$\mathcal{E}(\mathbb{T}^n)$-SUBSCALAR n-TUPLES AND THE CESARO OPERATOR ON H^p

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Abstract

An n-tuple $T \in L(X)^n$ of bounded linear operators on a Banach space X is defined to be $\mathcal{E}(\mathbb{T}^n)$-subscalar if there is a Banach space $\hat{X}$ and a commuting n-tuple $\hat{T} \in L(\hat{X})^n$ possessing a continuous functional calculus defined on the algebra $C^\infty(\mathbb{T}^n)$ of smooth functions on the n-torus such that T coincides (up to similarity) with the restriction of $\hat{T}$ to a joint invariant subspace.

In the following article, various characterizations of $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuples are obtained, inspired by the characterization of subscalar operators due to Eschmeier and Putinar, see [7].

In the single operator case, the methods used to solve the above characterization problem can be modified to obtain a sufficient condition for operators having property $(\beta)_\mathcal{E}$ modulo a compact subset of the unit circle. This can be applied to show that the Cesàro operator on the Hardy space H^p satisfies property $(\beta)_\mathcal{E}$ modulo $\{0\}$, for $1 < p < \infty$.

Most of the material contained in this paper is part of the author's master thesis [4], written at the Universität des Saarlandes, Saarbrücken, under the direction of Prof. Dr. J. Eschmeier. The author would like to thank him for many stimulating discussions and suggestions. Thanks are also due to Prof. Dr. T.L. Miller for answering some of the author's questions concerning the Cesàro operator during a stay at Saarbrücken in 1998.

Contents

1 Introduction

Let $\mathbb{T}$ denote the unit circle in the complex plane and let $n \geq 1$ be an integer. We write $\mathcal{E}(\mathbb{T}^n)$ for the algebra of all complex-valued C^∞-functions on $\mathbb{T}^n$, equipped with the topology of uniform convergence of all derivatives, which turns $\mathcal{E}(\mathbb{T}^n)$ into a Fréchet algebra.

A commuting n-tuple $T \in L(X)^n$ of bounded linear operators on a Banach space X is called $\mathcal{E}(\mathbb{T}^n)$-scalar if there is a continuous $\mathcal{E}(\mathbb{T}^n)$-functional calculus for T. If T is (up to similarity) the restriction of an $\mathcal{E}(\mathbb{T}^n)$-scalar n-tuple $\hat{T} \in L(\hat{X})^n$ to a joint invariant subspace, then T is said to be $\mathcal{E}(\mathbb{T}^n)$-subscalar. Using this terminology, the main concern of this paper can be described as follows:

Given an arbitrary commuting n-tuple $T \in L(X)^n$ of continuous linear operators on a Banach space X. Find necessary and sufficient conditions for T to be $\mathcal{E}(\mathbb{T}^n)$-subscalar.

It is a well-known fact that $\mathcal{E}(\mathbb{T}^n)$-scalar n-tuples can be characterized by the growth behaviour of their powers (see, for instance, [3]). More precisely, a single Banach-space operator $T \in L(X)$ is $\mathcal{E}(\mathbb{T})$-scalar if and only if T is invertible and there are constants $c > 0$ and $\kappa \in \mathbb{N}_0$ such that $\|T^m\| \leq c(1 + |m|)^\kappa$ for all $m \in \mathbb{Z}$. Furthermore, a commuting n-tuple of operators is $\mathcal{E}(\mathbb{T}^n)$-scalar if and only if its components are $\mathcal{E}(\mathbb{T})$-scalar operators, see [1]. As an analogue of the latter condition we obtain (Theorem 2.2.7):

A commuting n-tuple $T = (T_1, \cdots, T_n) \in L(X)^n$ is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if each T_i is an $\mathcal{E}(\mathbb{T})$-subscalar operator $(i = 1, \cdots, n)$ and, in addition, the product $T_1 \cdots T_n$ is $\mathcal{E}(\mathbb{T})$-subscalar.

This will be deduced from the following result (Theorem 2.2.3) which can be thought of as the counterpart of the above mentioned growth condition characterizing $\mathcal{E}(\mathbb{T})$-scalar operators.

A commuting n-tuple $T = (T_1, \cdots, T_n) \in L(X)^n$ is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if there is a constant $\kappa \in \mathbb{N}_0$ such that, for each $x' \in X'$, there is a sequence $(a_m)_{m\in\mathbb{Z}^n}$ in X' with $a_0 = x'$ satisfying the conditions

$$a_{k+e_i} = T_i' a_k \quad (k \in \mathbb{Z}^n, i = 1, \cdots, n) \quad \text{and} \quad \sup_{m\in\mathbb{Z}^n} \frac{\|a_m\|}{(1+|m|)^\kappa} < \infty,$$

where e_i denotes the canonical i-th unit vector in $\mathbb{R}^n$.

To prove these results we apply methods originally used by Eschmeier and Putinar to characterize the class of subscalar operators (see [7], Chapter 6).

In comparison with the theory of subscalar operators, one may conjecture that, for a single operator $T \in L(X)$ to be $\mathcal{E}(\mathbb{T})$-subscalar, it should be necessary and sufficient that the multiplication map $\mathcal{E}(\mathbb{T}, X) \longrightarrow \mathcal{E}(\mathbb{T}, X), f \mapsto (z - T)f$ is a topological monomorphism. (In this case T is said to have property $(\beta)_{\mathcal{E}(\mathbb{T})}$, in analogy with property $(\beta)_{\mathcal{E}}$, which characterizes the class of subscalar operators, see Corollary 6.4.9 in [7].) We will see in Section 2.4 that this condition is necessary but not sufficient, as easy counterexamples show. In Theorem 2.4.8 we will present a characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators which at least involves property $(\beta)_{\mathcal{E}(\mathbb{T})}$.

Another characterization relies on the existence of slowly growing local resolvents for the adjoint operator. In Section 2.3 we prove:

A Banach-space operator $T \in L(X)$ is $\mathcal{E}(\mathbb{T})$-subscalar if and only if there are constants $c > 0$ and $\kappa \in \mathbb{N}_0$ such that for each $x' \in X'$ there is an analytic function $f : \mathbb{C} \setminus \mathbb{T} \to X'$ satisfying $(z - T')f(z) = x'$ for $z \in \mathbb{C} \setminus \mathbb{T}$ and

$$\|f(z)\| \leq \frac{c}{|1 - |z||^{\kappa}} \quad (0 < |1 - |z|| < 1).$$

The basic motivation for the work of Chapter 3 is a result implicitly contained in a paper of T.L.Miller, V.G.Miller and R.C.Smith [10], which states that the Cesàro operator C_p on the Hardy space H^p over the unit disc, $1 < p < \infty$, possesses a left resolvent satisfying almost the same growth condition as the one formulated above for local resolvents (see Theorem 4.2.3). The aim of the third chapter is to modify the theory of $\mathcal{E}(\mathbb{T})$-subscalar operators in such a way that operators like the Cesàro operator can be treated. We obtain a sufficient criterion for an operator to have property $(\beta)_{\mathcal{E}}$ modulo a compact subset of the unit circle (see Theorem 3.2.2 and Corollary 3.2.3).

In the above cited paper [10] it is shown that, for $1 < p < \infty$, the Cesàro operator C_p on the Hardy space H^p satisfies Bishop's property (β) and hence is subdecomposable. In view of the fact that C_2 is subnormal (see [9]), the question arose whether C_p is subscalar, or equivalently, has property $(\beta)_{\mathcal{E}}$, for $1 < p < \infty$.

By combining the results of Chapter 3 and the growth estimates from [10] we will partially answer this question in Corollary 4.2.7. More precisely, we prove the following result:

For $1 < p < \infty$, the Cesàro operator on the Hardy space H^p has property $(\beta)_{\mathcal{E}}$ modulo $\{0\}$.

Since an operator which is decomposable modulo a single point is decomposable, this yields another proof of the fact that C_p $(1 < p < \infty)$ is subdecomposable. However, the question whether C_p is subscalar remains open.

2 $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuples

Given an integer $n \geq 1$ we denote by $\mathcal{E}(\mathbb{T}^n)$ the space

$$\{f : \mathbb{T}^n \to \mathbb{C} \quad | \quad ((t_1, \cdots, t_n) \mapsto f(e^{it_1}, \cdots, e^{it_n})) \in C^\infty(\mathbb{R}^n)\}$$

of all C^∞-functions on $\mathbb{T}^n$, equipped with its natural Fréchet-space topology. For a concrete system of generating seminorms, see 2.1.5.

2.1 Preliminaries concerning $\mathcal{E}(\mathbb{T}^n)$-scalar n-tuples

2.1.1 Definition. *Let X be a Banach space and $T = (T_1, \cdots, T_n) \in L(X)^n$ a commuting n-tuple of continuous linear operators on X.*

- *(a) Let us say that T is $\mathcal{E}(\mathbb{T}^n)$-scalar if T possesses a continuous $\mathcal{E}(\mathbb{T}^n)$-functional calculus. This means that there is a continuous unital algebra homomorphism $\Phi : \mathcal{E}(\mathbb{T}^n) \to L(X)$ mapping the i-th coordinate function z_i to T_i $(i = 1, \cdots, n)$.*
- *(b) The tuple T will be called $\mathcal{E}(\mathbb{T}^n)$-subscalar if there is a Banach space $\hat{X}$ and a topological monomorphism $i : X \to \hat{X}$ intertwining $T \in L(X)^n$ and an $\mathcal{E}(\mathbb{T}^n)$-scalar n-tuple $\hat{T} \in L(\hat{X})^n$ componentwise.*

2.1.2 Theorem. (Colojoară and Foiaş, [3]) *For a single operator $T \in L(X)$ on a Banach space X, the following conditions are equivalent:*

- *(a) T is $\mathcal{E}(\mathbb{T})$-scalar;*
- *(b) $\sigma(T) \subset \mathbb{T}$ and there are constants $c > 0$, $k \in \mathbb{N}_0$, such that*

$$\| R(\zeta, T) \| \leq \frac{c}{|1 - |\zeta||^k} \quad (0 < |1 - |\zeta|| < 1);$$

- *(c) T is invertible and there are $\tilde{c} > 0$, $\tilde{k} \in \mathbb{N}_0$ such that*

$$\| T^m \| \leq \tilde{c}(1 + |m|)^{\tilde{k}} \quad (m \in \mathbb{Z}).$$

Proof. We only give a sketch of the proof to demonstrate which techniques are involved here. Details may be found in the monograph of Colojoară and Foiaş. (a) $\Rightarrow$ (b). Fix a continuous $\mathcal{E}(\mathbb{T})$-functional calculus Φ for T, and observe that

$$R(\zeta, T) = \Phi(\frac{1}{\zeta - z}) \quad (\zeta \in \mathbb{C} \setminus \mathbb{T}).$$

