2}T_2^{*(i+1)},$$

where the series converges in the strong operator topology.

Theorem 4.3. *Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions such that T_1 is of class $C_{\cdot 0}$ and let $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$. Then, the transfer function $\tau_U : \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T_1})$ of U induces an isometric multiplication operator $M_{\tau_U} \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1}))$ such that*

$$(T_1, T_2) \cong P_{\mathcal{Q}} \left(M_z \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_{\tau_U} \right) |_{\mathcal{Q}}$$

for the joint $\left(M_z^* \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_{\tau_U}^* \right)$-invariant subspace $\mathcal{Q} = \Pi_{T_1}\mathcal{H} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1})$.

Proof. Since T_1 is of class $C_{\cdot 0}$, Theorem 3.7 yields

$$T_1 \cong P_{\mathcal{Q}} \left(M_z \otimes \mathrm{id}_{\mathcal{D}_{T_1}} \right) |_{\mathcal{Q}}$$

for the $\left(M_z^* \otimes \mathrm{id}_{\mathcal{D}_{T_1}}\right)$-invariant subspace $\mathcal{Q} = \Pi_{T_1}\mathcal{H}$ of $\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1})$. As mentioned before, the transfer function τ_U of U is inner and hence induces the isometry $M_{\tau_U} \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1}))$ by Lemma 2.22. Therefore it is enough to show that Π_{T_1} intertwines T_2^* and $M_{\tau_U}^*$. Let $h \in \mathcal{H}$, $n \geq 0$ and $\eta \in \mathcal{D}_{T_1}$. We obtain

$$\begin{aligned}
\left\langle M_{\tau_U}^* \Pi_{T_1} h, z^n \eta \right\rangle &= \left\langle \Pi_{T_1} h, M_{\tau_U}\left(z^n \eta\right)\right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} \left(D_{T_1} T_1^{*k} h\right) z^k, \left(A + zB\left(\mathrm{id}_{\mathcal{D}_{T_1}} - zD\right)^{-1} C\right)\left(z^n \eta\right)\right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} \left(D_{T_1} T_1^{*k} h\right) z^k, A\left(z^n \eta\right) + \sum_{i=0}^{\infty} \left(BD^i C\eta\right) z^{i+n+1}\right\rangle \\
&= \left\langle A^* D_{T_1} T_1^{*n} h, \eta\right\rangle + \sum_{i=0}^{\infty} \left\langle C^* D^{*i} B^* D_{T_1} T_1^{*(i+n+1)} h, \eta\right\rangle \\
&= \left\langle \left(A^* D_{T_1} + \sum_{i=0}^{\infty} C^* D^{*i} B^* D_{T_1} T_1^{*(i+1)}\right) T_1^{*n} h, \eta\right\rangle \\
&= \left\langle D_{T_1} T_2^* \left(T_1^{*n} h\right), \eta\right\rangle,
\end{aligned}$$

where we have used Lemma 4.1 in the last equation.

On the other hand, we observe

$$\begin{aligned}
\left\langle \Pi_{T_1} T_2^* h, z^n \eta\right\rangle &= \left\langle \sum_{k=0}^{\infty} \left(D_{T_1} T_1^{*k} T_2^* h\right) z^k, z^n \eta\right\rangle \\
&= \left\langle D_{T_1} T_1^{*n} T_2^* h, \eta\right\rangle \\
&= \left\langle D_{T_1} T_2^* \left(T_1^{*n} h\right), \eta\right\rangle
\end{aligned}$$

and hence $M_{\tau_U}^* \Pi_{T_1} = \Pi_{T_1} T_2^*$. □

A similar proof using Remark 4.2 instead of Lemma 4.1 yields the following remark.

Remark 4.4. *Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting contractions such that T_2 is of class $C_{\cdot 0}$ and let $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$. Then, the transfer function $\tau_{\tilde{U}} \colon \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T_2})$ of $\tilde{U}$ is induces an isometric multiplication operator $M_{\tau_{\tilde{U}}} \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_2}))$ such that*

$$(T_1, T_2) \cong P_{\mathcal{Q}}\left(M_{\tau_{\tilde{U}}}, M_z \otimes \mathrm{id}_{\mathcal{D}_{T_2}}\right)\big|_{\mathcal{Q}}$$

for the joint $\left(M_{\tau_{\tilde{U}}}^*, M_z^* \otimes \mathrm{id}_{\mathcal{D}_{T_2}}\right)$-invariant subspace $\mathcal{Q} = \Pi_{T_2}\mathcal{H} \subseteq \mathrm{H}^2\left(\mathbb{D}, \mathcal{D}_{T_2}\right)$.

4.2. Factorizations of Contractions

In this section, we consider pairs of commuting contractions $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ such that the product $T_1 T_2$ is of class $C_{\cdot 0}$. In particular, we do not assume the defect spaces of

T_1 or T_2 to be finite dimensional. The first theorem will be of central importance in the following. In fact, Theorem 4.5 will be used twice in the proof of Theorem 4.7.

Theorem 4.5. *Let $S_1, S_2 \in \mathcal{L}(\mathcal{H})$ be commuting contractions such that S_2 is of class $C_{\cdot 0}$ and let $V \in \mathcal{L}\left(\mathcal{D}_{S_2}, \mathcal{E}\right)$ be an isometry into a Hilbert space $\mathcal{E}$. Furthermore, let $\mathcal{F}$ be a Hilbert space and let*

$$W = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} : \mathcal{E} \oplus (\mathcal{F} \oplus \mathcal{D}_{S_1}) \to \mathcal{E} \oplus (\mathcal{F} \oplus \mathcal{D}_{S_1})$$

be a unitary operator such that

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} V D_{S_2} h \\ \begin{pmatrix} 0 \\ D_{S_1} S_2^* h \end{pmatrix} \end{pmatrix} = \begin{pmatrix} V D_{S_2} S_1^* h \\ \begin{pmatrix} 0 \\ D_{S_1} h \end{pmatrix} \end{pmatrix}$$

holds for all $h \in \mathcal{H}$. Then the transfer function

$$\tau_{W^*} : \mathbb{D} \longrightarrow \mathcal{L}(\mathcal{E}), \qquad z \longmapsto A^* + zC^*B^*$$

of W^* satisfies the identity

$$\Pi_{S_2}^V S_1^* = M_{\tau_{W^*}}^* \Pi_{S_2}^V.$$

Proof. By hypothesis, we have that

$$V D_{S_2} S_1^* h = A V D_{S_2} h + B \begin{pmatrix} 0 \\ D_{S_1} S_2^* h \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ D_{S_1} h \end{pmatrix} = C V D_{S_2} h$$

for all $h \in \mathcal{H}$ and hence

$$V D_{S_2} S_1^* = A V D_{S_2} + B C V D_{S_2} S_2^*. \tag{4.5}$$

For $h \in \mathcal{H}, n \geq 1, \eta \in \mathcal{E}$, we obtain

$$\begin{aligned}
\left\langle M_{\tau_{W^*}}^* \Pi_{S_2}^V h, z^n \eta \right\rangle &= \left\langle \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \sum_{k=0}^{\infty} \left(D_{S_2} S_2^{*k} h\right) z^k, \left(A^* + zC^*B^*\right) z^n \eta \right\rangle \\
&= \left\langle \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \sum_{k=0}^{\infty} \left(D_{S_2} S_2^{*k} h\right) z^k, A^* z^n \eta \right\rangle \\
&\quad + \left\langle \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \sum_{k=0}^{\infty} \left(D_{S_2} S_2^{*k} h\right) z^k, C^* B^* z^{n+1} \eta \right\rangle \\
&= \left\langle V D_{S_2} S_2^{*n} h, A^* \eta \right\rangle + \left\langle V D_{S_2} S_2^{*n+1} h, C^* B^* \eta \right\rangle \\
&= \left\langle A V D_{S_2} S_2^{*n} h + B C V D_{S_2} S_2^{*n+1} h, \eta \right\rangle \\
&= \left\langle \left(A V D_{S_2} + B C V D_{S_2} S_2^*\right) \left(S_2^{*n} h\right), \eta \right\rangle \\
&= \left\langle V D_{S_2} S_1^* (S_2^{*n} h), \eta \right\rangle,
\end{aligned}$$

where we used Equation 4.5 in the last step.

