Schatten-p-class perturbations of Toeplitz operators

Dominik Schillo

09.05.2016

Content

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ with the canonical probability measure m.

Definition

We call

$$H^2(m) = \left\{ f \in L^2(m) \; ; \; \hat{f}(n) = 0 \text{ for all } n < 0 \right\} \subset L^2(m)$$

the *Hardy space with respect to m*.

Definition

Let $f \in L^\infty(m)$. We call the compression of the multiplication operator

$$M_f \colon L^2(m) \to L^2(m),\ g \mapsto fg$$

to $H^2(m)$ the *Toeplitz operator with symbol f* and denote it by T_f, i.e.

$$T_f = P_{H^2(m)} M_f|_{H^2(m)},$$

where $P_{H^2(m)} \colon L^2(m) \to H^2(m)$ is the orthogonal projection onto $H^2(m)$.

Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^\infty(m)$ such that $X = T_f$) if and only if

$$T_z^* X T_z - X = 0,$$

where $z \in L^\infty(m)$ is the identity map.

Definition

Define

$$H^\infty(m) = L^\infty(m) \cap H^2(m) \subset L^\infty(m)$$

and we call

$$I_m = \{f \in H^\infty(m) \ ; \ |f| = 1 \ m\text{-a.e.}\}$$

the set of *inner functions with respect to* m.

Theorem (Brown-Halmos condition II)

An operator $X \in B(H^2(m))$ *is a Toeplitz operator if and only if*

$$T_u^* X T_u - X = 0$$

for all $u \in I_m$.

Theorem (Gu, 2004)

An operator $X \in B(H^2(m))$ is a finite rank Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $F \in F(H^2(m))$ such that $X = T_f + F$) if and only if

$$T_u^* X T_u - X \in F(H^2(m))$$

for all $u \in I_m$.

Theorem (Xia, 2009)

An operator $X \in B(H^2(m))$ is a compact Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $K \in K(H^2(m))$ such that $X = T_f + K$) if and only if

$$T_u^* X T_u - X \in K(H^2(m))$$

for all $u \in I_m$.

Definition

Let $p \in [1, \infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a *Schatten-p-class* operator if

$$\|S\|_p^p = \operatorname{tr}(|S|^p) = \sum_{e \in \mathcal{E}} \langle |S|^p e, e \rangle = \sum_{e \in \mathcal{E}} \left\langle (S^* S)^{\frac{p}{2}} e, e \right\rangle < \infty$$

for some orthonormal basis $\mathcal{E}$ of H. Furthermore, we set

$$\mathcal{S}_p(H) = \left\{ S \in B(H) \; ; \; \|S\|_p < \infty \right\}$$

equipped with $\|\cdot\|_p$ and

$$\mathcal{S}_0(H) = F(H) \quad \text{as well as} \quad \mathcal{S}_\infty(H) = K(H)$$

both equipped with the operator norm.

Theorem

Let $p \in \{0\} \cup [1, \infty]$. An operator $X \in B(H^2(m))$ is a $\mathcal{S}_p(H^2(m))$ Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $S \in \mathcal{S}_p(H^2(m))$ such that $X = T_f + S$) if and only if

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(m))$$

for all $u \in I_m$.

Definition

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on $\mathbb{D}$. We call

$$A(\mathbb{D}) = \left\{ f \in C(\overline{\mathbb{D}}) \; ; \; f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}) \right\}$$

the *disc algebra*.

Proposition

The following statements hold.

(i) $H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}}$.

(ii) $H^\infty(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}$.

(iii) $H^\infty(m) H^2(m) \subset H^2(m)$.

Let D be a bounded domain in $\mathbb{C}^n$.

Definition

We denote by

(i) $A(D) = \{f \in C(\overline{D})\ ;\ f|_D \in \mathcal{O}(D)\} \subset C(\overline{D})$ the *domain algebra* of D,

(ii) $S = \partial_{A(D)}$ the *Shilov boundary* of $A(D)$ (i.e. the smallest closed subset of $\overline{D}$ such that

$$\sup_{z \in \overline{D}} |f(z)| = \sup_{z \in S} |f(z)|$$

for all $f \in A(D)$).