Now the desired norm estimate of the resolvent easily follows by using the continuity of Φ expressed explicitly in terms of generating seminorms for the topology of $\mathcal{E}(\mathbb{T})$.
(b) $\Rightarrow$ (c). Using the holomorphic functional calculus for T, we obtain an integral representation of T^m $(m \in \mathbb{Z})$ involving the resolvent,

$$T^m = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} R(\zeta, T)\zeta^m d\zeta \quad (m \in \mathbb{Z}),$$

where γ_ε stands for the path $\partial D_{1+\varepsilon}(0) \ominus \partial D_{1-\varepsilon}(0)$ $(0 < \varepsilon < 1)$. Let $k \in \mathbb{N}_0$ denote the constant from (b) and choose $\varepsilon = \frac{k+1}{m}$, for $|m|$ sufficiently large. Now combine the given norm estimate of the resolvent and the standard estimate for path integrals to obtain that $(\|T^m\|)_m$ is slowly growing in the sense of (c).
(c) $\Rightarrow$ (a). It is an easy exercise to check that, under the growth condition made in (c), a continuous $\mathcal{E}(\mathbb{T})$-functional calulus for T is given by the formula

$$\mathcal{E}(\mathbb{T}) \ni \varphi \mapsto \sum_{m=-\infty}^{\infty} \hat{\varphi}(m) T^m \in L(X),$$

where $\hat{\varphi}(m) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) e^{-imt} dt$ denotes the m-th Fourier coefficient of φ. □

The method used to deduce (a) from (c) in the preceding proof can be carried over to the case of several commuting operators. Suppose that $\Phi : \mathcal{E}(\mathbb{T}^n) \to L(X)$ is a continuous functional calculus for the tuple $T = (T_1, \cdots, T_n) \in L(X)^n$. Since each $f \in \mathcal{E}(\mathbb{T}^n)$ allows a unique Fourier series representation of the form $f = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) z^k$, where the series converges in the topology of $\mathcal{E}(\mathbb{T}^n)$, the continuity and the algebraic properties of Φ imply that

$$\Phi(f) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k_1, \cdots, k_n) T_1^{k_1} \cdots T_n^{k_n}.$$

This observation shows that an $\mathcal{E}(\mathbb{T}^n)$-functional calculus for a tuple T is unique, if it exists; namely, it can be constructed by replacing the coordinate functions in the Fourier series of an $\mathcal{E}(\mathbb{T}^n)$-function by the components of T. Working out this idea leads to the following well-known result.

2.1.3 Theorem. (Albrecht, [1]) *A commuting tuple of Banach-space operators $(T_1, \cdots, T_n) \in L(X)^n$ is $\mathcal{E}(\mathbb{T}^n)$-scalar if and only if each $T_i \in L(X)$ is $\mathcal{E}(\mathbb{T})$-scalar $(i = 1, \cdots, n)$.* □

2.1.4 Lemma. *Let $T = (T_1, \cdots, T_n) \in L(X)^n$ be an $\mathcal{E}(\mathbb{T}^n)$-scalar n-tuple with corresponding functional calculus Φ.*

(a) *If $Y \subset X$ is a closed linear subspace invariant for T_i and T_i^{-1} $(i = 1, \cdots, n)$, then $T/Y = (T_1/Y, \cdots, T_n/Y) \in L(X/Y)^n$ is also $\mathcal{E}(\mathbb{T}^n)$-scalar.*

(b) *For an arbitrary Banach space Y, the tuple $T \otimes \mathbf{1} = (T_1 \otimes \mathbf{1}, \cdots, T_n \otimes \mathbf{1}) \in L(X \widehat{\otimes}_\varepsilon Y)^n$ is $\mathcal{E}(\mathbb{T}^n)$-scalar. The $\mathcal{E}(\mathbb{T}^n)$-functional calculus for $T \otimes \mathbf{1}$ is explicitly given by*

$$\hat{\Phi} : \mathcal{E}(\mathbb{T}^n) \to L(X \widehat{\otimes}_\varepsilon Y), \quad \varphi \mapsto \Phi(\varphi) \otimes \mathbf{1}.$$

Proof.

(a) It suffices to check that

$$\hat{\Phi} : \mathcal{E}(\mathbb{T}^n) \to L(X/Y), \quad \varphi \mapsto \Phi(\varphi)/Y$$

is well defined. (In this case $\hat{\Phi}$ obviously defines the desired functional calculus for T/Y.) This follows from the remarks preceding Theorem 2.1.3, since by hypothesis $T_1^{k_1} \cdots T_n^{k_n} Y \subset Y$ for all $k \in \mathbb{Z}^n$.

(b) Obviously, $\hat{\Phi}$ is well defined and linear. For the continuity, it suffices to observe that, for every $\varphi \in \mathcal{E}(\mathbb{T}^n)$, $\|\Phi(\varphi) \otimes \mathbf{1}\| = \|\Phi(\varphi)\|$. The multiplicativity follows from the fact that $(\Phi(\varphi) \circ \Phi(\psi)) \otimes \mathbf{1} = (\Phi(\varphi) \otimes \mathbf{1}) \circ (\Phi(\psi) \otimes \mathbf{1})$. Checking that $\hat{\Phi}$ maps 1 to $\mathbf{1} \otimes \mathbf{1}$ and z_i to $T_i \otimes \mathbf{1}$ finishes the proof. □

2.1.5 Example.

(a) For $\varphi \in \mathcal{E}(\mathbb{T}^n)$, the multiplication operator

$$M_\varphi : C^k(\mathbb{T}^n) \to C^k(\mathbb{T}^n), \quad f \mapsto \varphi f$$

is obviously linear, and an application of the product rule implies the existence of a constant $c > 0$, such that

$$\|M_\varphi\| \leq c\|\varphi\|_k \quad (\varphi \in \mathcal{E}(\mathbb{T}^n)),$$

where $\| \cdot \|_k$ is one of the generating seminorms for the topology of $\mathcal{E}(\mathbb{T}^n)$ defined by

$$\|\varphi\|_m = \sup_{\substack{\alpha \in \mathbb{N}_0^n, |\alpha| \leq m \\ t \in [0, 2\pi]^n}} |D^\alpha \varphi(e^{it_1}, \cdots, e^{it_n})| \quad (\varphi \in \mathcal{E}(\mathbb{T}^n), m \in \mathbb{N}_0).$$

Hence the algebra homomorphism

$$\Phi : \mathcal{E}(\mathbb{T}^n) \to L(C^k(\mathbb{T}^n)), \quad \varphi \mapsto M_\varphi$$

turns out to be continuous. Since it also maps 1 to $\mathbf{1}$ and z_i to M_{z_i} ($i = 1, \cdots, n$), the tuple

$$(M_{z_1}, \cdots, M_{z_n}) \in L(C^k(\mathbb{T}^n))^n$$

is $\mathcal{E}(\mathbb{T}^n)$-scalar with corresponding functional calculus Φ.

(b) Let us fix an arbitrary Banach space X. As a consequence of part (b) of the above lemma, the tuple

$$(M_{z_1} \otimes \mathbf{1}, \cdots, M_{z_n} \otimes \mathbf{1}) \in L(C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X)^n$$

is $\mathcal{E}(\mathbb{T}^n)$-scalar.

Given a single operator $T \in L(X)$, we simply write T instead of $\mathbf{1} \otimes T \in L(C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X)$, if there is no danger of confusion. In the same manner, we use φ as an abbreviation for the operator $M_\varphi \otimes \mathbf{1} \in L(C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X)$, $\varphi \in \mathcal{E}(\mathbb{T}^n)$.

Let us fix an arbitrary n-tuple $T = (T_1, \cdots, T_n) \in L(X)^n$. Having in mind the abbreviations from above, it is obvious that z_j and T_i commute as operators on $C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$ whenever $1 \leq i, j \leq n$. Hence, it is easy to see that the subspace

$$\vee_{i=1}^n (z_i - T_i) C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$$

is invariant for z_j and z_j^{-1} $(j = 1, \cdots, n)$. Now, the first part of the preceding lemma implies that the quotient of the tuple $(z_1, \cdots, z_n) \in L(C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X)^n$ acting on

$$C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X / \vee_{i=1}^n (z_i - T_i) C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$$

is $\mathcal{E}(\mathbb{T}^n)$-scalar.

2.2 A characterization of $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuples

2.2.1 Lemma. *Fix an integer $n \geq 1$. For $j = 1, \cdots, n$ let $(E_k \xrightarrow{T_k^{(j)}} F_k)_{k \in \mathbb{N}_0}$ be a morphism of countable inverse systems of Fréchet spaces, and denote by $T^{(j)} : E \to F$ the induced mapping between the corresponding inverse limits $E = \varprojlim E_k$, $F = \varprojlim F_k$. Suppose that the system $(E_k)_k$ is reduced and that the density relations*

$$\vee_{j=1}^n T_k^{(j)} E_k = F_k \quad (k \geq 0)$$

hold. Then we have

$$\vee_{j=1}^n T^{(j)} E = F.$$

Proof. By Hahn-Banach it suffices to check that the only continuous linear form y' on F satisfying

$$y' \circ T^{(j)} = 0 \quad (j = 1, \cdots, n)$$

is the trivial one. Expressing the continuity of y' in terms of seminorms, we obtain that there exists an integer $m \geq 0$ such that

$$|y'(x)| \leq p(\pi_m^F(x)) \quad (x \in F),$$

where $\pi_m^F : F \to F_m$ denotes the canonical projection and p is a continuous seminorm on F_m. This implies that we can define a continuous linear form y'_m on $\pi_m^F F$ by setting

$$y'_m(\pi_m^F x) = y'(x) \quad (x \in F).$$

By Hahn-Banach there is a continuous linear extension of y'_m to F_m again denoted by y'_m, satisfying $y'_m \circ \pi_m^F = y'$. Now using the relations

$$y'_m \circ T_m^{(j)} \circ \pi_m^E = y' \circ T^{(j)} = 0 \quad (j = 1, \cdots, n),$$

the fact that the natural projections $\pi_k^E : E \to E_k$ of the reduced inverse system $(E_k)_k$ have dense range (see [7], Remark 3.2.5 (b)), and the density relation $\vee_{j=1}^n T_k^{(j)} E_k = F_k \quad (k \geq 0)$, which is valid by hypothesis, we can deduce that y'_m and hence $y' = y'_m \circ \pi_m^F$ vanishes identically. □

2.2.2 Lemma. *Given any Banach space X and $n \in \mathbb{N}$, the following topological identification holds:*

$$\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X/ \vee_{i=1}^n (z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X = \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X/ \vee_{i=1}^n (z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X.$$

Proof. Step (1). Because of the canonical identification $\mathcal{E}(\mathbb{T}^n) = \varprojlim C^k(\mathbb{T}^n)$, we can write

$$\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X = (\varprojlim C^k(\mathbb{T}^n))\widehat{\otimes}_\varepsilon X = \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X,$$

for the last equality, see Jarchow [8], Corollary 16.3.2. Note that the structural maps

$$C^l(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \stackrel{inj\otimes\mathbf{1}}{\longrightarrow} C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \quad (l \geq k)$$

corresponding to the inverse limit on the right have dense range. To see this, it suffices to observe that their images contain the set $\mathcal{E}(\mathbb{T}) \otimes X$, which is a dense subspace of $C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$ $(k \in \mathbb{N})$.