On the other hand, for $h \in \mathcal{H}$, $n \geq 1$ and $\eta \in \mathcal{E}$ we find that

$$\begin{aligned}
\langle \Pi_{S_2}^V S_1^* h, z^n \eta \rangle &= \left\langle \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V \right) \sum_{k=0}^{\infty} \left(D_{S_2} S_2^{*k} S_1^* h \right) z^k, z^n \eta \right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} \left(V D_{S_2} S_1^* S_2^{*k} h \right) z^k, z^n \eta \right\rangle \\
&= \langle V D_{S_2} S_1^* S_2^{*n} h, \eta \rangle \,.
\end{aligned}$$

Thus we have proved the identity $\Pi_{S_2}^V S_1^* = M_{\tau_{W^*}}^* \Pi_{S_2}^V$. □

Remark 4.6. *In the setting of Theorem 4.5, we have*

$$S_1^* \cong M_{\tau_{W^*}}^* |_{\mathcal{Q}}$$

and, according to Theorem 3.7,

$$S_2^* \cong \left(M_z^* \otimes \mathrm{id}_{\mathcal{E}} \right) |_{\mathcal{Q}},$$

where $\mathcal{Q} = \Pi_{S_2}^V \mathcal{H} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{E})$.

Theorem 4.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $C_{\cdot 0}$-contraction and let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be commuting contractions. Then the following are equivalent:*

(i) $T = T_1 T_2$.

(ii) There exist a Hilbert space $\mathcal{E}$, a unitary operator $U \in \mathcal{L}(\mathcal{E})$ and an orthogonal projection $P \in \mathcal{L}(\mathcal{E})$ such that the operator-valued mappings $\Phi, \Psi \colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ defined by

$$\Phi(z) = (P + zP^{\perp})U^* \qquad \text{and} \qquad \Psi(z) = U(P^{\perp} + zP) \qquad (z \in \mathbb{D})$$

induce a pure pair $(M_\Phi, M_\Psi) \in \mathcal{L}\left(\mathrm{H}^2(\mathbb{D}, \mathcal{E})\right)^2$ of isometric multiplication operators with

$$M_\Phi M_\Psi = M_\Psi M_\Phi = M_z \otimes \mathrm{id}_{\mathcal{E}} \,.$$

Moreover, there exists an isometry $V \in \mathcal{L}(\mathcal{D}_T, \mathcal{E})$ such that $\mathcal{Q} = \Pi_T^V \mathcal{H}$ is a joint $(M_\Phi^*, M_\Psi^*, M_z^* \otimes \mathrm{id}_{\mathcal{E}})$-invariant subspace of $\mathrm{H}^2(\mathbb{D}, \mathcal{E})$ and such that

$$(T_1, T_2, T) \cong P_{\mathcal{Q}} \left(M_\Phi, M_\Psi, M_z \otimes \mathrm{id}_{\mathcal{E}} \right) |_{\mathcal{Q}}$$

via the unitary operator $\Pi_T^V \colon \mathcal{H} \to \mathcal{Q}$.

(iii) There exist $\mathcal{L}(\mathcal{D}_T)$-valued polynomials φ and ψ of degree at most 1 such that $\mathcal{Q} = \Pi_T\mathcal{H}$ is a joint (M_φ^*, M_ψ^*)-invariant subspace of $\mathrm{H}^2(\mathbb{D}, \mathcal{D}_T)$,

$$P_\mathcal{Q}(M_z \otimes \mathrm{id}_{\mathcal{D}_T})|_\mathcal{Q} = P_\mathcal{Q} M_{\varphi\psi}|_\mathcal{Q} = P_\mathcal{Q} M_{\psi\varphi}|_\mathcal{Q},$$

and such that

$$(T_1, T_2) \cong P_\mathcal{Q}(M_\varphi, M_\psi)|_\mathcal{Q}$$

via the unitary operator $\Pi_T \colon \mathcal{H} \to \mathcal{Q}$.

Proof. Suppose that *(ii)* holds. Then there exists a unitary operator $\Pi \colon \mathcal{Q} \to \mathcal{H}$ such that

$$(T_1, T_2, T) = \Pi P_\mathcal{Q} \left(M_\Phi, M_\Psi, M_z \otimes \mathrm{id}_\mathcal{E}\right)|_\mathcal{Q} \Pi^*$$

and hence

$$\begin{aligned} T_1 T_2 &= \Pi P_\mathcal{Q} M_\Phi|_\mathcal{Q} \Pi^* \Pi P_\mathcal{Q} M_\Psi|_\mathcal{Q} \Pi^* \\ &= \Pi P_\mathcal{Q} M_\Phi|_\mathcal{Q} P_\mathcal{Q} M_\Psi|_\mathcal{Q} \Pi^* \\ &= \Pi P_\mathcal{Q} M_\Phi M_\Psi|_\mathcal{Q} \Pi^* \\ &= \Pi P_\mathcal{Q} \left(M_z \otimes \mathrm{id}_\mathcal{E}\right)|_\mathcal{Q} \Pi^* = T. \end{aligned}$$

Thus, *(ii)* implies *(i)*.

Suppose that $T = T_1 T_2$. Let $\mathcal{D}$ be the Hilbert space constructed in Proposition 3.4 and define

$$\mathcal{E} = (\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}.$$

Let furthermore $U \colon \mathcal{E} \to \mathcal{E}$ be the operator from Proposition 3.4 satisfying

$$U \begin{pmatrix} \begin{pmatrix} 0 \\ D_{T_1} T_2^* h \end{pmatrix} \\ D_{T_2} h \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ D_{T_1} h \end{pmatrix} \\ D_{T_2} T_1^* h \end{pmatrix}$$

for all $h \in \mathcal{H}$. Let $\iota_1 \colon \mathcal{D} \oplus \mathcal{D}_{T_1} \to \mathcal{E}$ and $\iota_2 \colon \mathcal{D}_{T_2} \to \mathcal{E}$ be the inclusion mappings defined by

$$\iota_1 (h, h_1) = (h, h_1, 0) \qquad \text{and} \qquad \iota_2 h_2 = (0, 0, h_2)$$

for $h \in \mathcal{D}$, $h_1 \in \mathcal{D}_{T_1}$ and $h_2 \in \mathcal{D}_{T_2}$. Then

$$P = \iota_2 \iota_2^* \colon \mathcal{E} \longrightarrow \mathcal{E}, \qquad (h, h_1, h_2) \longmapsto (0, 0, h_2)$$

is the orthogonal projection onto $\mathcal{D}_{T_2}$ and the orthogonal projection $P^\perp$ onto $\mathcal{D} \oplus \mathcal{D}_{T_1}$ is given by

$$\iota_1 \iota_1^* \colon \mathcal{E} \longrightarrow \mathcal{E}, \qquad (h, h_1, h_2) \longmapsto (h, h_1, 0).$$

Thus

$$\begin{pmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{pmatrix} : \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{T_1}) \longrightarrow \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{T_1})$$

is unitary since

$$\begin{pmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{pmatrix} \begin{pmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{pmatrix} = \begin{pmatrix} P + P^\perp & P\iota_1 \\ \iota_1^* P & \iota_1^* \iota_1 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\mathcal{E}} & 0 \\ 0 & \mathrm{id}_{\mathcal{D} \oplus \mathcal{D}_{T_1}} \end{pmatrix}.$$

But then also

$$W_1 = \begin{pmatrix} U & 0 \\ 0 & \mathrm{id}_{\mathcal{D} \oplus \mathcal{D}_{T_1}} \end{pmatrix} \begin{pmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{pmatrix} = \begin{pmatrix} UP & U\iota_1 \\ \iota_1^* & 0 \end{pmatrix} \in \mathcal{L}\left(\mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{T_1})\right)$$

defines a unitary operator. By Remark 3.3, there is a unique isometry $V : \mathcal{D}_T \to \mathcal{E}$ with

$$V(D_T h) = (0, D_{T_1} h, D_{T_2} T_1^* h) \qquad (h \in \mathcal{H}).$$

By construction, we have

$$\begin{aligned} W_1 \left(V\left(D_T h\right), \left(0, D_{T_1} T^* h\right)\right) &= W_1 \left(\left(0, D_{T_1} h, D_{T_2} T_1^* h\right), \left(0, D_{T_1} T^* h\right)\right) \\ &= \left(UP\left(0, D_{T_1} h, D_{T_2} T_1^* h\right)\right. \\ &\qquad \left. + U\iota_1 \left(0, D_{T_1} T^* h\right), \iota_1^* \left(0, D_{T_1} h, D_{T_2} T_1^* h\right)\right) \\ &= \left(U\left(0, 0, D_{T_2} T_1^* h\right) + U\left(0, D_{T_1} T^* h, 0\right), \left(0, D_{T_1} h\right)\right) \\ &= \left(U\left(0, D_{T_1} T_2^* T_1^* h, D_{T_2} T_1^* h\right), \left(0, D_{T_1} h\right)\right) \\ &= \Big(\left(0, D_{T_1} T_1^* h, D_{T_2} T_1^{*2} h\right), \left(0, D_{T_1} h\right)\Big) \\ &= \left(V\left(D_T T_1^* h\right), \left(0, D_{T_1} h\right)\right) \end{aligned}$$

for all $h \in \mathcal{H}$. Applying Theorem 4.5 to the commuting contractions T_1, T and the unitary operator W_1, we conclude

$$\Pi_T^V T_1^* = M_\Phi^* \Pi_T^V, \tag{4.6}$$

where

$$\Phi(z) = \tau_{W_1^*}(z) = PU^* + z\iota_1 \iota_1^* U^* = \left(P + zP^\perp\right) U^* \in \mathcal{L}(\mathcal{E}) \qquad (z \in \mathbb{D})$$

is the transfer function of W_1^*.

