Schatten-p-class perturbations of Toeplitz operators

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Definition

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ with the canonical probability measure m. The *Hardy space with respect to m* will be denoted by

$$H^2(m) = \left\{ f \in L^2(m) \ ; \ \hat{f}(n) = 0 \text{ for all } n < 0 \right\} \subset L^2(m).$$

Define

$$H^\infty(m) = L^\infty(m) \cap H^2(m) \subset L^\infty(m)$$

and we call

$$I_m = \{ f \in H^\infty(m) \ ; \ |f| = 1 \ m\text{-a.e.} \}$$

the set of *inner functions with respect to m*.

Definition

Let $f \in L^\infty(m)$. We call the compression of the multiplication operator

$$M_f \colon L^2(m) \to L^2(m),\ g \mapsto fg$$

to $H^2(m)$ the *Toeplitz operator with symbol f* and denote it by T_f, i.e.,

$$T_f = P_{H^2(m)} M_f|_{H^2(m)},$$

where $P_{H^2(m)} \colon L^2(m) \to H^2(m)$ is the orthogonal projection onto $H^2(m)$.

Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^\infty(m)$ such that $X = T_f$) if and only if

$$T_z^* X T_z - X = 0,$$

where $z \in L^\infty(m)$ is the identity map.

Theorem (Brown-Halmos condition II)

An operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_m$.

Theorem (Gu, 2004)

An operator $X \in B(H^2(m))$ is a finite rank Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $F \in F(H^2(m))$ such that $X = T_f + F$) if and only if

$$T_u^* X T_u - X \in F(H^2(m))$$

for all $u \in I_m$.

Theorem (Xia, 2009)

An operator $X \in B(H^2(m))$ is a compact Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $K \in K(H^2(m))$ such that $X = T_f + K$) if and only if

$$T_u^* X T_u - X \in K(H^2(m))$$

for all $u \in I_m$.

Definition

Let $p \in [1, \infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a *Schatten-p-class* operator if

$$\|S\|_p^p = \operatorname{tr}(|S|^p) = \sum_{e \in \mathcal{E}} \langle |S|^p e, e \rangle = \sum_{e \in \mathcal{E}} \left\langle (S^* S)^{\frac{p}{2}} e, e \right\rangle < \infty$$

for some orthonormal basis $\mathcal{E}$ of H. Furthermore, we set

$$\mathcal{S}_p(H) = \left\{ S \in B(H) \ ; \ \|S\|_p < \infty \right\}$$

equipped with $\|\cdot\|_p$ and

$$\mathcal{S}_0(H) = F(H) \quad \text{as well as} \quad \mathcal{S}_\infty(H) = K(H)$$

both equipped with the operator norm.

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with the canonical probability measure $m \in M_1^+(\mathbb{T})$.

Theorem (Gu, Xia, Didas-Eschmeier-S.)

Let $p \in \{0\} \cup [1, \infty]$. An operator $X \in B(H^2(m))$ is a $\mathcal{S}_p(H^2(m))$ Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $S \in \mathcal{S}_p(H^2(m))$ such that $X = T_f + S$) if and only if

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(m))$$

for all $u \in I_m$.

Definition

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on $\mathbb{D}$. We call

$$A(\mathbb{D}) = \left\{ f \in C(\overline{\mathbb{D}}) \; ; \; f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}) \right\}$$

the *disc algebra* and denote by $\partial_{A(\mathbb{D})}$ the *Shilov boundary* of $A(\mathbb{D})$ (i.e. the smallest closed subset of $\overline{\mathbb{D}}$ such that

$$\sup_{z \in \overline{\mathbb{D}}} |f(z)| = \sup_{z \in \partial_{A(\mathbb{D})}} |f(z)| \quad \text{for all } f \in A(\mathbb{D})).$$

Proposition

We have

$$\partial_{A(\mathbb{D})} = \partial \mathbb{D} = \mathbb{T}.$$

Definition

Let $D \subset \mathbb{C}^n$ be a bounded domain. We denote by

(i) $A(D) = \{f \in C(\overline{D}) \; ; \; f|_D \in \mathcal{O}(D)\} \subset C(\overline{D})$ the *domain algebra* of D,

(ii) $S = \partial_{A(D)}$ the *Shilov boundary* of $A(D)$ (i.e. the smallest closed subset of $\overline{D}$ such that

$$\sup_{z \in \overline{D}} |f(z)| = \sup_{z \in S} |f(z)|$$

for all $f \in A(D)$).

Let $D \subset \mathbb{C}^n$ be a stricly pseudoconvex or a bounded symmetric and circled domain. We denote by $\mu \in M_1^+(S)$ the canonical probability measure on S.

Examples

(i) $D = \mathbb{B}_n$: $S = \partial\mathbb{B}_n$ and $\mu = \sigma$.

(ii) $D = \mathbb{D}^n$: $S = \mathbb{T}^n$ and $\mu = \otimes_n m$.

We have

$$H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}} \quad \text{and} \quad H^\infty(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}.$$

Definition

We define

$$H^2(\mu) = \overline{A(D)|_S}^{\tau_{\|\cdot\|_{L^2(\mu)}}} \subset L^2(\mu)$$

and

$$H^\infty(\mu) = \overline{A(D)|_S}^{\tau_{w^*}} \subset L^\infty(\mu).$$

Furthermore, we denote by

$$I_\mu = \{f \in H^\infty(\mu)\ ;\ |f| = 1\ \mu\text{-a.e.}\}$$

the set of inner functions with respect to μ.