Notice that for all $j = 1, \cdots, n$ the following diagram commutes

$$\begin{array}{ccccc}
\varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X & \xrightarrow{z_j - T_j} & \varprojlim \vee_{i=1}^n (z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X & \xrightarrow{inj} & \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \\
\imath \uparrow & & \imath_0 \uparrow & & \imath \uparrow \\
\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X & \xrightarrow{z_j - T_j} & \vee_{i=1}^n (z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X & \xrightarrow{inj} & \mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X,
\end{array}$$

where we denote by $\imath : \mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X \to \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$ the topological isomorphism obtained above and $\imath_0$ stands for the restriction of $\imath$ to $\vee_{i=1}^n (z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X$. Using the right half of the diagram, we deduce that $\imath_0$ must be a topological monomorphism, whereas the left half combined with the preceding lemma yields the surjectivity of $\imath_0$. Hence, modulo $\imath_0$, we have the equality

$$\vee_{i=1}^{n}(z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X = \varprojlim \vee_{i=1}^{n}(z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$$

in the category of locally convex spaces.

Step (2). Using the fact that the inverse system corresponding to

$$\varprojlim \vee_{i=1}^{n}(z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$$

is reduced, we obtain the exactness of the sequence

$$\begin{aligned} 0 \quad \longrightarrow \quad & \varprojlim \vee_{i=1}^{n}(z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \longrightarrow \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \\ \longrightarrow \quad & \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X / \vee_{i=1}^{n} (z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \longrightarrow 0 \end{aligned}$$

as an application of Theorem 3.2.4 in Eschmeier and Putinar, [7]. Now, a look at the topological identifications established at the beginning and the end of step (1) yields the exactness of

$$\begin{aligned} 0 \quad \longrightarrow \quad & \vee_{i=1}^{n}(z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X \longrightarrow \mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X \\ \longrightarrow \quad & \varprojlim C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X / \vee_{i=1}^{n} (z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X \longrightarrow 0, \end{aligned}$$

which proves the lemma. □

2.2.3 Theorem. *Let X be a Banach space and $T = (T_1, \cdots, T_n) \in L(X)^n$ a commuting n-tuple of continuous linear operators on X. Then the following conditions are equivalent:*

- *(a) T is $\mathcal{E}(\mathbb{T}^n)$-subscalar;*
- *(b) the mapping $j : X \to \mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X / \vee_{i=1}^{n} (z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes}X$, $\quad x \mapsto [1 \otimes x]$, is a topological monomorphism;*
- *(c) for each $x' \in X'$, there is a distribution $u \in L(\mathcal{E}(\mathbb{T}^n), X')$ satisfying*

  $$u(1) = x' \quad \text{and} \quad u(z_i\varphi) = T_i' u(\varphi) \text{ for all } \varphi \in \mathcal{E}(\mathbb{T}^n),\ i = 1, \cdots, n;$$

- *(d) there is a constant $\alpha \in \mathbb{N}_0$ such that, for each $x' \in X'$, there is a sequence $(a_k)_{k\in\mathbb{Z}^n}$ in X' satisfying the growth condition*

  $$\sup_{m\in\mathbb{Z}^n} \frac{\|a_m\|}{(1 + |m|)^\alpha} < \infty$$

  as well as the algebraic relations

  $$a_0 = x' \quad \text{and} \quad a_{k+e_i} = T_i' a_k \text{ for all } k \in \mathbb{Z}^n, i \in \{1, \cdots, n\},$$

  where e_i stands for the canonical i-th unit vector in $\mathbb{R}^n$.

Proof. (b) $\Rightarrow$ (a). Using the identification established in the preceding lemma, one obtains that, for a suitable constant $k \in \mathbb{N}_0$, the mapping

$$j : X \to C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X / \vee_{i=1}^n (z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X, \quad x \mapsto [1 \otimes x]$$

is a topological monomorphism. From

$$[1 \otimes T_i x] = [1 \otimes T_i x + (z_i \otimes x - 1 \otimes T_i x)] = [z_i \otimes x] \quad (x \in X, i = 1, \cdots, n)$$

we deduce that j intertwines $T \in L(X)^n$ and the quotient of $(z_1, \cdots, z_n)$ on $C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X / \vee_{i=1}^n (z_i - T_i)C^k(\mathbb{T}^n)\widehat{\otimes}_\varepsilon X$ componentwise. According to the definition of an $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuple it remains to check that this latter tuple is $\mathcal{E}(\mathbb{T}^n)$-scalar. This is in fact the case, as we have shown in 1.1.5. So (b) implies (a).

(c) $\Rightarrow$ (b). Recall that, since $\mathcal{E}(\mathbb{T}^n)$ is a nuclear Fréchet space, there is a canonical algebraic (even topological) isomorphism

$$\psi : L(\mathcal{E}(\mathbb{T}^n), X') \to (\mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X)', \quad u \mapsto \psi(u),$$

where $\psi(u)$ stands for the unique continuous linear form on $\mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X$ mapping $\varphi \otimes x$ to $\langle x, u(\varphi)\rangle$ for all $\varphi \in \mathcal{E}(\mathbb{T}^n), x \in X$ (see Jarchow [8], Theorem 21.5.9). It is an easy exercise to show that, modulo ψ, the adjoint of the map

$$z_i - T_i : \mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X \longrightarrow \mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X$$

becomes the mapping

$$z_i - T_i' : L(\mathcal{E}(\mathbb{T}^n), X') \longrightarrow L(\mathcal{E}(\mathbb{T}^n), X'), \quad u \mapsto u(z_i \cdot) - T_i' u.$$

Hence the dual of the quotient space

$$\mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X / \vee_{i=1}^n (z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X$$

can be canonically identified with

$$\bigcap_{i=1}^n \ker z_i - T_i' = \{u \in L(\mathcal{E}(\mathbb{T}^n), X') : u(z_i \varphi) = T_i' u(\varphi) \quad \forall \varphi \in \mathcal{E}(\mathbb{T}^n), i = 1, \cdots, n\}$$

in the category of vector spaces. Since for $u \in L(\mathcal{E}(\mathbb{T}^n), X')$,

$$\langle 1 \otimes x, \psi(u)\rangle = \langle x, u(1)\rangle \quad (x \in X),$$

the adjoint of the continuous linear map

$$j : X \to \mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X / \vee_{i=1}^n (z_i - T_i)\mathcal{E}(\mathbb{T}^n)\widehat{\otimes} X, \quad x \mapsto [1 \otimes x]$$

can be identified via ψ with the mapping

$$\bigcap_{i=1}^{n} \ker \Big(L(\mathcal{E}(\mathbb{T}^n), X') \overset{z_i - T_i'}{\longrightarrow} L(\mathcal{E}(\mathbb{T}^n), X') \Big) \longrightarrow X', \quad u \mapsto u(1).$$

Now using the well-known fact that a continuous linear map j between Fréchet spaces is a topological monomorphism if and only if its adjoint j' is surjective, one easily sees that (c) and (b) are in fact equivalent.

(d) $\Rightarrow$ (c). Given a sequence $(a_k)_{k\in\mathbb{Z}^n}$ in X' with the desired properties, one simply defines

$$u : \mathcal{E}(\mathbb{T}^n) \to X', \quad \varphi \mapsto \sum_{m\in\mathbb{Z}^n} \hat{\varphi}(m) a_m.$$

The growth condition on (a_k) implies that, for each $\varphi \in \mathcal{E}(\mathbb{T}^n)$,

$$\sum_{m\in\mathbb{Z}^n} |\hat{\varphi}(m)| \|a_m\| \leq \Big(\sup_{m\in\mathbb{Z}^n} \frac{\|a_m\|}{(1+|m|)^\alpha} \Big) \sum_{m\in\mathbb{Z}^n} |\hat{\varphi}(m)| (1+|m|)^\alpha.$$

Notice that the sum on the right can be interpreted as a generating seminorm for the locally convex topology of the space $s(\mathbb{Z}^n)$ of rapidly decreasing sequences, which is topologically isomorphic to $\mathcal{E}(\mathbb{T}^n)$ via Fourier-transformation. From this observation we deduce that u is well defined and that, moreover, $u : \mathcal{E}(\mathbb{T}^n) \to X'$ is a continuous linear map. Direct computation shows that $u(1) = a_0$ and that $u(z_i\varphi) = T_i' u(\varphi)$ for all $\varphi \in \mathcal{E}(\mathbb{T}^n)$, $i = 1, \cdots, n$.