Analogously, we obtain the unitary operator

$$W_2 = \begin{pmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & \mathrm{id}_{\mathcal{D}_{T_2}} \end{pmatrix} = \begin{pmatrix} P^\perp U^* & \iota_2 \\ \iota_2^* U^* & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{E} \oplus \mathcal{D}_{T_2}).$$

Again, by construction we have

$$\begin{aligned}
W_2\left(V\left(D_T h\right), D_{T_2}T^*h\right) &= W_2\left(\left(0, D_{T_1}h, D_{T_2}T_1^*h\right), D_{T_2}T^*h\right) \\
&= \Big(P^\perp U^*\left(0, D_{T_1}h, D_{T_2}T_1^*h\right) \\
&\qquad + \iota_2\left(D_{T_2}T^*h\right), \iota_2^* U^*\left(0, D_{T_1}h, D_{T_2}T_1^*h\right)\Big) \\
&= \Big(P^\perp\left(0, D_{T_1}T_2^*h, D_{T_2}h\right) \\
&\qquad + \left(0, 0, D_{T_2}T^*h\right), \iota_2^*\left(0, D_{T_1}T_2^*h, D_{T_2}h\right)\Big) \\
&= \left(\left(0, D_{T_1}T_2^*h, 0\right) + \left(0, 0, D_{T_2}T^*h\right), D_{T_2}h\right) \\
&= \left(\left(0, D_{T_1}T_2^*h, D_{T_2}T_1^*T_2^*h\right), D_{T_2}h\right) \\
&= \left(V\left(D_T T_2^*h\right), D_{T_2}h\right)
\end{aligned}$$

for all $h \in \mathcal{H}$. Theorem 4.5 applied to the commuting contractions T_2, T and the unitary operator W_2 yields

$$\Pi_T^V T_2^* = M_\Psi^* \Pi_T^V, \tag{4.7}$$

where

$$\Psi(z) = \tau_{W_2^*}(z) = UP^\perp + zU\iota_2\iota_2^* = U(P^\perp + zP) \qquad (z \in \mathbb{D})$$

is the transfer function of W_2^*. Furthermore, since T is of class $C_{\cdot 0}$, Theorem 3.7 yields

$$\Pi_T^V T^* = (M_z^* \otimes \mathrm{id}_{\mathcal{E}})\Pi_T^V. \tag{4.8}$$

Summing up Equations 4.6, 4.7, 4.8, we derive

$$(T_1, T_2, T) \cong P_{\mathcal{Q}}\left(M_\Phi, M_\Psi, M_z \otimes \mathrm{id}_{\mathcal{E}}\right)|_{\mathcal{Q}}$$

via the unitary operator $\Pi_T^V \colon \mathcal{H} \to \mathcal{Q}$, where $\mathcal{Q} = \Pi_T^V \mathcal{H}$ is a joint $(M_\Phi^*, M_\Psi^*, M_z^* \otimes \mathrm{id}_{\mathcal{E}})$-invariant subspace of $\mathrm{H}^2(\mathbb{D}, \mathcal{E})$.

Moreover, we have

$$\begin{aligned}
\Phi(z)\Psi(z) &= \left(P + zP^\perp\right) U^*U \left(P^\perp + zP\right) \\
&= PP^\perp + zP + zP^\perp + z^2 P^\perp P \\
&= z\left(P + P^\perp\right) = z\,\mathrm{id}_{\mathcal{E}}
\end{aligned}$$

and analogously

$$\begin{aligned}
\Psi(z)\Phi(z) &= U\left(P^\perp + zP\right)\left(P + zP^\perp\right) U^* \\
&= U\left(P^\perp P + zP^\perp + zP + z^2 PP^\perp\right) U^* \\
&= U\left(z\left(P^\perp + P\right)\right) U^* = z\,\mathrm{id}_{\mathcal{E}}
\end{aligned}$$

for all $z \in \mathbb{D}$. Hence

$$M_\Phi M_\Psi = M_\Psi M_\Phi = M_z \otimes \mathrm{id}_{\mathcal{E}} \,.$$

Since Φ and Ψ are inner by Proposition 2.23, we conclude that M_Φ and M_Ψ are isometries by Lemma 2.22. By Example 2.10, we know that $M_z \otimes \mathrm{id}_{\mathcal{E}}$ is of class $C_{\cdot 0}$. Hence, $M_\Phi M_\Psi = M_z \otimes \mathrm{id}_{\mathcal{E}}$ is pure by Proposition 2.11. Thus *(i)* implies *(ii)*.

Let $T = T_1 T_2$. Then with the notations from above

$$\Pi_T^V T_1^* = M_\Phi^* \Pi_T^V \qquad \text{and} \qquad \Pi_T^V T_2^* = M_\Psi^* \Pi_T^V,$$

where $V \colon \mathcal{D}_T \to \mathcal{E}$ is an isometry and $\Phi, \Psi \colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ are inner $\mathcal{L}(\mathcal{E})$-valued polynomials of degree at most 1. Since $\Pi_T^V = \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \Pi_T$, we find that

$$\Pi_T T_1^* = (\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V^*) M_\Phi^* (\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V) \Pi_T,$$

or that

$$\Pi_T T_1^* = M_\varphi^* \Pi_T$$

with

$$\varphi \colon \mathbb{D} \longrightarrow \mathcal{L}(\mathcal{D}_T), \qquad z \longmapsto V^* \Phi(z) V.$$

Similarly, we obtain that

$$\Pi_T T_2^* = \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V^*\right) M_\Psi^* \left(\mathrm{id}_{\mathrm{H}^2(\mathbb{D})} \otimes V\right) \Pi_T,$$

or

$$\Pi_T T_2^* = M_\psi^* \Pi_T$$

with

$$\psi \colon \mathbb{D} \longrightarrow \mathcal{L}(\mathcal{D}_T), \qquad z \longmapsto V^* \Psi(z) V.$$

In particular, φ and ψ are $\mathcal{L}(\mathcal{D}_T)$-valued polynomials of degree at most 1 such that

$$(T_1, T_2) \cong P_{\mathcal{Q}} \left(M_\varphi, M_\psi\right)|_{\mathcal{Q}}$$

via the unitary operator $\Pi_T \colon \mathcal{H} \to \mathcal{Q}$.

Furthermore, $\mathcal{Q} = \Pi_T \mathcal{H}$ is a joint $\left(M_\varphi^*, M_\psi^*\right)$-invariant subspace. Since

$$\Pi_T T^* = \left(M_z^* \otimes \mathrm{id}_{\mathcal{D}_T}\right) \Pi_T,$$

$\mathcal{Q}$ is also a $(M_z^* \otimes \mathrm{id}_{\mathcal{D}_T})$-invariant subspace of $\mathrm{H}^2(\mathbb{D}, \mathcal{D}_T)$. To sum up, we know that

$$\begin{aligned} M_{\varphi\psi}^* \Pi_T &= M_\psi^* M_\varphi^* \Pi_T = M_\psi^* \Pi_T T_1^* \\ &= \Pi_T T_2^* T_1^* = \Pi_T T_1^* T_2^* \\ &= M_\varphi^* \Pi_T T_2^* = M_\varphi^* M_\psi^* \Pi_T = M_{\psi\varphi}^* \Pi_T \end{aligned}$$

and

$$\Pi_T T_1^* T_2^* = \Pi_T T^* = (M_z^* \otimes \mathrm{id}_{\mathcal{D}_T}) \Pi_T.$$

Hence

$$M_{\varphi\psi}^*|_{\mathcal{Q}} = (M_z^* \otimes \mathrm{id}_{\mathcal{D}_T})\,|_{\mathcal{Q}} = M_{\psi\varphi}^*|_{\mathcal{Q}}$$

and therefore

$$P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}\left(M_z \otimes \mathrm{id}_{\mathcal{D}_T}\right)|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\psi\varphi}|_{\mathcal{Q}}.$$

Thus, *(i)* implies *(iii)*.

On the other hand, if polynomials $\varphi, \psi\colon \mathbb{D} \to \mathcal{L}(\mathcal{D}_T)$ are given such that

$$P_{\mathcal{Q}}\left(M_z \otimes \mathrm{id}_{\mathcal{D}_T}\right)|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\psi\varphi}|_{\mathcal{Q}}$$

and

$$(T_1, T_2) \cong P_{\mathcal{Q}}\left(M_\varphi, M_\psi\right)|_{\mathcal{Q}}$$

holds for the joint $\left(M_\varphi^*, M_\psi^*\right)$-invariant subspace $\mathcal{Q} = \Pi_T \mathcal{H} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{D}_T)$, then it follows exactly as at the beginning of the proof that $T = T_1 T_2$. $\square$

In the following, we give applications of the main result proven in this section. We first note that Theorem 4.7 is a generalization of the dilation theorem of Berger, Coburn and Lebow in [5], which reads as follows.