Definition

Let $f \in L^\infty(\mu)$. We call

$$T_f \colon H^2(\mu) \to H^2(\mu),\ g \mapsto P_{H^2(\mu)}(fg),$$

where $P_{H^2(\mu)} \colon L^2(\mu) \to H^2(\mu)$ is the orthogonal projection onto $H^2(\mu)$, the *Toeplitz operator with symbol f*.

Theorem (Didas-Eschmeier-Everard, 2011)

An operator $X \in B(H^2(\mu))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_\mu$.

Definition

We denote by $H^\infty(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ the set of all bounded analytic functions on $\mathbb{D}$.

Theorem

The map

$$r_m\colon H^\infty(\mathbb{D}) \to H^\infty(m),\ \theta \mapsto \tau_{w^*}\text{-}\lim_{r\to 1}[\theta(r\cdot)] =: \theta^*$$

is an isometric algebra isomorphism and weak homeomorphism with* $r_m(\theta|_{\mathbb{D}}) = [\theta|_{\mathbb{T}}]$ *for all* $\theta \in A(\mathbb{D})$.

Definition

We denote by $H^\infty(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Theorem

There exists a map

$$r_\mu \colon H^\infty(D) \to H^\infty(\mu), \ \theta \mapsto r_\mu(\theta) =: \theta^*,$$

which is an isometric algebra isomorphism and weak homeomorphism with* $r_\mu(\theta|_D) = [\theta|_S]$ *for all* $\theta \in A(D)$.

Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex or bounded symmetric and circled domain and $\mu \in M_1^+(S)$ be the probability measure on the Shilov boundary $S = \partial_{A(D)}$ obtained before.

Theorem (Didas-Eschmeier-S., 2016)

Let $p \in [1, \infty)$. An operator $X \in B(H^2(\mu))$ is a $\mathcal{S}_p(H^2(\mu))$ Toeplitz perturbation (i.e. there exists a $f \in L^\infty(\mu)$ and $S \in \mathcal{S}_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$.

Proposition

Let $(\alpha_k)_{k\in\mathbb{N}}$ *be a sequence in* $H^\infty(\mu)$ *with*

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k = \alpha \in [0,1)$$

and $X \in B(H^2(\mu))$ *an operator such that*

$$Y = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{\alpha_k}^* X T_{\alpha_k} \in B(H^2(\mu))$$

exists. If $T_u^* X T_u - X \in \mathcal{S}_\infty(H^2(\mu))$ *for all* $u \in I_\mu$*, then there exists a function* $f \in L^\infty(\mu)$ *such that*

$$X = T_f + \frac{1}{1-\alpha^2}(X - Y).$$

Proposition (Hiai, 1997)

The map

$$\|\cdot\|_p : (B(H^2(\mu)), \tau_{\mathrm{WOT}}) \to [0,\infty],\ S \mapsto \|S\|_p$$

is lower semi-continuous.

$$\left\| \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} X - T^*_{\alpha_k} X T_{\alpha_k} \right\|_p \leq \liminf_{k\to\infty} \left\| X - T^*_{\alpha_k} X T_{\alpha_k} \right\|_p$$

We denote by

$$I_D = \{\theta \in H^\infty(D) \; ; \; \theta^* \in I_\mu\}$$

the set of inner functions with respect to D and μ.

Proposition (Aleksandrov, 1984)

Let $\alpha \in [0,1)$. Then there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in I_D such that

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha.$$

Proposition (Xia, 2009)

Let $p \in [1, \infty]$*. Suppose that* $X \in B(H^2(\mu))$ *is an operator such that*

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$*. Then, for all* $\varepsilon > 0$*, there is a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\|T_{\theta^*}^* X T_{\theta^*} - X\|_p \leq \varepsilon$$

for all $\theta \in I_D$ *with* $\left|\int_S 1 - \theta^* \, \mathrm{d}\mu\right| \leq \delta$*.*

Proof.

There exists $0 < \delta < 1$ such that $\| T_{\theta^*}^* X T_{\theta^*} - X \|_p \leq 1$ for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$.

$\implies$ There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with $\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha$.

By passing to a subsequence we can achieve that $\left| \int_D 1 - \alpha_k^* \, \mathrm{d}\mu \right| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\mathrm{WOT}}\text{-} \lim_{k\to\infty} T_{\alpha_k^*}^* X T_{\alpha_k^*} \in B(H^2(\mu))$$

exists. Hence

$$\left\| \tau_{\mathrm{WOT}}\text{-} \lim_{k\to\infty} X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \leq \liminf_{k\to\infty} \left\| X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \leq 1.$$

$\implies X - Y \in \mathcal{S}_p(H^2(\mu))$. □

Remark

The following ingredients are essential for the proof.

(i) The triple $(A(D)|_S, S, \mu)$ is *regular (in the sense of Aleksandrov)*.

(ii) The measure μ is a *faithful Henkin measure*.

Question

What about $p = 0, \infty$?