(a) $\Rightarrow$ (d). Suppose that $\hat{T} \in L(\hat{X})^n$ is an $\mathcal{E}(\mathbb{T}^n)$-scalar extension of T and let $i : X \to \hat{X}$ be the intertwining topological monomorphism. Fix an $x' \in X'$ and let $\hat{x}'$ be an element of $\hat{X}'$ satisfying $i'\hat{x}' = x'$. Define

$$a_k = i' \hat{T}_1'^{k_1} \cdots \hat{T}_n'^{k_n} \hat{x}' \quad (k \in \mathbb{Z}^n).$$

Obviously $a_0 = i'\hat{x}' = x'$ and

$$a_{k+e_i} = i' \hat{T}_1'^{k_1} \cdots \hat{T}_i'^{k_i+1} \cdots \hat{T}_n'^{k_n} \hat{x}' = T_i' a_k \quad (k \in \mathbb{Z}^n).$$

So it remains to prove that $(a_k)_k$ is slowly growing in the sense of (d). But recall that, since $\hat{T}_i$ is $\mathcal{E}(\mathbb{T})$-scalar ($i = 1, \cdots, n$), there are constants $c > 0$ and $\kappa \in \mathbb{N}_0$ such that

$$\|\hat{T}_i^m\| \leq c(1+|m|)^\kappa \quad (m \in \mathbb{Z}, i = 1, \cdots, n).$$

So we have

$$\|a_m\| \leq c^n \|i'\| \Pi_{i=1}^n (1+|m_i|)^\kappa \leq c^n \|i'\| (1+|m|)^{n\kappa} \quad (m \in \mathbb{Z}^n),$$

which is the desired growth condition. □

2.2.4 Corollary. *Let $n \geq 1$ be an integer. A commuting n-tuple of continuous linear operators $T = (T_1, \cdots, T_n) \in L(X)^n$ on a Banach space X is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if there exist constants $c > 0, \kappa \in \mathbb{N}_0$ such that the following conditions are satisfied:*

(a) $\|T_i^m\| \leq c(1+m)^\kappa \quad (m \in \mathbb{N}_0, i = 1, \cdots, n)$;

(b) for each $x' \in X'$, there is a sequence $(d_m)_{m \leq 0}$ in X' satisfying

$$(T_n' \cdots T_1')d_m = d_{m+1} \quad (m \leq -1) \quad \text{and} \quad d_0 = x',$$

which is slowly growing in the sense that

$$\sup_{m \in \mathbb{N}_0} \frac{\|d_{-m}\|}{(1+m)^\kappa} < \infty.$$

Proof. To see that the two conditions are sufficient, it is enough to check that they imply the validity of condition (d) of the preceding theorem. For this aim, fix an arbitrary $x' \in X'$. We want to define a corresonding sequence $(a_k)_{k \in \mathbb{Z}^n}$ in X' as required in 2.2.3 (d) in the following way: For a given $k \in \mathbb{Z}^n$, choose an arbitrary integer $m \leq 0$ satisfying $m \leq \min\{k_1, \cdots, k_n\}$ and set

$$a_k = (\prod_{i=1}^n (T_i')^{k_i - m}) d_m.$$

Note that this definition is independent of the special choice of m, because if both m and $\tilde{m}$ are as above and $\tilde{m} \geq m$, we have

$$(\Pi_{i=1}^n (T_i')^{k_i - m}) d_m = (\Pi_{i=1}^n (T_i')^{k_i - \tilde{m}})(\Pi_{i=1}^n (T_i'))^{\tilde{m} - m} d_m = (\Pi_{i=1}^n (T_i')^{k_i - \tilde{m}}) d_{\tilde{m}}.$$

As an easy consequence, we obtain that for any $j \in \{1, \cdots, n\}$ and any $k \in \mathbb{Z}^n$ we have

$$T_j' a_k = T_j' (\Pi_{i=1}^n (T_i')^{k_i - m}) d_m = (\Pi_{i=1}^n (T_i')^{k_i + \delta_{ij} - m}) d_m = a_{k + e_j},$$

where e_j stands for the canonical j-th unit vector in $\mathbb{R}^n$ and $m \leq 0$ is an integer satisfying

$$m \leq \min\{k_1, \cdots, k_n\} \leq \min\{k_1, \cdots, k_j + 1, \cdots, k_n\}.$$

Furthermore, we have $a_0 = x'$, and we conclude that the algebraic relations stated in 2.2.3 (d) are valid.

Using the growth restriction of $(\|T_i^m\|)_{m \in \mathbb{N}_0}$ $(i = 1, \cdots, n)$ we deduce that, for any fixed $k \in \mathbb{Z}^n$ and any $\hat{k} \in \mathbb{Z}, \hat{k} \leq \min\{0, k_1, \cdots, k_n\}$, we have

$$\|a_k\| \leq c^n \Pi_{i=1}^n (1 + k_i - \hat{k})^\kappa \|d_{\hat{k}}\|.$$

Choosing $\hat{k} = -\max_{i=1,\cdots,n} |k_i| \le \min\{0, k_1, \cdots, k_n\}$ we obtain the estimate

$$\begin{aligned} \|a_k\| &\le (1+2|k|)^{n\kappa} c^n \|d_{\hat{k}}\| \\ &\le (1+|k|)^{n\kappa} 2^{n\kappa} c^n \|d_{\hat{k}}\| \quad (k \in \mathbb{Z}^n). \end{aligned}$$

Consequently,

$$\frac{\|a_k\|}{(1+|k|)^{(n+1)\kappa}} \le 2^{n\kappa} c^n \frac{\|d_{\hat{k}}\|}{(1+|k|)^{\kappa}} \le 2^{n\kappa} c^n \frac{\|d_{\hat{k}}\|}{(1+|\hat{k}|)^{\kappa}} \quad (k \in \mathbb{Z}),$$

and the term on the right-hand side is bounded by hypothesis. Therefore the growth condition formulated in Theorem 2.2.3 (d) holds.

To check the necessity of conditions (a) and (b), note that if $T = (T_1, \cdots, T_n) \in L(X)^n$ is $\mathcal{E}(\mathbb{T}^n)$-subscalar, then the product $T_1 \cdots T_n \in L(X)$ is $\mathcal{E}(\mathbb{T})$-subscalar, for if $(\hat{T}_1, \cdots, \hat{T}_n) \in L(\hat{X})^n$ is an $\mathcal{E}(\mathbb{T}^n)$-scalar extension of T on a Banach space $\hat{X}$, then obviously $\hat{T}_1 \cdots \hat{T}_n$ is an extension of $T_1 \cdots T_n$, which is $\mathcal{E}(\mathbb{T})$-scalar in view of Theorem 2.1.2. Now by part (d) of the preceding theorem, we know that condition (b) must be satisfied. Finally, since each component of an $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuple is $\mathcal{E}(\mathbb{T})$-subscalar (see Theorem 2.1.3), the following lemma tells us that (a) holds. □

2.2.5 Lemma. *If $T \in L(X)$ is an $\mathcal{E}(\mathbb{T})$-subscalar operator, then there are constants $c > 0$, $\kappa \in \mathbb{N}_0$, such that*

$$\frac{\|x\|}{c(1+m)^{\kappa}} \le \|T^m x\| \le c(1+m)^{\kappa} \|x\| \quad (x \in X, m \in \mathbb{N}_0).$$

Proof. Let us choose an $\mathcal{E}(\mathbb{T})$-scalar operator $\hat{T} \in L(\hat{X})$ and a topological monomorphism $i : X \to \hat{X}$ intertwining T and $\hat{T}$, i.e. $i \circ T = \hat{T} \circ i$. We know that there are constants $\hat{c} > 0$ and $\kappa \in \mathbb{N}_0$ satisfying

$$\|\hat{T}^m\| \le \hat{c}(1+|m|)^{\kappa} \quad (m \in \mathbb{Z}).$$

Furthermore, since i is injective with closed range, we can fix a constant $\gamma > 0$ with the property that

$$\gamma\|x\| \le \|ix\| \quad (x \in X).$$

From these two observations we deduce that for any $x \in X$ we have

$$\gamma\|x\| \le \|\hat{T}^{-m}\hat{T}^m ix\| = \|\hat{T}^{-m} i T^m x\| \le \hat{c}\|i\|(1+m)^{\kappa}\|T^m x\| \quad (m \in \mathbb{N}_0),$$

as well as

$$\gamma\|T^m x\| \le \|iT^m x\| = \|\hat{T}^m ix\| \le \hat{c}\|i\|(1+m)^{\kappa} \quad (m \in \mathbb{N}_0).$$

Thus the assertion of the lemma holds with $c = \frac{\hat{c}\|i\|}{\gamma}$. □

The following criterion obviously applies to commuting n-tuples of isometries.

2.2.6 Corollary. *Let $T = (T_1, \cdots, T_n) \in L(X)^n$ be a commuting n-tuple of bounded linear operators on a Banach space. Suppose that there are constants $c > 0$ and $\kappa \in \mathbb{N}_0$ such that the estimates*

$$\|x\| \leq \|T_i^m\| \leq c(1+m)^\kappa \|x\| \quad (x \in X, m \in \mathbb{N}_0, i = 1, \cdots, n)$$

hold. Then T is $\mathcal{E}(\mathbb{T}^n)$-subscalar.

Proof. Since condition (a) of Corollary 2.2.4 is clearly satisfied by hypothesis, it suffices to check the validity of 2.2.4 (b). For this aim, note that $A = T_1 \cdots T_n$ satisfies $\|Ax\| \geq \|x\|$ for all $x \in X$.

Suppose that $d_0 \in X'$ is an arbitrary continuous linear functional on X. The above norm estimate for A implies that we can define a continuous linear functional d_{-1} of norm at most $\|d_0\|$ on AX by setting

$$d_{-1}(Ax) = d_0(x) \quad (x \in X).$$

By the theorem of Hahn-Banach we obtain an extension of d_{-1} to a continuous linear form on X, again denoted by d_{-1}, which satisfies

$$(T_1' \cdots T_n')d_{-1} = d_0 \quad \text{and} \quad \|d_{-1}\| \leq \|d_0\|.$$

Repeating this argument yields a sequence $(d_m)_{m\leq 0}$ in X' satisfying all the requirements of Corollary 2.2.4 (b). □

As another immediate consequence of Corollary 2.2.4 we obtain the following result.

2.2.7 Theorem. *A commuting n-tuple $(T_1, \cdots, T_n) \in L(X)^n$ of continuous linear operators on a Banach space is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if the following two conditions hold:*

(a) each T_i is $\mathcal{E}(\mathbb{T})$-subscalar $(i = 1, \cdots, n)$;

(b) the product $T_1 \cdots T_n$ is $\mathcal{E}(\mathbb{T})$-subscalar.