Theorem 4.8. *Let $V \in \mathcal{L}(\mathcal{H})$ be a pure isometry and let $V_1, V_2 \in \mathcal{L}(\mathcal{H})$ be commuting isometries. Then the following are equivalent:*

(i) $V = V_1 V_2$.

(ii) *There exist a Hilbert space $\mathcal{E}$, a unitary operator $U \in \mathcal{L}(\mathcal{E})$ and an orthogonal projection $P \in \mathcal{L}(\mathcal{E})$ such that the operator-valued functions $\Phi, \Psi\colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ defined by*

$$\Phi(z) = \left(P + zP^\perp\right) U^* \qquad \textit{and} \qquad \Psi(z) = U\left(P^\perp + zP\right) \qquad (z \in \mathbb{D})$$

induce a pure pair $(M_\Phi, M_\Psi) \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{E}))^2$ of isometric multiplication operators with

$$M_\Phi M_\Psi = M_\Psi M_\Phi = M_z \otimes \mathrm{id}_{\mathcal{E}}$$

and such that

$$(V_1, V_2, V) \cong (M_\Phi, M_\Psi, M_z \otimes \mathrm{id}_{\mathcal{E}}).$$

Proof. If *(ii)* holds, then *(i)* follows exactly as in the proof of Theorem 4.7.

Suppose that $V = V_1V_2$ and let $\Phi, \Psi, \mathcal{E}$ and $\mathcal{Q}$ be as in Theorem 4.7. Then it suffices to show that

$$\mathcal{Q} = \mathrm{H}^2(\mathbb{D}, \mathcal{M})$$

with a closed subspace $\mathcal{M} \subseteq \mathcal{E}$ that is reducing for U and P. The existence of $\mathcal{M}$ can be shown as follows.

Since the multiplication operators M_Φ, M_Ψ and $M_z \otimes \mathrm{id}_\mathcal{E}$ as well as their compressions $P_\mathcal{Q} M_\Phi|_\mathcal{Q}, P_\mathcal{Q} M_\Psi|_\mathcal{Q}$ and $P_\mathcal{Q}(M_z \otimes \mathrm{id}_\mathcal{E})|_\mathcal{Q}$ to $\mathcal{Q}$ are isometric, it follows by the Pythagorean theorem that $\mathcal{Q}$ is a reducing subspace for M_Φ, M_Ψ and $M_z \otimes \mathrm{id}_\mathcal{E}$. Since $\mathcal{Q}$ is reducing for $M_z \otimes \mathrm{id}_\mathcal{E}$, it follows that there is a closed subspace $\mathcal{M} \subseteq \mathcal{E}$ with $\mathcal{Q} = \mathrm{H}^2(\mathbb{D}, \mathcal{M})$. This is a particular case of [12, Lemma 4.1.6].

Since

$$PU^*x = \Phi(0)x \in \mathcal{M}$$

for all $x \in \mathcal{M}$, the space $\mathcal{M}$ is invariant for PU^*. Since

$$PU^*x + zP^\perp U^*x \in \mathcal{Q} = \mathrm{H}^2(\mathbb{D}, \mathcal{M})$$

for all $x \in \mathcal{M}$, it follows that $\mathcal{M}$ is also invariant for $P^\perp U^*$. Playing the same game with Ψ instead of Φ, one obtains that $\mathcal{M}$ is also invariant for $UP^\perp$ and UP. But then $\mathcal{M}$ is invariant for

$$U^* = PU^* + P^\perp U^*, \qquad U = UP^\perp + UP \qquad \text{and} \qquad P = (PU^*)(UP).$$

Thus

$$(V_1, V_2, V) \cong (M_\alpha, M_\beta, M_z \otimes \mathrm{id}_\mathcal{M}),$$

where $\alpha, \beta \colon \mathbb{D} \to \mathcal{L}(\mathcal{M})$ are defined by

$$\alpha(z) = \left(P|_\mathcal{M} + zP|_\mathcal{M}^\perp\right) U|_\mathcal{M}^* \qquad \text{and} \qquad \beta(z) = U|_\mathcal{M} \left(P|_\mathcal{M}^\perp + zP|_\mathcal{M}\right)$$

for all $z \in \mathbb{D}$. □

Note that Theorem 4.7 is a sharper version of Andô's dilation theorem for a pair of commuting contractions, whose product has the $C_{\cdot 0}$-property. In fact, Theorem 4.7 gives a proof of the following theorem.

Theorem 4.9 (Andô)**.** *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be commuting contractions such that T_1T_2 is a $C_{\cdot 0}$-contraction. Then there exist commuting unitary operators $U_1, U_2 \in \mathcal{L}(\tilde{\mathcal{H}})$ on a Hilbert space $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ such that*

$$T_1^n T_2^m h = P_\mathcal{H} U_1^n U_2^m h$$

for all $h \in \mathcal{H}$ and $n, m \in \mathbb{N}$.

Proof. Theorem 4.7 provides a Hilbert space $\mathcal{E}$ as well as a pair of commuting isometries $(M_\Phi, M_\Psi) \in \mathcal{L}(\mathrm{H}^2(\mathbb{D}, \mathcal{E}))^2$ such that

$$(T_1, T_2) \cong P_{\mathcal{Q}}\,(M_\Phi, M_\Psi)\,|_{\mathcal{Q}}$$

for the joint (M_Φ^*, M_Ψ^*)-invariant subspace $\mathcal{Q} = \Pi_T^V \mathcal{H} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{E})$. Furthermore, we have $\mathcal{H} \cong \mathcal{Q}$ via the unitary operator $\Pi_T^V \colon \mathcal{H} \to \mathcal{Q}$. By [11, Lemma 4], (M_Φ, M_Ψ) extends to a commuting pair $(U_1, U_2) \in \mathcal{L}(\tilde{\mathcal{H}})^2$ of unitary operators on a Hilbert space $\tilde{\mathcal{H}} \supseteq \mathcal{Q}$. By construction, the unitary operators $U_1, U_2 \in \mathcal{L}(\tilde{\mathcal{H}})$ have the desired property. □

5. Von Neumann Inequality

In this final chapter, we will prove two sharper versions of von Neumann's inequality for certain pairs of commuting contractions using Theorem 4.3 and Theorem 4.7. We start with the construction of the joint spectrum of a commuting tuple of normal operators on $\mathcal{H}$.

Let $\mathcal{A}$ be a commutative unital Banach algebra. For $a = (a_1, ..., a_n) \in \mathcal{A}^n$, the *joint spectrum of* a is defined as

$$\sigma_{\mathcal{A}}(a) = \left\{ z \in \mathbb{C}^n \mid \sum_{i=1}^{n} (z_i - a)\mathcal{A} \neq \mathcal{A} \right\}.$$

One can show (cf. [9, Korollar 12.3]) that $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{C}^n$ is a non-empty compact set such that the polynomial spectral mapping theorem

$$\sigma_{\mathcal{A}}((p_1(a), ..., p_m(a))) = p(\sigma_{\mathcal{A}}(a))$$

holds for each tuple $p = (p_1, ..., p_m) \in \mathbb{C}[z_1, ..., z_n]^m$.

Let $N = (N_1, ..., N_n) \in \mathcal{L}(\mathcal{H})^n$ be a commuting tuple of normal operators. By the Putnam-Fuglede theorem, the $(2n)$-tuple $(N, N^*) = (N_1, ..., N_n, N_1^*, ..., N_n^*) \in \mathcal{L}(\mathcal{H})^{2n}$ is commuting again. Hence also the C^*-algebra

$$C^*(N) = \overline{\{p(N, N^*) \mid p \in \mathbb{C}[z_1, ..., z_{2n}]\}} \subseteq \mathcal{L}(\mathcal{H})$$

generated by N is commutative. We call the set

$$\sigma(N) = \sigma_{C^*(N)}(N)$$

the *joint spectrum of* N. Note that, for each single operator $A \in C^*(N)$, the set $\sigma_{C^*(N)}(A) = \sigma_{C^*(A)}(A)$ coincides with the usual spectrum of the bounded linear operator $A \in \mathcal{L}(\mathcal{H})$. Moreover, we obtain the inclusion

$$\sigma(N) \subseteq \sigma(N_1) \times ... \times \sigma(N_n).$$

Lemma 5.1. *Let $N = (N_1, ..., N_n) \in \mathcal{L}(\mathcal{H})^n$ be a commuting tuple of normal operators. Then*

$$\begin{aligned} \sigma(N_1 \cdot ... \cdot N_n) &= \{z_1 \cdot ... \cdot z_n \mid (z_1, ..., z_n) \in \sigma(N)\} \\ &\subseteq \{z_1 \cdot ... \cdot z_n \mid z_i \in \sigma(N_i) \text{ for } i = 1, ..., n\}. \end{aligned}$$

Proof. It suffices to apply the polynomial spectral mapping theorem to the polynomial $p(z_1, ..., z_n) = z_1 \cdot ... \cdot z_n$. $\square$

We continue with the definition of distinguished varieties and observe some of their properties.