Examples

(i) $D = \mathbb{B}_n$: $S = \partial\mathbb{B}_n$

(ii) $D = \mathbb{D}^n$: S$= \mathbb{T}^n$

(iii) D strictly pseudoconvex: $S = \partial D$.

Let $\mu \in M^+(S)$ be a positive Borel measure on S.

Definition

We define

$$H^2(\mu) = \overline{A(D)|_S}^{\tau_{\|\cdot\|_{L^2(\mu)}}} \subset L^2(\mu)$$

and

$$H^\infty(\mu) = \overline{A(D)|_S}^{\tau_{w^*}} \subset L^\infty(\mu).$$

Furthermore, we denote by

$$I_\mu = \{f \in H^\infty(\mu)\ ;\ |f| = 1\ \mu\text{-a.e.}\}$$

the set of inner functions with respect to μ.

Definition

Let $f \in L^\infty(\mu)$. We call

$$T_f : H^2(\mu) \to H^2(\mu),\ g \mapsto P_{H^2(\mu)}(fg),$$

where $P_{H^2(\mu)} : L^2(\mu) \to H^2(\mu)$ is the orthogonal projection onto $H^2(\mu)$, the *Toeplitz operator with symbol* f.

Definition

Let $K \subset \mathbb{C}^n$ be a compact set, $A \subset C(K)$ be a closed subalgebra and $\nu \in M^+(K)$ a positive Borel measure.

(i) The triple (A, K, ν) is called *regular (in the sense of Aleksandrov)* if for every $\varphi \in C(K)$ with $\varphi > 0$, there exists a sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ in A with $|\varphi_k| < \varphi$ on K for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} |\varphi_k| = \varphi$$

ν-almost everywhere on K.

(ii) The measure ν is called *continuous* if every one-point set has ν-measure zero

Examples

(i) $A = A(\mathbb{B}_n)|_{\partial\mathbb{B}_n}, K = \partial\mathbb{B}_n, \nu = \sigma.$

(ii) $A = A(\mathbb{D}^n)|_{\mathbb{T}^n}, K = \mathbb{T}^n, \nu = \otimes_n m.$

Theorem (Aleksandrov, 1984)

Let (A, K, ν) be a regular triple with a continuous measure ν in $M^+(K)$. Then the weak sequential closure of the set I_ν contains all $L^\infty(\nu)$-equivalence classes of functions $f \in A$ with $|f| \leq 1$ on K.*

Let $\mu \in M_1^+(S)$ and $(A(D)|_S, S, \mu)$ be a regular triple.

Theorem

An operator $X \in B(H^2(\mu))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_\mu$.

Definition

We denote by $H^\infty(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Proposition

The following statements hold:

(i) The space $H^\infty(D) \subset L^\infty(D) = (L^1(D))'$ is weak closed.*

(ii) The space $L^1(D)/^\perp H^\infty(D)$ is separable with

$$H^\infty(D) \cong \left(L^1(D)/^\perp H^\infty(D)\right)'.$$

(iii) The closed unit ball $\overline{B}_1^{H^\infty(D)}(0)$ equipped with the relative topology of the weak topology of $H^\infty(D)$ is a compact metrizable space.*

Theorem

The map

$$r_m\colon H^\infty(\mathbb{D}) \to H^\infty(m),\ \theta \mapsto \tau_{w^*}\text{-}\lim_{r\to 1}[\theta(r\cdot)] =: \theta^*$$

is an isometric isomorphism and weak homeomorphism with* $r_m(\theta|_{\mathbb{D}}) = [\theta|_{\mathbb{T}}]$ *for all* $\theta \in A(\mathbb{D})$.