Proof. Obviously, (a) and (b) are necessary. To prove that they are sufficient, use the criterion established in Corollary 2.2.4. By applying 2.2.3 (d) to the product $T_1 \cdots T_n$, we obtain the lifting property as desired in 2.2.4 (b), whereas Lemma 2.2.5 implies that 2.2.4 (a) is fulfilled. □

2.2.8 Corollary. *Let $T \in L(X)^n$ be an $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuple on a Banach space X, and suppose that $S \in L(X)$ is $\mathcal{E}(\mathbb{T})$-scalar and commutes with every component of T. Then, the $(n+1)$-tuple $(T, S) \in L(X)^{n+1}$ is $\mathcal{E}(\mathbb{T}^{n+1})$-subscalar.*

Proof. In view of the preceding corollary, the problem can be reduced to the question whether the product TS of a single $\mathcal{E}(\mathbb{T})$-subscalar operator $T \in L(X)$ and a single $\mathcal{E}(\mathbb{T})$-scalar operator $S \in L(X)$ commuting with T is $\mathcal{E}(\mathbb{T})$-subscalar. By Lemma 2.2.5, we only have to check the validity of 2.2.4 (b). For this aim, fix an arbirary $x' \in X'$. Then choose a sequence $(c_m)_{m\leq 0}$ satisfying $T'c_m = c_{m+1}$ $(m \leq -1)$, $c_0 = x'$ and $\sup_{m\in\mathbb{N}} \frac{\|c_{-m}\|}{(1+m)^{\kappa_T}} < \infty$, with a suitable constant $\kappa_T \in \mathbb{N}_0$. Set

$$d_m = (S')^m c_m \quad (m \leq 0)$$

to obtain a sequence $(d_m)_{m\leq 0}$, satisfying $T'S'd_m = (S')^{m+1}T'c_m = d_{m+1}$ $(m \leq -1)$, $d_0 = x'$. Since S is $\mathcal{E}(\mathbb{T})$-scalar, there are constants $\kappa_S \in \mathbb{N}_0$, $c > 0$ such that

$$\|d_m\| \leq \|S^m\|\|c_m\| \leq c(1 + |m|)^{\kappa_S}\|c_m\| \quad (m \leq 0).$$

Therefore, we have $\sup_{m\in\mathbb{N}} \frac{\|d_{-m}\|}{(1+m)^{(\kappa_T+\kappa_S)}} < \infty$, as desired. □

2.2.9 Problem. Let $T \in L(X)^n$ be a commuting n-tuple of continuous linear operators. If T is $\mathcal{E}(\mathbb{T}^n)$-subscalar, then each component of T is an $\mathcal{E}(\mathbb{T})$-subscalar operator by Theorem 2.2.7. Is the converse also true? In other words: Is the condition (b) of Theorem 2.2.7 actually superfluous?

2.2.10 Problem. Is the necessary condition for an operator to be $\mathcal{E}(\mathbb{T})$-subscalar given in Lemma 2.2.5 also sufficient? If it is, this would immediately imply an affirmative answer to the preceding question about the multi-variable case: Indeed, a growth condition like the one considered in Lemma 2.2.5 is inherited by products.

2.2.11 Theorem. *Suppose that $T \in L(H)$ is a continuous linear operator on a Hilbert space H satisfying*

$$T^*T^{m+1}H \subset T^m H \quad (m \in \mathbb{N}_0).$$

Then the following two conditions are equivalent:

(a) T is $\mathcal{E}(\mathbb{T})$-subscalar;

(b) there are constants $c > 0$ and $\kappa \in \mathbb{N}_0$ such that

$$\frac{\|x\|}{c(1+m)^\kappa} \leq \|T^m x\| \leq c(1+m)^\kappa\|x\| \quad (x \in X, m \in \mathbb{N}_0).$$

Proof. According to Lemma 2.2.5, it suffices to check that (b) implies (a). Making use of Theorem 2.2.4, the non-trivial part of the proof consists in checking the validity of 2.2.4 (b), where we can substitute H' by H and T' by T^*. Fix an arbitrary $a_0 \in H$ to start with. We want to construct a sequence $(a_k)_{k\le 0}$ in H satisfying

$$T^*a_k = a_{k+1} \quad (k \le -1) \quad \text{and} \quad \sup_{k\in\mathbb{N}_0} \frac{\|a_{-k}\|}{(1+k)^\kappa} < \infty,$$

with κ as in the statement of the theorem. Towards this end, note that the left inequality of (b) guarantees that T^m is injective and has closed range for all $m \in \mathbb{N}_0$.

Hence, for each $k \in \mathbb{N}$, we are able to choose a vector $a_{-k} \in T^kH$ satisfying

$$\langle T^kx, a_{-k}\rangle = \langle x, a_0\rangle \quad (x \in H).$$

Again applying the left estimate of part (b), we obtain that

$$|\langle T^kx, a_{-k}\rangle| \le \|x\|\|a_0\| \le c(1+k)^\kappa\|T^kx\|\|a_0\| \quad (x \in H, k \in \mathbb{N}).$$

Hence

$$\|a_{-k}\| \le c(1+k)^\kappa\|a_0\| \quad (k \in \mathbb{N}),$$

which is the desired growth behaviour.

To finish the proof, it suffices to check that

$$\langle x, T^*a_{-k}\rangle = \langle x, a_{-k+1}\rangle \quad (k \in \mathbb{N}, x \in H).$$

From the condition on the invariant subspace lattice of T^*T we know that

$$T(T^kH)^\perp \subset (T^{k+1}H)^\perp$$

for all $k \in \mathbb{N}_0$.

So if $k \in \mathbb{N}$ and $x \in (T^{k-1}H)^\perp$, we have

$$\langle x, T^*a_{-k}\rangle = \langle Tx, a_{-k}\rangle = 0 = \langle x, a_{-k+1}\rangle.$$

On the other hand, if $x = T^{k-1}y$ for some vector $y \in H$, then

$$\langle x, T^*a_{-k}\rangle = \langle T^ky, a_{-k}\rangle = \langle y, a_0\rangle = \langle T^{k-1}y, a_{-k+1}\rangle = \langle x, a_{-k+1}\rangle,$$

and the assertion follows. □

2.2.12 Corollary. *Suppose that $T = (T_1, \cdots, T_n) \in L(H)^n$ is a commuting tuple of operators on a Hilbert space H such that the product $A = T_1 \cdots T_n$ satisfies*

$$A^*A^{m+1}H \subset A^mH \quad (m \in \mathbb{N}_0).$$

Then T is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if each component of T is $\mathcal{E}(\mathbb{T})$-subscalar.

Proof. Suppose that each T_i is $\mathcal{E}(\mathbb{T})$-subscalar ($i = 1, \cdots, n$). By Lemma 2.2.5 there are constants $c > 0$, $\kappa \in \mathbb{N}_0$ such that

$$\frac{\|x\|}{c(1+m)^\kappa} \leq \|T_i^m x\| \leq c(1+m)^\kappa \|x\| \quad (x \in X, m \in \mathbb{N}_0, i = 1, \cdots, n).$$

Since the T_i's commute, the powers of the product satisfy the estimates

$$\frac{\|x\|}{c^n(1+m)^{n\kappa}} \leq \|A^m x\| \leq c^n(1+m)^{n\kappa} \|x\| \quad (x \in X, m \in \mathbb{N}_0).$$

Hence, by the hypothesis on A, we are allowed to apply Theorem 2.2.11 to conclude that T is $\mathcal{E}(\mathbb{T}^n)$-subscalar. This proves the non-trivial implication. □

2.3 A local resolvent condition in the single operator case

In the present section, we want to show how the resolvent estimate contained in Theorem 2.1.2 can be used to deduce a characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators.

2.3.1 Theorem. *For a single operator $T \in L(X)$, the following conditions are equivalent:*

(a) T is $\mathcal{E}(\mathbb{T})$-subscalar;

(b) there are constants $c > 0$ and $k \in \mathbb{N}_0$ such that for any given $x' \in X'$ there is a function $f \in \mathcal{O}(\mathbb{C} \setminus \mathbb{T}, X')$ satisfying $(z - T')f(z) = x'$ for $z \in \mathbb{C} \setminus \mathbb{T}$ and

$$\|f(z)\| \leq \frac{c}{|1 - |z||^k} \quad (z \in U_1(\mathbb{T})),$$

where $U_1(\mathbb{T}) = \{z \in \mathbb{C} : 0 < |1 - |z|| < 1\}$.

Proof. (a) ⇒ (b). Let $\hat{T} \in L(\hat{X})$ be an $\mathcal{E}(\mathbb{T})$-scalar extension of T and let $i : X \to \hat{X}$ be the corresponding intertwining map, which is by definition a topological monomorphism. Suppose that $x' \in X'$ is arbitrary. Using the fact that i' is surjective we can choose an element $\hat{x}' \in \hat{X}'$ in such a way that $i'\hat{x}' = x'$. Using the resolvent of $\hat{T}'$, we define

$$f : \mathbb{C} \setminus \mathbb{T} \to X', \quad z \mapsto i'R(z, \hat{T}')\hat{x}'$$

which is easily shown to be a local resolvent for T' at x'. So it remains to estimate the growth behaviour of f. But $\hat{T}'$ is $\mathcal{E}(\mathbb{T})$-scalar. Hence an application of Theorem 2.1.2 completes the proof.

(b) $\Rightarrow$ (a). Fix an arbitrary vector $x' \in X'$ and a corresponding local resolvent $f \in \mathcal{O}(\mathbb{C} \setminus \mathbb{T}, X')$ as described in part (b). We are looking for a sequence $(a_k)_{k\in\mathbb{Z}}$ in X' as required for an application of 2.2.3 (d). For this aim, note that $f|_{\mathbb{D}}$ has a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n \quad (z \in \mathbb{D})$ with coefficients in X'. Since $(z - T')f(z) = x'$ $(z \in \mathbb{D})$, we obtain that

$$\sum_{n=0}^{\infty}(c_n - T'c_{n+1})z^{n+1} - (T'c_0 + x') = 0 \quad (z \in \mathbb{D}),$$

or equivalently, that

$$T'c_{n+1} = c_n \quad (n \in \mathbb{N}_0) \quad \text{and} \quad T'c_0 = -x'.$$

By Cauchy's theorem the formula

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{\partial D_{(1-\varepsilon)}(0)} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \quad (n \in \mathbb{N}_0)$$

holds for every $0 < \varepsilon < 1$. It follows that

$$\|c_n\| \leq (1-\varepsilon)\sup_{|\zeta|=1-\varepsilon}\frac{\|f(\zeta)\|}{|\zeta|^{n+1}} \leq \frac{1-\varepsilon}{(1-\varepsilon)^{n+1}}\frac{c}{\varepsilon^k} = \frac{c}{(1-\varepsilon)^n\varepsilon^k} \quad (n \in \mathbb{N}_0),$$

where we have used the growth behaviour of f to obtain the second inequality. Taking $\varepsilon = \frac{k}{n}$ for $n > k$ leads us to

$$\frac{\|c_n\|}{n^k} \leq \frac{c}{(1-\frac{k}{n})^n k^k} \quad (n > k).$$

The fact that the right-hand side of this inequality tends to $\frac{c}{e^{-k}k^k}$ (as $n \to \infty$) shows that $\sup_{n\in\mathbb{N}_0}\frac{\|c_n\|}{(1+n)^k} < \infty$.