Definition 5.2. A non-empty set $V \subseteq \mathbb{D}^2$ is called a *distinguished variety* if

$$V = \left\{ (z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U(z_1) - z_2 \operatorname{id}_{\mathbb{C}^n}\right) = 0 \right\},$$

where $\tau_U \colon \mathbb{D} \to \mathcal{L}(\mathbb{C}^n)$ is the transfer function of a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^m \longrightarrow \mathbb{C}^n \oplus \mathbb{C}^m.$$

Lemma 5.3. *Let U and V be as in Definition 5.2. Then, for the unitary operator $\tilde{U} = \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m \oplus \mathbb{C}^n)$ and*

$$\tilde{V} = \left\{ (z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_{\tilde{U}}(z_2) - z_1 \operatorname{id}_{\mathbb{C}^m}\right) = 0 \right\},$$

it holds $V = \tilde{V}$. In particular, $\tilde{V}$ is a distinguished variety.

Proof. Let $(z_1, z_2) \in \mathbb{D}^2$. Then we have $(z_1, z_2) \in V$ if and only if

$$\left[A + z_1 B \left(\operatorname{id}_{\mathbb{C}^m} - z_1 D\right)^{-1} C \right] v_1 = z_2 v_1 \quad \text{for a vector } v_1 \in \mathbb{C}^n \backslash \{0\}. \tag{5.1}$$

We will show that Property 5.1 holds if and only if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ z_1 v_2 \end{pmatrix} = \begin{pmatrix} z_2 v_1 \\ v_2 \end{pmatrix} \quad \text{for some } v_1 \in \mathbb{C}^n \backslash \{0\} \text{ and } v_2 \in \mathbb{C}^m \backslash \{0\}. \tag{5.2}$$

If Property 5.2 holds, then solving gives Property 5.1. On the other hand, if Property 5.1 holds, then Property 5.2 follows for

$$v_2 = \left(\operatorname{id}_{\mathbb{C}^m} - z_1 D\right)^{-1} C v_1.$$

Assume that $v_2 = 0$. Then $v_1 \in \ker C$ and $A v_1 = z_2 v_1$. Since

$$\|Ax\|^2 + \|Cx\|^2 = \langle (A^*A + C^*C)x, x \rangle = \|x\|^2$$

for all $x \in \mathbb{C}^n$, it follows

$$\|v_1\|^2 = \|Av_1\|^2 = \|z_2 v_1\|^2$$

and thus $z_2 \in \mathbb{T}$, contradicting the fact that $(z_1, z_2) \in \mathbb{D}^2$.

Applying this to $\tilde{U}$ and $\tilde{V}$, we obtain that $(z_1, z_2) \in \tilde{V}$ if and only if

$$\begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \begin{pmatrix} v_2 \\ z_2 v_1 \end{pmatrix} = \begin{pmatrix} z_1 v_2 \\ v_1 \end{pmatrix} \quad \text{for some } v_1 \in \mathbb{C}^n \backslash \{0\} \text{ and } v_2 \in \mathbb{C}^m \backslash \{0\}. \tag{5.3}$$

Interchanging coordinates, Property 5.3 becomes

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} z_2 v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ z_1 v_2 \end{pmatrix} \quad \text{for some } v_1 \in \mathbb{C}^n \backslash \{0\} \text{ and } v_2 \in \mathbb{C}^m \backslash \{0\}. \tag{5.4}$$

Since U is unitary, Property 5.2 and Property 5.4 are equivalent. $\square$

Lemma 5.4. *A distinguished variety $V \subseteq \mathbb{D}^2$ satisfies the identity*

$$\overline{V} \cap \partial(\mathbb{D}^2) = \overline{V} \cap \mathbb{T}^2.$$

Moreover, we have $\overline{V} \cap \mathbb{T}^2 \neq \mathbb{T}^2$.

Proof. Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}\left(\mathbb{C}^n \oplus \mathbb{C}^m\right)$ be unitary and let $A = A_0 \oplus A_1$ on $\mathbb{C}^n = \mathcal{H}^0 \oplus \mathcal{H}^1$ be the decomposition as in Theorem 2.6. By Proposition 2.20, we have

$$\tau_U(z) = \begin{pmatrix} \tau_U^0(z) & 0 \\ 0 & \tau_U^1(z) \end{pmatrix} \in \mathcal{L}\left(\mathcal{H}^0 \oplus \mathcal{H}^1\right) \qquad (z \in \mathbb{D}),$$

where

$$\tau_U^0(z) = A_0 \in \mathcal{L}\left(\mathcal{H}^0\right) \qquad (z \in \mathbb{D})$$

and

$$\tau_U^1(z) = \tau_{U_1}(z) = A_1 + zB\left(\mathrm{id}_{\mathbb{C}^m} - zD\right)^{-1} C|_{\mathcal{H}^1} \in \mathcal{L}\left(\mathcal{H}^1\right) \qquad (z \in \mathbb{D})$$

is the transfer function of the unitary

$$U_1 = \begin{pmatrix} A_1 & B \\ C|_{\mathcal{H}^1} & D \end{pmatrix} \in \mathcal{L}\left(\mathcal{H}^1 \oplus \mathbb{C}^m\right).$$

With respect to this decomposition, the distinguished variety

$$V = \left\{(z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U(z_1) - z_2\, \mathrm{id}_{\mathbb{C}^n}\right) = 0\right\}$$

is given by $V = V_0 \cup V_1$, where

$$V_0 = \left\{(z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U^0(z_1) - z_2\, \mathrm{id}_{\mathcal{H}^0}\right) = 0\right\}$$

and

$$V_1 = \left\{(z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U^1(z_1) - z_2\, \mathrm{id}_{\mathcal{H}^1}\right) = 0\right\}.$$

As $\tau_U^0(z) = A_0 \in \mathcal{L}\left(\mathcal{H}_0\right)$ is unitary for all $z \in \mathbb{D}$, we have $\sigma\left(\tau_U^0(z)\right) \subseteq \mathbb{T}$ for all $z \in \mathbb{D}$ and hence

$$V = \left\{(z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U^1(z_1) - z_2\, \mathrm{id}_{\mathcal{H}^1}\right) = 0\right\}.$$

Since $A_1 \in \mathcal{L}(\mathcal{H}^1)$ is completely non-unitary, $\tau_U^1(z)$ does not have any unimodular eigenvalues for $z \in \mathbb{D}$ by Proposition 2.19. Due to Remark 2.15, we obtain that

$$\sigma\left(\tau_U^1(z)\right) \subseteq \mathbb{D} \qquad (z \in \mathbb{D}).$$

By Proposition 2.23, both transfer functions τ_U, τ_U^1 admit continuous extensions to the closed unit disc $\overline{\mathbb{D}}$ which we again denote by τ_U and τ_U^1. In particular, the boundary values $\tau_U(z) \in \mathcal{L}(\mathcal{D}_{T_1})$ and $\tau_U^1(z) \in \mathcal{L}(\mathcal{H}^1)$ are unitary for $z \in \mathbb{T}$ and hence

$$\sigma(\tau_U(z)) \subseteq \mathbb{T} \qquad \text{and} \qquad \sigma(\tau_U^1(z)) \subseteq \mathbb{T} \qquad (z \in \mathbb{T}).$$

Let $(z_1, z_2) \in \overline{V} \subseteq \overline{\mathbb{D}^2}$. Since $\sigma(\tau_U^1(z)) \subseteq \mathbb{D}$ for $z \in \mathbb{D}$ and $\sigma\left(\tau_U^1(z)\right) \subseteq \mathbb{T}$ for $z \in \mathbb{T}$, we conclude that $z_1 \in \mathbb{T}$ if and only if $z_2 \in \mathbb{T}$. Since $\tau_U^1(z)$ are matrices for all $z \in \mathbb{T}^2$, the remaining assertion follows. $\square$

Proposition 5.5. *Let* $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathbb{C}^n \oplus \mathbb{C}^m)$ *be a unitary operator and let*

$$V = \left\{(z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U(z_1) - z_2 \operatorname{id}_{\mathbb{C}^n}\right) = 0\right\}.$$

If $\sigma(\tau_U(z)) \subseteq \mathbb{D}$ *for all* $z \in \mathbb{D}$*, then the identity*

$$\overline{V} = \left\{(z_1, z_2) \in \overline{\mathbb{D}^2} \mid \det\left(\tau_U(z_1) - z_2 \operatorname{id}_{\mathbb{C}^n}\right) = 0\right\}$$

holds.