Definition

We call μ a *(faithful) Henkin measure* if there is a contractive (isometric) weak* continuous algebra homomorphism

$$r_\mu \colon H^\infty(D) \to L^\infty(\mu),\ \theta \mapsto r_\mu(\theta) =: \theta^*$$

with $r_\mu(\theta|_D) = [\theta|_S]$ for all $\theta \in A(D)$.

Let $\mu \in M_1^+(S)$ be a faithful Henkin measure.

Remark

The map $r_\mu \colon H^\infty(D) \to \mathrm{Im}(r_\mu)$ is an isometric isomorphism and weak* homeomorphism with weak* closed range.

Proposition

We have

$$H^\infty(\mu) \subset \operatorname{Im}(r_\mu).$$

Examples

(i) $D = \mathbb{B}_n, \mu = \sigma$

(ii) $D = \mathbb{D}^n, \mu = \otimes_n m$

Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\mu \in M_1^+(S)$ be a continuous faithful Henkin probability measure such that $(A(D)|_S, S, \mu)$ is a regular triple in the sense of Aleksandrov.

Theorem

Let $p \in [1, \infty)$. An operator $X \in B(H^2(\mu))$ is a $\mathcal{S}_p(H^2(\mu))$ Toeplitz perturbation (i.e. there exists a $f \in L^\infty(\mu)$ and $S \in \mathcal{S}_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$.

Proposition

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $H^\infty(\mu)$ with

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k = \alpha \in \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$$

and $X \in B(H^2(\mu))$ an operator such that

$$Y = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{\alpha_k}^* X T_{\alpha_k} \in B(H^2(\mu))$$

exists. If $T_u^* X T_u - X \in \mathcal{S}_\infty(H^2(\mu))$ for all $u \in I_\mu$, then there exists a function $f \in L^\infty(\mu)$ such that

$$X = T_f + \frac{1}{1-|\alpha|^2}(X - Y).$$

Proposition (Hiai, 1997)

The map

$$\|\cdot\|_p : (B(H^2(\mu)), \tau_{\mathrm{WOT}}) \to [0, \infty], \ S \mapsto \|S\|_p$$

is lower semi-continuous.

We denote by

$$I_D = \{\theta \in H^\infty(D)\ ;\ \theta^* \in I_\mu\}$$

the set of inner functions with respect to D.

Proposition

Let $p \in [1, \infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$. Then, for all sequences $(\theta_k)_{k\in\mathbb{N}}$ in I_D with

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \theta_k^* = 1,$$

we have

$$\tau_{\|\cdot\|_p}\text{-}\lim_{k\to\infty} T_{\theta_k^*}^* X T_{\theta_k^*} - X = 0.$$

Proposition

Let $(\theta_k)_{k\in\mathbb{N}}$ *be a sequence in* I_D. *Then the following statements are equivalent:*

(i) $\tau_{w^*}\text{-}\lim_{k\to\infty}\theta_k^* = 1$,

(ii) $\lim_{k\to\infty}\int_S \theta_k^*\, \mathrm{d}\mu = 1$,

(iii) There exists $w \in D$ *such that* $\lim_{k\to\infty}\theta_k(w) = 1$.

Proposition

Let $p \in [1, \infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$. Then, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\| T_{\theta^*}^* X T_{\theta^*} - X \|_p \leq \varepsilon$$

for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$.

Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\mu \in M_1^+(S)$ be a continuous faithful Henkin probability measure such that $(A(D)|_S, S, \mu)$ is a regular triple in the sense of Aleksandrov.

Theorem

Let $p \in [1, \infty)$. An operator $X \in B(H^2(\mu))$ is a $\mathcal{S}_p(H^2(\mu))$ Toeplitz perturbation (i.e. there exists a $f \in L^\infty(\mu)$ and $S \in \mathcal{S}_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$.

Questions

(i) For which regular triple (A, K, μ) does the result holds?

(ii) Are there other ideals for which the result holds?

(iii) What about $p = 0, \infty$?