The fact that f is a local resolvent implies that $f(z) \to 0$ as $|z| \to \infty$. Hence, the function $\mathbb{D} \setminus \{0\} \to X', z \mapsto f(\frac{1}{z})$ has a removable singularity at 0 and extends to a holomorphic function $\tilde{f} \in \mathcal{O}(\mathbb{D}, X')$ with $\tilde{f}(0) = 0$. Note that, since f is a local resolvent of T' at x', we have

$$(\frac{1}{z} - T')\tilde{f}(z) = (\frac{1}{z} - T')f(\frac{1}{z}) = x' \quad (z \in \mathbb{D} \setminus \{0\}).$$

Using the power series expansion $\tilde{f}(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{D})$, we derive the equation

$$\sum_{n=2}^{\infty} b_n z^n - \sum_{n=2}^{\infty} T'b_{n-1}z^n + (b_1 - x')z = 0 \quad (z \in \mathbb{D}),$$

showing that the coefficients $b_n \in X'$ ($n \in \mathbb{N}$) satisfy the relations

$$b_1 = x' \quad \text{and} \quad T'b_{n-1} = b_n \quad (n \geq 2).$$

The growth behaviour of the b_k's can be computed by the same argument as above: replacing there f by $\tilde{f}$ yields $\sup_{n\in\mathbb{N}_0} \frac{\|b_n\|}{(1+n)^k} < \infty$.

A moment's thought shows that the definition

$$a_n = \begin{cases} -c_{n-1} & (n \geq 1) \\ b_{-n+1} & (n \leq 0). \end{cases}$$

yields a sequence as required in part (d) of Theorem 2.2.3. Indeed,

$$T'a_{n+1} = \left\{ \begin{array}{ccc} -T'c_n & = & -c_{n-1} \quad (n \geq 1) \\ -T'c_0 & = & x' = b_1 \quad (n = 0) \\ T'b_{-n} & = & b_{-n+1} \quad (n \leq -1) \end{array} \right\} = a_n,$$

as desired. □

2.4 The property $(\beta)_{\mathcal{E}(\mathbb{T})}$

In Eschmeier and Putinar [7], Section 6.4, it is shown that an operator $T \in L(X)$ is subscalar if and only if the mapping

$$\mathcal{E}(\mathbb{C}, X) \xrightarrow{z-T} \mathcal{E}(\mathbb{C}, X)$$

is a topological monomorphism, where $\mathcal{E}(\mathbb{C}, X)$ is defined to be the space of all X-valued C^∞-functions on the complex plane, equipped with its natural Fréchet-space topology.

In our context, a natural question is the following: Replace in the above definition of the mapping $z - T$ the space $\mathcal{E}(\mathbb{C}, X)$ by $\mathcal{E}(\mathbb{T}, X)$, the space of all smooth X-valued functions on the unit circle (again carrying its natural Fréchet-space topology). Does one obtain a characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators in this way?

The following simple counterexample tells us that the answer is negative: Let $T \in L(X)$ be an operator whose spectrum is disjoint from the closed unit disc. In this case, the map $\mathcal{E}(\mathbb{T}, X) \xrightarrow{z-T} \mathcal{E}(\mathbb{T}, X)$ is obviously a topological isomorphism, but T can never be $\mathcal{E}(\mathbb{T})$-subscalar, for the following reason:

2.4.1 Lemma. *Suppose that $T \in L(X)$ is $\mathcal{E}(\mathbb{T})$-subscalar. Then one of the following alternatives holds:*

(a) $\sigma(T) \subset \mathbb{T}$, *and in this case* T *is* $\mathcal{E}(\mathbb{T})$*-scalar;*

(b) $\sigma(T) = \overline{\mathbb{D}}$.

Proof. Using the well-known formula for the spectral radius $r(T)$ and the norm estimates of the positive powers of T given in Lemma 2.2.5, we see that $r(T) \leq 1$, hence $\sigma(T) \subset \overline{\mathbb{D}}$.

Now suppose that $\sigma(T) \subset \mathbb{T}$. Then T is invertible, and by Lemma 2.2.5 we know that there are constants $c > 0, \kappa \in \mathbb{N}_0$ such that T satisfies the estimates

$$\frac{\|x\|}{c(1+m)^\kappa} \leq \|T^m x\| \leq c(1+m)^\kappa \|x\| \quad (x \in X, m \in \mathbb{N}_0).$$

By replacing there x by $T^{-m}x$ one obtains that the same estimates are valid with T^{-1} instead of T. Applying 2.1.2 (c), we see that T is in fact $\mathcal{E}(\mathbb{T})$-scalar.

Assume that (a) does not hold. Then we have $\sigma(T) \cap \mathbb{D} \neq \emptyset$. Since T is similar to the restriction of an operator having spectrum contained in $\mathbb{T}$, we conclude that $\sigma(T) \supset \mathbb{D}$, which yields the assertion. □

Before we go further, it may be useful to remember some facts about Fourier series. Recall that each function $f \in \mathcal{E}(\mathbb{T}, X)$ has a representation of the form

$$f = \sum_{k=-\infty}^{\infty} \hat{f}(k) z^k \quad \text{with} \quad \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-ikt} dt,$$

where the series converges in $\mathcal{E}(\mathbb{T}, X)$.

Now let s_X denote the vector space of all sequences $(a_n)_{n \in Z}$ in X satisfying $\sum_{n \in \mathbb{Z}} \|a_n\|(1+|n|)^k < \infty$ for all $k \in \mathbb{N}_0$. Then s_X becomes a Fréchet space if it is equipped with the locally convex topology generated by the seminorms

$$q_k((a_n)_n) = \sum_{n \in \mathbb{Z}} \|a_n\|(1+|n|)^k \quad (k \in \mathbb{N}_0).$$

Using integration by parts one can show that the Fourier coefficients of each function in $\mathcal{E}(\mathbb{T}, X)$ form a sequence in s_X. Moreover, the mapping

$$\mathcal{E}(\mathbb{T}, X) \longrightarrow s_X, \quad f \mapsto (\hat{f}(n))_{n \in \mathbb{Z}}$$

which is obviously linear, becomes a topological isomorphism. Complete proofs of these facts can be found for instance in [4].

2.4.2 Theorem. *Let* $T \in L(X)$ *be an* $\mathcal{E}(\mathbb{T})$*-subscalar operator on a Banach space* X*. Then the multiplication operator* $\mathcal{E}(\mathbb{T}, X) \xrightarrow{z-T} \mathcal{E}(\mathbb{T}, X)$ *is a topological monomorphism.*

Proof. First observe that we can restrict ourselves to the case where T is $\mathcal{E}(\mathbb{T})$-scalar, since the restriction of a topological monomorphism remains a topological monomorphsim.

Now suppose that $T \in L(X)$ is $\mathcal{E}(\mathbb{T})$-scalar, and fix an integer $N \in \mathbb{N}_0$. Using the growth behaviour of the powers of an $\mathcal{E}(\mathbb{T})$-scalar operator, we obtain that for each $f \in \mathcal{E}(\mathbb{T}, X)$ the estimate

$$\sum_{n\in\mathbb{Z}} \|T^{n-N}\hat{f}(n)\|(1+|n|)^k \leq c(1+N)^{k_0} \sum_{n\in\mathbb{Z}}(1+|n|)^{k_0+k}\|\hat{f}(n)\|$$

holds with suitable constants $c > 0, k_0 \in \mathbb{N}$ only depending on T. Hence for each $N \in \mathbb{N}_0$ the map

$$\psi_N : \mathcal{E}(\mathbb{T}, X) \to \mathcal{E}(\mathbb{T}, X), \quad f \mapsto \sum_{n\in\mathbb{Z}} T^{n-N}\hat{f}(n)z^n$$

is a well defined continuous linear operator. Replacing T by T^{-1} (which is also $\mathcal{E}(\mathbb{T})$-scalar) we obtain the continuous linear operator

$$\tilde{\psi}_N : \mathcal{E}(\mathbb{T}, X) \to \mathcal{E}(\mathbb{T}, X), \quad f \mapsto \sum_{n\in\mathbb{Z}} T^{N-n}\hat{f}(n)z^n.$$

Obviously, $\tilde{\psi}_N \circ \psi_N = \psi_N \circ \tilde{\psi}_N = \mathbf{1}$, which tells us that ψ_N is a topological isomorphism for each $N \in \mathbb{N}_0$. The observation that

$$\psi_1(z-T)f = \sum_{n\in\mathbb{Z}}(T^{n-1}\hat{f}(n-1) - T^n\hat{f}(n))z^n = (z-\mathbf{1})\psi_0 f$$

shows that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{E}(\mathbb{T}, X) & \xrightarrow{z-T} & \mathcal{E}(\mathbb{T}, X) \\
\psi_0 \big\downarrow & & \big\downarrow \psi_1 \\
\mathcal{E}(\mathbb{T}, X) & \xrightarrow{z-\mathbf{1}} & \mathcal{E}(\mathbb{T}, X).
\end{array}$$

To complete the proof, it suffices to check that the lower horizontal map is a topological monomorphism. This was proved in [7] (proof of Theorem 6.4.11).

□

Note that there is a canonical way to identify $\mathcal{E}(\mathbb{T}, X)$ and $\mathcal{E}(\mathbb{T})\widehat{\otimes}X$. Namely, the continuous bilinear map $\mathcal{E}(\mathbb{T}) \times X \to \mathcal{E}(\mathbb{T}, X), \quad (f, x) \mapsto f \cdot x$ induces a uniquely determined continuous linear map

$$\mathcal{E}(\mathbb{T})\widehat{\otimes}X \longrightarrow \mathcal{E}(\mathbb{T}, X),$$

mapping $f \otimes x$ to $f \cdot x$ ($f \in \mathcal{E}(\mathbb{T}), x \in X$), which can be shown to be a topological isomorphism (see e.g. [4]).

Moreover, the reader should recall the fact that if E and F are both Fréchet spaces, and E is nuclear, then there is a canonical identification of $(E\widehat{\otimes}F)'$ and $E'\widehat{\otimes}F'$ in the category of locally convex spaces (see [7], Theorem. A1.12). Here, as in the sequel, all the dual spaces are always equipped with their strong topology.