Proof. The inclusion $\subseteq$ is obvious. By Proposition 2.23, there exists an open disc $D_R(0)$ with radius $R > 1$ such that τ_U extends holomorphically to a function $\tau_U \colon D_R(0) \to \mathcal{L}(\mathbb{C}^n)$. In particular, $\tau_U(z)$ is unitary for $z \in \mathbb{T}$ and hence

$$\sigma\left(\tau_U(z)\right) \subseteq \mathbb{T} \qquad (z \in \mathbb{T}).$$

By Lemma 5.4 it is enough to show that

$$\left\{(z_1, z_2) \in \mathbb{T}^2 \mid \det\left(\tau_U(z_1) - z_2 \operatorname{id}_{\mathbb{C}^n}\right) = 0\right\} \subseteq \overline{V}.$$

Let $(\tilde{z}, \tilde{w}) \in \mathbb{T}^2$ such that

$$\det(\tau_U(\tilde{z}) - \tilde{w} \operatorname{id}_{\mathbb{C}^n}) = 0.$$

Consider the analytic function

$$f \colon D_R(0) \times \mathbb{C} \longrightarrow \mathbb{C}, \qquad (z, w) \longmapsto \det(\tau_U(z) - w \operatorname{id}_{\mathbb{C}^n}).$$

An application of Rouché's theorem allows one to show (see [9, Lemma 4.3]) that there is a real number $r_0 > 0$ such that, for every $n \in \mathbb{N}$ with $\frac{1}{n} \in (0, r_0)$, there is a $\delta_n > 0$ with $\delta_n < \frac{1}{n}$ and

$$\overline{D_{\delta_n}}(\tilde{z}) \times \overline{D_{\frac{1}{n}}}(\tilde{w}) \subseteq D_R(0) \times \mathbb{C}.$$

Furthermore, for each $z \in D_{\delta_n}(\tilde{z}) \cap \mathbb{D}$, the function $f(z, \cdot)$ has at least one zero in $D_{\frac{1}{n}}(\tilde{w}) \cap \mathbb{D}$. It follows that there are sequences $(z_n)_{n \in \mathbb{N}}$ in $\mathbb{D}$ and $(w_n)_{n \in \mathbb{N}}$ in $\mathbb{D}$ such that $\det(\tau_U(z_n) - w_n \operatorname{id}_{\mathbb{C}^n}) = 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} z_n = \tilde{z} \qquad \text{and} \qquad \lim_{n \to \infty} w_n = \tilde{w}. \qquad \square$$

In the following, we consider pairs of commuting $C_{\cdot 0}$-contractions $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ with finite dimensional defect spaces. Using Theorem 4.3, we are able to prove the following version of von Neumann's inequality.

Theorem 5.6. *Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ be a pair of commuting $C_{\cdot 0}$-contractions with $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$. Then there exists a distinguished variety $V \subseteq \mathbb{D}^2$ such that*

$$\|p(T_1, T_2)\| \leq \|p\|_V$$

for all polynomials $p \in \mathbb{C}[z_1, z_2]$.

Proof. By Theorem 4.3 it follows that

$$(T_1, T_2) \cong P_{\mathcal{Q}} \left(M_z \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_{\tau_U} \right) |_{\mathcal{Q}}$$

for the inner multiplier $\tau_U \colon \mathbb{D} \to \mathcal{L}\left(\mathcal{D}_{T_1}\right)$ and the joint $\left(M_z^* \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_{\tau_U}^*\right)$-invariant subspace $\mathcal{Q} = \Pi_{T_1} \mathcal{H} \subseteq \mathrm{H}^2\left(\mathbb{D}, \mathcal{D}_{T_1}\right)$. Note that

$$\tau_U(z) = A + zB \left(\mathrm{id}_{\mathcal{D}_{T_2}} - zD\right)^{-1} C \in \mathcal{L}(\mathcal{D}_{T_1}) \qquad (z \in \mathbb{D})$$

is the transfer function of the unitary $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}\left(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}\right)$ as defined in the beginning of Section 4.1. Let $A = A_0 \oplus A_1 \in \mathcal{L}(\mathcal{H}^0 \oplus \mathcal{H}^1)$ on $\mathcal{D}_{T_1} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be the orthogonal decomposition of A into the unitary part A_0 on $\mathcal{H}^0$ and the completely non-unitary part A_1 on $\mathcal{H}^1$ as in Theorem 2.6. By Proposition 2.20 it follows that the transfer function τ_U of U decomposes into

$$\tau_U(z) = \begin{pmatrix} \tau_U^0(z) & 0 \\ 0 & \tau_U^1(z) \end{pmatrix} \in \mathcal{L}\left(\mathcal{H}^0 \oplus \mathcal{H}^1\right) \qquad (z \in \mathbb{D}),$$

where

$$\tau_U^0(z) = A_0 \in \mathcal{L}\left(\mathcal{H}^0\right) \qquad (z \in \mathbb{D})$$

and

$$\tau_U^1(z) = \tau_{U_1}(z) = A_1 + zB \left(\mathrm{id}_{\mathcal{D}_{T_2}} - zD\right)^{-1} C|_{\mathcal{H}_1} \in \mathcal{L}\left(\mathcal{H}^1\right) \qquad (z \in \mathbb{D})$$

is the transfer function of the unitary operator

$$U_1 = \begin{pmatrix} A_1 & B \\ C|_{\mathcal{H}^1} & D \end{pmatrix} \in \mathcal{L}(\mathcal{H}^1 \oplus \mathcal{D}_{T_2}).$$

We first want to prove that

$$\sigma(\tau_U(z)) \subseteq \mathbb{D}$$

for all $z \in \mathbb{D}$. By Proposition 2.19 this will follow if we can show that the contraction $A \in \mathcal{L}(\mathcal{D}_{T_1})$ is completely non-unitary, that is, if $A_0 \in \mathcal{L}(\mathcal{H}^0)$ vanishes. With respect to the decomposition

$$\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1}) = \mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^0\right) \oplus \mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^1\right),$$

the multiplication operator $M_{\tau_U} \in \mathcal{L}\left(\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1})\right)$ acts as the direct sum

$$M_{\tau_U} = A_0 \oplus M_{\tau_U^1}.$$

Let $f \in \mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^0\right)$. Define $f_n = M_{\tau_U}^{*n} f = A_0^{*n} f$ for $n \in \mathbb{N}$. Since

$$\begin{aligned} \left| \langle f, \Pi_{T_1} h \rangle \right| &= \left| \left\langle M_{\tau_U}^n f_n, \Pi_{T_1} h \right\rangle \right| = \left| \left\langle f_n, M_{\tau_U}^{*n} \Pi_{T_1} h \right\rangle \right| = \left| \langle f_n, \Pi_{T_1} {T_2}^{*n} h \rangle \right| \\ &\leq \|f_n\| \, \|\Pi_{T_1} T_2^{*n} h\| \leq \|f\| \, \|T_2^{*n} h\| \xrightarrow{(n \to \infty)} 0 \end{aligned}$$

for all $h \in \mathcal{H}$, it follows that $\mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^0\right) \subseteq (\operatorname{Im} \Pi_{T_1})^{\perp}$ or, equivalently,

$$\mathcal{Q} = \operatorname{Im} \Pi_{T_1} \subseteq \mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^1\right).$$

Since $\Pi_{T_1} : \mathcal{H} \to \mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1})$ is a minimal dilation of T_1 and $\mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^1\right)$ is a $(M_z \otimes \mathrm{id}_{\mathcal{H}^1})$-reducing subspace of $\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1})$, it follows that

$$\mathrm{H}^2(\mathbb{D}, \mathcal{D}_{T_1}) = \bigvee_{k \in \mathbb{N}} z^k \, \mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^1\right) = \mathrm{H}^2\left(\mathbb{D}, \mathcal{H}^1\right)$$

(cf. [6, Section 2]). This shows $\mathcal{H}^0 = \{0\}$. Therefore $A = A_1$ is completely non-unitary.

Consider the distinguished variety

$$V = \left\{ (z_1, z_2) \in \mathbb{D}^2 \mid \det(\tau_U(z_1) - z_2 \, \mathrm{id}_{\mathcal{D}_{T_1}}) = 0 \right\}.$$

By Proposition 2.23, the transfer function τ_U admits a continuous extentions to the closed unit disc $\overline{\mathbb{D}}$ which we again denote by τ_U. In particular, the boundary values $\tau_U(z) \in \mathcal{L}(\mathcal{D}_{T_1})$ are unitary for $z \in \mathbb{T}$.