Since the adjoint of a topological monomorphism between Fréchet spaces is a surjection, dualizing the assertion of the preceding theorem immediately yields the next result:

2.4.3 Corollary. *For each $\mathcal{E}(\mathbb{T})$-subscalar operator $T \in L(X)$ on a Banach space, the map $\mathcal{E}'(\mathbb{T})\widehat{\otimes}X' \xrightarrow{z-T'} \mathcal{E}'(\mathbb{T})\widehat{\otimes}X'$ is onto.* □

Following Eschmeier and Putinar [7], we say that a continuous linear map $\alpha : E \to F$ between locally convex spaces allows the lifting of bounded sets if each bounded set in F is the image (under α) of a bounded set in E.

2.4.4 Theorem. *Suppose that $T \in L(X)$ is $\mathcal{E}(\mathbb{T})$-scalar. Then the mapping $\mathcal{E}'(\mathbb{T})\widehat{\otimes}X \xrightarrow{z-T} \mathcal{E}'(\mathbb{T})\widehat{\otimes}X$ is onto and allows the lifting of bounded sets.*

Proof. Let $\Phi : \mathcal{E}(\mathbb{T}) \to L(X)$ denote the functional calculus of T. In the following, we write $W^k(\mathbb{T})$ for the usual Sobolev space of order $k \in \mathbb{N}_0$. The canonical representation $\mathcal{E}(\mathbb{T}) = \varprojlim W^k(\mathbb{T})$ (see [4], Appendix C, for details) allows us to choose a $k \in \mathbb{N}_0$ in such a way that Φ has a continuous linear extension $\Phi_0 : W^k(\mathbb{T}) \to L(X)$. Using the fact that $\mathcal{E}(\mathbb{T})$ is dense in $W^k(\mathbb{T})$, we deduce that $\Phi_0(zf) = T\Phi_0(f)$ for all $f \in W^k(\mathbb{T})$. Furthermore, the mapping

$$W^k(\mathbb{T}) \times X \to X, \quad (f, x) \mapsto \Phi_0(f)x$$

is obviously continuous bilinear, and hence possesses a continuous linear factorization

$$\Psi : W^k(\mathbb{T})\widehat{\otimes}_\pi X \to X \quad \text{satisfying} \quad \Psi(f \otimes x) = \Phi_0(f)x \quad (f \in W^k(\mathbb{T}), x \in X).$$

Note that a continuous linear right inverse for Ψ is given by

$$R : X \to W^k(\mathbb{T})\widehat{\otimes}X, \quad x \mapsto 1 \otimes x.$$

Using the following commutative diagram

$$\begin{array}{ccc}
\mathcal{E}'(\mathbb{T})\widehat{\otimes}W^k(\mathbb{T})\widehat{\otimes}_\pi X & \xrightarrow{(M_z'\otimes\mathbf{1}-\mathbf{1}\otimes M_z)\otimes\mathbf{1}} & \mathcal{E}'(\mathbb{T})\widehat{\otimes}W^k(\mathbb{T})\widehat{\otimes}_\pi X \\
\downarrow{\scriptstyle \mathbf{1}\otimes\Psi} & & \downarrow{\scriptstyle \mathbf{1}\otimes\Psi} \\
\mathcal{E}'(\mathbb{T})\widehat{\otimes}X & \xrightarrow{z-T} & \mathcal{E}'(\mathbb{T})\widehat{\otimes}X
\end{array}$$

and the fact that $\mathbf{1}\otimes R$ is a continuous linear right inverse for the vertical maps, we conclude that the lower horizontal map is onto and allows the lifting of bounded sets if the upper one does.

Since $\mathcal{E}'(\mathbb{T})$ is a complete nuclear (DF)-space (as the strong dual of a nuclear Fréchet space) and $W^k(\mathbb{T})$ is a reflexive Banach space, we can apply Lemma 6.4.5 from [7] which tells us that it suffices to check the surjectivity of

$$\mathcal{E}'(\mathbb{T})\widehat{\otimes}W^k(\mathbb{T}) \xrightarrow{M_z'\otimes\mathbf{1}-\mathbf{1}\otimes M_z} \mathcal{E}'(\mathbb{T})\widehat{\otimes}W^k(\mathbb{T})\cdot$$

But $M_z : W^k(\mathbb{T}) \to W^k(\mathbb{T})$ is the adjoint of the $\mathcal{E}(\mathbb{T})$-scalar operator $M_{\bar{z}} : W^k(\mathbb{T}) \to W^k(\mathbb{T})$, and the assertion follows as an application of Corollary 2.4.3.

□

The last result can be generalized to quotients of $\mathcal{E}(\mathbb{T})$-scalar operators. For this aim, we establish the following lifting property.

2.4.5 Lemma. *Let X, Y be Banach spaces and E a complete nuclear (DF)-space. If $T \in L(X,Y)$ is surjective, then the map*

$$E\widehat{\otimes}X \xrightarrow{\mathbf{1}\otimes T} E\widehat{\otimes}Y$$

allows the lifting of bounded sets.

Proof. Fix a bounded set $W \subset E\widehat{\otimes}Y$. By Corollary A1.11 in [7] we are able to choose absolutely convex, closed and bounded sets $A \subset E$, $B \subset Y$ such that W is contained in the image of the closed unit ball under the canonical map

$$E_A\widehat{\otimes}_\pi Y_B \xrightarrow{i\otimes j} E\widehat{\otimes}Y.$$

Here E_A denotes the subspace $\bigcup_{n\in\mathbb{N}} nA$ of E equipped with the Minkowski functional of A as norm (which turns E_A into a Banach space), and Y_B has to be understood in the same manner. By $E_A \xrightarrow{i} E$ and $Y_B \xrightarrow{j} Y$ we mean the corresponding inclusion maps.

As a consequence of the last observation we deduce that the right vertical map in the following commutative diagram allows the lifting of bounded sets:

$$\begin{array}{ccc} E_A\widehat{\otimes}_\pi Y_B & \xrightarrow{\mathbf{1}\otimes j} & E_A\widehat{\otimes}_\pi Y \\ {\scriptstyle i\otimes j}\downarrow & & \downarrow{\scriptstyle i\otimes\mathbf{1}} \\ E\widehat{\otimes}Y & = & E\widehat{\otimes}Y. \end{array}$$

From this we conclude that if the upper horizontal map in the following commutative diagram allows the lifting of bounded sets, then so does the lower one:

$$\begin{array}{ccc} E_A\widehat{\otimes}_\pi X & \xrightarrow{\mathbf{1}\otimes T} & E_A\widehat{\otimes}_\pi Y \\ {\scriptstyle i\otimes\mathbf{1}}\downarrow & & \downarrow{\scriptstyle i\otimes\mathbf{1}} \\ E\widehat{\otimes}X & \xrightarrow{\mathbf{1}\otimes T} & E\widehat{\otimes}Y. \end{array}$$

But the upper horizontal map is a surjection between Banach spaces and hence allows the lifting of bounded sets by the open mapping theorem. □

2.4.6 Corollary. *If $T \in L(X)$ is the quotient of an $\mathcal{E}(\mathbb{T})$-scalar operator, then $\mathcal{E}'(\mathbb{T})\widehat{\otimes}X \xrightarrow{z-T} \mathcal{E}'(\mathbb{T})\widehat{\otimes}X$ is onto and allows the lifting of bounded sets.*

Proof. By hypothesis, there is an $\mathcal{E}(\mathbb{T})$-scalar operator $\hat{T} \in L(\hat{X})$ on a Banach space $\hat{X}$ and a continuous linear surjection $q : \hat{X} \to X$ intertwining $\hat{T}$ and T. To conclude the proof, it suffices to apply Theorem 2.4.4 to the upper horizontal map and the preceding lemma to the right vertical map of the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}'(\mathbb{T})\widehat{\otimes}\hat{X} & \xrightarrow{z-\hat{T}} & \mathcal{E}'(\mathbb{T})\widehat{\otimes}\hat{X} \\ {\scriptstyle \mathbf{1}\otimes q}\downarrow & & \downarrow{\scriptstyle \mathbf{1}\otimes q} \\ \mathcal{E}'(\mathbb{T})\widehat{\otimes}X & \xrightarrow{z-T} & \mathcal{E}'(\mathbb{T})\widehat{\otimes}X. \end{array}$$

□

Let $T \in L(X)$ be a continuous linear operator on a Banach space. Let us say that T satisfies property $(\beta)_{\mathcal{E}(\mathbb{T})}$ if and only if the multiplication map

$$\mathcal{E}(\mathbb{T}, X) \xrightarrow{z-T} \mathcal{E}(\mathbb{T}, X)$$

is a topological monomorphism.

At the beginning of this section, we have shown that each $\mathcal{E}(\mathbb{T})$-subscalar operator satisfies property $(\beta)_{\mathcal{E}(\mathbb{T})}$. We have also seen that the converse is not true in general. However, it turns out to be true under some additional assumptions on T. This result, which will be formulated in Theorem 2.4.8, is based on the following observation:

2.4.7 Lemma. *Let $T \in L(X)$ be an operator satisfying the following two conditions:*

(a) $\|T^n\| \leq c(1+n)^{k_0}$ for all $n \in \mathbb{N}_0$ (with suitable constants $c > 0, k_0 \in \mathbb{N}_0$);

(b) $z - T : \mathcal{E}(\mathbb{T}, X) \longrightarrow \mathcal{E}(\mathbb{T}, X)$ *is a topological monomorphism.*

Then, given any zero-sequence in $\mathcal{E}(\mathbb{T}, X)$ *of the form* $(x_m + (z-T)f_m)_m$*, where* $(x_m)_m$ *denotes a sequence in* X *and* $(f_m)_m$ *is a sequence in* $\mathcal{E}(\mathbb{T}, X)$*, we have*

$$\hat{f}_m(0) \to 0 \quad \text{in } X.$$

Proof. In terms of Fourier series, the condition that $x_m + (z-T)f_m \to 0$ in $\mathcal{E}(\mathbb{T}, X)$ means exactly that, for all $k \in \mathbb{N}_0$,

$$\sum_{n \neq 0} \|\hat{f}_m(n-1) - T\hat{f}_m(n)\|(1+|n|)^k + \|x_m + (\hat{f}_m(-1) - T\hat{f}_m(0))\| \overset{m\to\infty}{\longrightarrow} 0.$$

In particular, for each fixed $k \in \mathbb{N}_0$, the value of the sum on the left tends to zero as $m \to \infty$.