Let $p \in \mathbb{C}[z_1, z_2]$ be an arbitrary polynomial. Using Proposition 1.15 and the maximum principle for holomorphic functions with values in Banach spaces, we obtain the estimation

$$\begin{aligned} \|p(T_1, T_2)\| &= \left\| P_{\mathcal{Q}} p\left(M_z \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_{\tau_U}\right) |_{\mathcal{Q}} \right\| \\ &\leq \|P_{\mathcal{Q}}\| \left\| p\left(M_z \otimes \mathrm{id}_{\mathcal{D}_{T_1}}, M_{\tau_U}\right) \right\| \\ &= \left\| M_{p\left(z \, \mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U\right)} \right\| \\ &= \sup_{z \in \mathbb{D}} \left\| p\left(z \, \mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right) \right\| \\ &= \sup_{z \in \mathbb{T}} \left\| p\left(z \, \mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right) \right\| \end{aligned}$$

Note that the norm and spectral radius of the normal operators $p\left(z\,\mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right) \in \mathcal{L}(\mathcal{D}_{T_1})$ coincide for $z \in \mathbb{T}$. We obtain that

$$\begin{aligned}
\sup_{z\in\mathbb{T}} \left\| p\left(z\,\mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right)\right\| &= \sup_{z\in\mathbb{T}} \sup \left\{ |\lambda| \mid \lambda \in \sigma\left(p\left(z\,\mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right)\right)\right\} \\
&= \sup_{z\in\mathbb{T}} \sup \left\{ |p(\lambda_1,\lambda_2)| \mid (\lambda_1,\lambda_2) \in \sigma\left(z\,\mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right)\right\} \\
&\le \sup_{z\in\mathbb{T}} \sup \left\{ |p(\lambda_1,\lambda_2)| \mid \lambda_1 \in \sigma\left(z\,\mathrm{id}_{\mathcal{D}_{T_1}}\right), \lambda_2 \in \sigma\left(\tau_U(z)\right)\right\} \\
&= \sup_{z\in\mathbb{T}} \sup \left\{ |p(z,\lambda_2)| \mid \lambda_2 \in \sigma\left(\tau_U(z)\right)\right\}
\end{aligned}$$

by applying the spectral mapping theorem for the joint spectrum as well as Lemma 5.1 to the commuting normal pairs $\left(z\,\mathrm{id}_{\mathcal{D}_{T_1}}, \tau_U(z)\right) \in \mathcal{L}(\mathcal{H})$ for $z \in \mathbb{T}$. Since $\sigma(\tau_U(z)) \subseteq \mathbb{D}$ for $z \in \mathbb{D}$, Proposition 5.5 shows that

$$\{(z_1,z_2) \mid z_1 \in \mathbb{T}, z_2 \in \sigma(\tau_U(z_1))\} \subseteq \overline{V}.$$

In particular, since $\overline{V} \subseteq \overline{\mathbb{D}^2}$ is a compact set und p is continuous, we have proven that

$$\|p(T_1,T_2)\| \le \|p\|_{\overline{V}} = \|p\|_V. \qquad \square$$

Lemma 5.4 shows that Theorem 5.6 is an improvement of von Neumann's inequality for pairs of commuting contractions.

Theorem 5.6 in the case that T_1, T_2 are commuting matrices was first proved with a different method by Agler and McCarthy (see [2, Theorem 3.1]).

Corollary 5.7. *Let $T_1, T_2 \in \mathcal{L}(\mathbb{C}^n)$ be commuting contractions neither of which has eigenvalues of modulus one. Then there is a distinguished variety $V \subseteq \mathbb{D}^2$ such that*

$$\|p(T_1,T_2)\| \le \|p\|_V$$

for all polynomials $p \in \mathbb{C}[z_1,z_2]$.

Proof. By the spectral radius formula a contraction $T \in \mathcal{L}(\mathbb{C}^n)$ has no unimodular eigenvalues if and only if T is of class $C_{\cdot 0}$. Hence the result follows immediately from Theorem 5.6. $\square$

Using Remark 4.4 instead of Theorem 4.3 one obtains a similar proof of Theorem 5.6. More precisely, one finds that

$$\|p(T_1,T_2)\| \le \|p\|_{\tilde{V}} \qquad (p \in \mathbb{C}[z_1,z_2])$$

for

$$\tilde{V} = \left\{(z_1,z_2) \in \mathbb{D}^2 \;\middle|\; \det\left(\tau_{\tilde{U}}(z_2) - z_1\,\mathrm{id}_{\mathcal{D}_{T_2}}\right) = 0\right\},$$

where

$$\tau_{\tilde{U}}(z) = D^* + zB^* \left(\mathrm{id}_{\mathcal{D}_{T_1}} - zA^*\right)^{-1} C^* \in \mathcal{L}(\mathcal{D}_{T_2}) \qquad (z \in \mathbb{D})$$

is the transfer function of the unitary operator $\tilde{U} = \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \in \mathcal{L}(\mathcal{D}_{T_2} \oplus \mathcal{D}_{T_1})$ as defined in the beginning of Section 4.1. In particular, Lemma 5.3 shows that $\tilde{V}$ is a distinguished variety and coincides with the distinguished variety

$$V = \left\{(z_1, z_2) \in \mathbb{D}^2 \mid \det\left(\tau_U(z_1) - z_2 \,\mathrm{id}_{\mathcal{D}_{T_1}}\right) = 0\right\}$$

constructed in the proof of Theorem 5.6.

Example 5.8. *Let $\mathcal{H} = \mathrm{H}^2(\mathbb{D})$ and let $(T_1, T_2) \in \mathcal{L}\left(\mathrm{H}^2(\mathbb{D})\right)^2$ be given by*

$$(T_1, T_2) = (M_z, M_z)$$

(cf. Example 2.10). Since $D_{M_z} = P_{\mathbb{C}}$ and thus $\mathcal{D}_{M_z} = \mathbb{C}$, we obtain that the unitary operator U defined in the beginning of Section 4.1 has the form

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2.$$

Thus the transfer function $\tau_U \colon \mathbb{D} \to \mathcal{L}(\mathbb{C})$ is given by

$$\tau_U(z) = z$$

for all $z \in \mathbb{D}$ and we observe that

$$\det(\tau_U(z_1) - z_2 \,\mathrm{id}_{\mathbb{C}}) = z_1 - z_2$$

for all $(z_1, z_2) \in \mathbb{D}^2$. Hence, the distinguished variety $V \subseteq \mathbb{D}^2$ from the proof of Theorem 5.6 is given by

$$V = \{(z, z) \mid z \in \mathbb{D}\}.$$

Before we are able to prove another version of von Neumann's inequality, we recall the definition of algebraic varieties.

Definition 5.9. A non-empty set $V \subseteq \mathbb{C}^2$ is called an *algebraic variety* if there is a subset $F \subseteq \mathbb{C}[z_1, z_2]$ such that

$$V = V(F) = \{(z_1, z_2) \in \mathbb{C}^2 \mid p(z_1, z_2) = 0 \text{ for all } p \in F\}.$$

It is easy to see that $V(F \cup G) = V(F) \cap V(G)$ holds for any given subsets $F, G \subseteq \mathbb{C}[z_1, z_2]$. Furthermore, by Hilbert's basis theorem, for every $F \subseteq \mathbb{C}[z_1, z_2]$ there exist polynomials $p_1, ..., p_m \in \mathbb{C}[z_1, z_2]$ such that $V(F) = V(\{p_1, ..., p_m\})$.

Theorem 5.10. *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be commuting contractions such that T_1T_2 is a $C_{\cdot 0}$-contraction and $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$. Then there exists an algebraic variety $V \subseteq \mathbb{C}^2$ with $V \cap \mathbb{T}^2 \neq \mathbb{T}^2$ such that*

$$\|p(T_1, T_2)\| \leq \|p\|_{V \cap \mathbb{T}^2}$$

for all polynomials $p \in \mathbb{C}[z_1, z_2]$.

Proof. The proof of Theorem 4.7 provides a pure pair of commuting isometries $(M_\Phi, M_\Psi) \in \mathcal{L}\left(\mathrm{H}^2(\mathbb{D}, \mathcal{E})\right)^2$ as well as a joint (M_Φ^*, M_Ψ^*)-invariant subspace $\mathcal{Q} \subseteq \mathrm{H}^2(\mathbb{D}, \mathcal{E})$ such that

$$(T_1, T_2) \cong P_{\mathcal{Q}} \left(M_\Phi, M_\Psi\right)|_{\mathcal{Q}},$$

where the multipliers $\Phi, \Psi \colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ are commuting transfer functions defined as

$$\Phi(z) = \left(P + zP^\perp\right) U^* \qquad \text{and} \qquad \Psi(z) = U\left(P^\perp + zP\right) \qquad (z \in \mathbb{D})$$

with a suitable unitary operator $U \in \mathcal{L}(\mathcal{E})$ and a suitable orthogonal projection $P \in \mathcal{L}(\mathcal{E})$. Since $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$, one can choose

$$\mathcal{E} = \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \cong \mathbb{C}^m$$

with $m = \dim(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2})$ (see the proofs of Theorem 4.7 and Proposition 3.4).

Both Φ and Ψ can be extended to all of $\mathbb{C}$ by the same formulas. We denote these extensions again by Φ and Ψ. Due to Proposition 2.23, $\Phi(z)$ and $\Psi(z)$ are unitary matrices for all $z \in \mathbb{T}$ since $\dim \mathcal{D}_{T_i} < \infty$ for $i = 1, 2$.

Let $p \in \mathbb{C}[z_1, z_2]$ be a polynomial. Using Proposition 1.15 and the maximum principle one obtains as before that

$$\begin{aligned}\|p(T_1, T_2)\| &= \|P_{\mathcal{Q}} p\left(M_\Phi, M_\Psi\right)|_{\mathcal{Q}}\| \\ &\leq \|P_{\mathcal{Q}}\| \|p(M_\Phi, M_\Psi)\| \\ &= \left\|M_{p(\Phi, \Psi)}\right\| \\ &= \sup_{z \in \mathbb{D}} \|p\left(\Phi(z), \Psi(z)\right)\| \\ &= \sup_{z \in \mathbb{T}} \|p\left(\Phi(z), \Psi(z)\right)\|\end{aligned}$$

Again, we use that the norm and spectral radius of the normal operators $p\left(\Phi(z), \Psi(z)\right) \in \mathcal{L}(\mathcal{E})$ $(z \in \mathbb{T})$ coincide:

$$\begin{aligned}\sup_{z \in \mathbb{T}} \|p(\Phi(z), \Psi(z))\| &= \sup_{z \in \mathbb{T}} \sup\left\{|\lambda| \mid \lambda \in \sigma(p(\Phi(z), \Psi(z)))\right\} \\ &= \sup_{z \in \mathbb{T}} \sup\{|p(\lambda_1, \lambda_2)| \mid (\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z))\}\end{aligned}$$

In the last step we have used the polynomial spectral mapping theorem for the joint spectrum of the commuting normal pairs $(\Phi(z), \Psi(z)) \in \mathcal{L}(\mathcal{E})^2$ $(z \in \mathbb{T})$.

For $z \in \mathbb{T}$ and $(\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z))$, it follows by Lemma 5.1 that

$$\lambda_1 \lambda_2 \in \sigma(\Phi(z)\Psi(z)) = \sigma(z \, \mathrm{id}_{\mathcal{E}}) = \{z\}$$

and hence that $\lambda_1 \lambda_2 = z$.

Since $\Phi(z)$ and $\Psi(z)$ are matrices for all $z \in \mathbb{C}$, the non-empty sets

$$V_1 = \left\{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \det(\Phi(\lambda_1 \lambda_2) - \lambda_1 \, \mathrm{id}_{\mathcal{E}}) = 0 \right\}$$

and

$$V_2 = \left\{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \det(\Psi(\lambda_1 \lambda_2) - \lambda_2 \, \mathrm{id}_{\mathcal{E}}) = 0 \right\}$$

define algebraic varieties. Thus, the set

$$V = V_1 \cap V_2$$

is again an algebraic variety and we find that

$$\bigcup_{z \in \mathbb{T}} \sigma(\Phi(z), \Psi(z)) \subseteq V.$$

In particular, since $\Phi(z), \Psi(z)$ are unitary for $z \in \mathbb{T}$, we find that

$$\bigcup_{z \in \mathbb{T}} \sigma(\Phi(z), \Psi(z)) \subseteq V \cap \mathbb{T}^2.$$

Hence we have proven that

$$\|p(T_1, T_2)\| \leq \|p\|_{V \cap \mathbb{T}^2}$$

for every $p \in \mathbb{C}[z_1, z_2]$. An elementary argument shows that $V \cap \mathbb{T}^2 \neq \mathbb{T}^2$. Indeed, otherwise $V_1 \cap \mathbb{T}^2 = \mathbb{T}^2$ and one would obtain the contradiction that

$$\lambda \in \sigma\left(\Phi\left(\lambda \frac{1}{\lambda}\right)\right) = \sigma(\Phi(1))$$

for all $\lambda \in \mathbb{T}$. □

Example 5.11. *Let $\mathcal{H} = \mathrm{H}^2(\mathbb{D})$ and let $(T_1, T_2) \in \mathcal{L}\left(\mathrm{H}^2(\mathbb{D})\right)^2$ be given by*

$$(T_1, T_2) = (M_z, M_z).$$

With the observations from Example 5.8 we find that $\mathcal{E} = \mathbb{C}^2$ and the orthogonal projections $P, P^\perp \in \mathcal{L}(\mathbb{C}^2)$ from the proof of Theorem 4.7 have the form

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2) \qquad \text{and} \qquad P^\perp = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2)$$

With $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2)$ *as in Example 5.8 we obtain that*

$$\Phi(z) = \left(P + zP^{\perp}\right) U^* = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$$

and

$$\Psi(z) = U\left(P^{\perp} + zP\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$$

for all $z \in \mathbb{C}$. *Since*

$$\det\left(\Phi(z_1 z_2) - z_1 \operatorname{id}_{\mathbb{C}^2}\right) = \det \begin{pmatrix} -z_1 & z_1 z_2 \\ 1 & -z_1 \end{pmatrix} = z_1(z_1 - z_2) \qquad (z_1, z_2 \in \mathbb{C}),$$

and

$$\det\left(\Psi(z_1 z_2) - z_2 \operatorname{id}_{\mathbb{C}^2}\right) = \det \begin{pmatrix} -z_2 & z_1 z_2 \\ 1 & -z_2 \end{pmatrix} = z_2(z_1 - z_2) \qquad (z_1, z_2 \in \mathbb{C}),$$

we conclude

$$V_1 = \{0\} \times \mathbb{C} \cup \{(z, z) \mid z \in \mathbb{C}\} \quad \textit{and} \quad V_2 = \mathbb{C} \times \{0\} \cup \{(z, z) \mid z \in \mathbb{C}\}.$$

Hence the algebraic variety from the proof of Theorem 5.10 has the form

$$V = V_1 \cap V_2 = \{(z, z) \mid z \in \mathbb{C}\}.$$

A. Cardinal Numbers

In this section we recall the concept of cardinal numbers and give some basic properties. For the proofs of the following results and a more detailed approach to this topic, see [10, Chapter I.4].

With every set A we associate a symbol, called the *cardinal number of* A, such that two sets A and B have the same symbol attached to them if and only if there exists a bijection $f\colon A \to B$. We will write $A \sim B$ if such a bijection exists. In this case, we say that A and B have the *same cardinality*. We write $|A|$ to denote the cardinal number of A.

The main purpose of cardinal numbers is to compare the cardinality of different sets. This leads to the following definitions.

Definition A.1. Let $|A| = \mathfrak{a}$ and $|B| = \mathfrak{b}$ for cardinal numbers $\mathfrak{a}, \mathfrak{b}$. We write $\mathfrak{a} \leq \mathfrak{b}$ to mean that there is a subset $U \subseteq B$ of B such that $|U| = \mathfrak{a}$.

Definition A.2. A set A is said to be *finite* if there is a natural number $n \in \mathbb{N}$ such that $A \sim \{k \in \mathbb{N} \mid k \leq n\}$. A set that is not finite is called *infinite.*

Lemma A.3. *Let A be an infinite set and let B be a finite set. If $A \cap B = \emptyset$, then $|A| = |A \cup B|$.*

The sum of two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ is defined as the cardinal number of the union of two disjoint representatives of $\mathfrak{a}$ and $\mathfrak{b}$.

Definition A.4. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers. Define $\mathfrak{a} + \mathfrak{b} = |A \cup B|$ if A and B are sets with $\mathfrak{a} = |A|$, $\mathfrak{b} = |B|$ and $A \cap B = \emptyset$.

It is important to notice that $\mathfrak{a} + \mathfrak{b}$ is defined for all cardinal numbers $\mathfrak{a}, \mathfrak{b}$, since it is always possible to find appropriate sets A and B. In fact, if $A \cap B \neq \emptyset$, then define

$$A_0 = \{(a, 0) \mid a \in A\} \qquad \text{and} \qquad B_0 = \{(b, 1) \mid b \in B\}$$

to obtain $A \sim A_0$, $B \sim B_0$ and $A_0 \cap B_0 = \emptyset$.

For the purpose of this thesis, we are particularly interested in the following result.

Theorem A.5. *Let $\mathfrak{a}$ be an infinite cardinal number. Then $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$. In particular, if $\mathfrak{b}$ is a cardinal number such that $\mathfrak{b} \leq \mathfrak{a}$, then $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$.*

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