Let us define

$$a_m(n) = \begin{cases} \hat{f}_m(n) & n \geq 0 \\ T^{-2n}\hat{f}_m(-n) & n < 0. \end{cases}$$

Using part (a) of the hypothesis, we obtain the estimate

$$\sum_{n=1}^{\infty} \|T^{2n}\hat{f}_m(n)\|(1+n)^k \leq 2^{k_0}c \sum_{n=1}^{\infty} \|\hat{f}_m(n)\|(1+n)^{k_0+k} \quad (k \in \mathbb{N}_0),$$

wich tells us that for any fixed $m \in \mathbb{N}_0$, the sum $a_m = \sum_{n\in\mathbb{Z}} a_m(n)z^n$ defines a function $a_m \in \mathcal{E}(\mathbb{T}, X)$.

To prove the lemma, it is enough to show that $(z-T)a_m$ is a zero-sequence. Indeed, using (b), this would immediately imply that a_m tends to zero as well, forcing $\hat{a}_m(0) = \hat{f}_m(0)$ to be a zero-sequence, as desired.

To ensure that $(z-T)a_m$ tends to zero, we use the following estimate

$$\begin{aligned} & \sum_{n\in\mathbb{Z}} \|a_m(n-1) - Ta_m(n)\|(1+|n|)^k \\ = & \sum_{n\geq 1} \|\hat{f}_m(n-1) - T\hat{f}_m(n)\|(1+n)^k + \|T^2\hat{f}_m(1) - T\hat{f}_m(0)\| \\ & + \sum_{n\leq -1} \|T^{-2(n-1)}\hat{f}_m(-n+1) - T^{-2n+1}\hat{f}_m(-n)\|(1+|n|)^k \\ \leq & (1+\|T\|)\sum_{n\neq 0} \|\hat{f}_m(n-1) - T\hat{f}_m(n)\|(1+|n|)^k \\ & + \sum_{n\geq 1} \|T^{2n+1}\|\|T\hat{f}_m(n+1) - \hat{f}_m(n)\|(1+n)^k \quad (k \in \mathbb{N}_0). \end{aligned}$$

As we have seen at the beginning of the proof, the value of the first sum tends to 0 as $m \to \infty$. The following calculation shows how to handle the second sum:

$$\begin{aligned} & \sum_{n\geq 2} \|\hat{f}_m(n-1) - T\hat{f}_m(n)\| \|T^{2n-1}\| n^k \\ \leq\ & c \sum_{n\geq 2} \|\hat{f}_m(n-1) - T\hat{f}_m(n)\| (2n)^{k_0} (1+n)^k \\ \leq\ & 2^{k_0} c \sum_{n\neq 0} \|\hat{f}_m(n-1) - T\hat{f}_m(n)\| (1+|n|)^{k_0+k} \quad (k \in \mathbb{N}_0). \end{aligned}$$

The fact that the value of the last series tends to zero follows again by the observations made at the beginning of the proof. □

2.4.8 Theorem. *An operator $T \in L(X)$ on a Banach space X is $\mathcal{E}(\mathbb{T})$-subscalar if and only if the following conditions hold:*

(a) $\|T^n\| \leq c(1+n)^{k_0}$ *for all* $n \in \mathbb{N}_0$ *(with suitable constants* $c > 0, k_0 \in \mathbb{N}_0$*);*

(b) $z - T : \mathcal{E}(\mathbb{T}, X) \longrightarrow \mathcal{E}(\mathbb{T}, X)$ *is a topological monomorphism;*

(c) *given a zero-sequence of the form* $(x_m + (z-T)f_m)_m$ *in* $\mathcal{E}(\mathbb{T}, X)$*, we have* $\hat{f}_m(-1) \to 0$ *in* X *as* $m \to \infty$.

Proof. "⇒": Suppose that $T \in L(X)$ is $\mathcal{E}(\mathbb{T})$-subscalar. By Theorem 2.4.2 and Lemma 2.2.5, all that remains to check is condition (c). So, take any zero sequence in $\mathcal{E}(\mathbb{T}, X)$ of the form $(x_m + (z-T)f_m)$ with $x_m \in X, f_m \in \mathcal{E}(\mathbb{T}, X)$ $(m \in \mathbb{N}_0)$. From the very definition of the quotient topology, we deduce that the sequence of equivalence classes $[x_m]$ in $\mathcal{E}(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}(\mathbb{T}, X)}$ is a zero-sequence. This implies that $x_m \to 0$ in X, since for an $\mathcal{E}(\mathbb{T})$-subscalar operator T the mapping $j : X \to \mathcal{E}(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}(\mathbb{T}, X)}$, $\quad x \mapsto [x]$ is a topological monomorphism (as was shown in Theorem 2.2.3). By condition (b) the sequence (f_m) converges to zero. Thus condition (c) holds.

"⇐": It suffices to show that the validity of (a) - (c) forces the map

$$j : X \to \mathcal{E}(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}(\mathbb{T}, X)}, \quad x \mapsto [x]$$

to be a topological monomorphism. Starting with a zero-sequence $([x_m])_m$ in the above quotient, we can choose functions $f_m \in \mathcal{E}(\mathbb{T}, X)$ $(m \in \mathbb{N}_0)$ in such a way that $(x_m + (z-T)f_m)_m$ tends to zero in $\mathcal{E}(\mathbb{T}, X)$. (Note that the quotient space under consideration is a quotient of Fréchet spaces.) This implies that $x_m + \hat{f}_m(-1) - T\hat{f}_m(0) \to 0$ $(m \to \infty)$. Hence

$$x_m = \Big(x_m + \hat{f}_m(-1) - T\hat{f}_m(0)\Big) - \hat{f}_m(-1) - T\hat{f}_m(0)$$

tends to zero by condition (c) and the preceding lemma. □

Finally, let us remark that the conditions (a) - (c) in the above theroem are not independent of each other. For instance, we can state the following result.

2.4.9 Lemma. *If the powers of an operator $T \in L(X)$ have a growth behaviour as described in condition (a) of the above theorem, then the map $\mathcal{E}(\mathbb{T}, X) \xrightarrow{z-T} \mathcal{E}(\mathbb{T}, X)$ is injective.*

Proof. Otherwise we could choose a non-zero element $f \in \mathcal{E}(\mathbb{T}, X)$ which is mapped to zero by $(z - T)$. That f is non-zero implies that there exists an integer $l \in \mathbb{Z}$ such that $\hat{f}(l) \neq 0$, while $(z - T)f = 0$ means precisely that

$$\hat{f}(n-1) - T\hat{f}(n) = 0 \quad (n \in \mathbb{Z})$$

in terms of Fourier-coefficients. By induction we obtain that

$$T^n \hat{f}(l+n) = \hat{f}(l) \quad (n \in \mathbb{N}_0).$$

Using the norm estimates for the powers of T we deduce that

$$\|\hat{f}(l)\| \leq c(1+n)^{k_0} \|\hat{f}(l+n)\| \quad (n \in \mathbb{N}_0).$$

Therefore

$$\sum_{n \geq |l|} \|\hat{f}(n+l)\|(1+n+l)^{k_0} \geq \frac{\|\hat{f}(l)\|}{c} \sum_{n \geq |l|} \frac{(1+n+l)^{k_0}}{(1+n)^{k_0}} = +\infty,$$

contradicting the fact that the series $\sum_{n \in \mathbb{Z}} \hat{f}(n)(1+|n|)^{k_0}$ must converge. □

It would be interesting to know whether there are other relationships between the three conditions of Theorem 2.4.8.

Maybe it is worth mentioning that property $(\beta)_{\mathcal{E}(\mathbb{T})}$ can be formulated as a transversality relation in the context of $\mathcal{O}(\mathbb{C})$-modules. Using the topologically free resolution

$$K_\bullet(z - w, \mathcal{O}(\mathbb{C}) \widehat{\otimes} \mathcal{E}(\mathbb{T})) \longrightarrow \mathcal{E}(\mathbb{T}) \longrightarrow 0$$

(cf. [7], p.115), one can show that $T \in L(X)$ has property $(\beta)_{\mathcal{E}(\mathbb{T})}$ if and only if

$$\mathcal{E}(\mathbb{T}) \perp_{\mathcal{O}(\mathbb{C})} X,$$

where X is equipped with the canonical $\mathcal{O}(\mathbb{C})$-module structure determined by T (see [4] for details).

3 $\mathcal{E}_W(\mathbb{T})$-subscalar operators

3.1 A characterization of $\mathcal{E}_W(\mathbb{T})$-subscalar operators

3.1.1 Definition. *(a) For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and $0 < \varepsilon < 1$, define*

$$Q_\varepsilon(\alpha, \beta) = \{re^{i\theta} : |r - 1| < \varepsilon,\ \alpha < \theta < \beta\}.$$

(b) An open set $W \subset \mathbb{C}$ will be called segmented if and only if it allows a decomposition

$$W = \dot\cup_{i=1}^n Q_\varepsilon(\alpha_i, \beta_i),$$

where $n \in \mathbb{N}$, $\alpha_i \in [0, 2\pi)$, $\alpha_i < \alpha_{i+1}$, $\alpha_i < \beta_i < \alpha_i + 2\pi$ and $0 < \varepsilon < 1$.

When we represent a segmented set W in the sequel as $\dot\cup_{i=1}^n Q_\varepsilon(\alpha_i, \beta_i)$, then this representation is always assumed to satisfy the conditions formulated in part (b) of the above definition. In particular, this means

$$0 \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_n < \beta_n \le \alpha_1 + 2\pi.$$

Furthermore, a single segment $Q_\varepsilon(\alpha, \beta)$ (as defined in part (a) above) is segmented if and only if $\alpha \in [0, 2\pi)$ and $\beta < \alpha + 2\pi$.

The following lemma, the proof of which is left to the reader, contains all the facts about segmented sets that we shall need in the sequel.

3.1.2 Lemma. *(a) For $\theta \in \mathbb{R}$, the sets $Q_{\frac{1}{n}}(\theta - \frac{1}{n}, \theta + \frac{1}{n})$ $(n \in \mathbb{N})$ form a neighborhood basis of $e^{i\theta}$.*

(b) If $S \subsetneq \mathbb{T}$ is a non-empty compact set and $U \supset S$ is an open subset of $\mathbb{C}$, then there is a segmented set $W \subset \mathbb{C}$ satisfying $S \subset W \subset U$.

(c) If $W \subset \mathbb{C}$ is segmented, then $\mathbb{D} \cap (\mathbb{C} \setminus \overline{W})$ is a star-shaped domain with center 0.

(d) Each segmented set $W \subset \mathbb{C}$ can be written as the countable union of segmented sets W_k $(k \in \mathbb{N}_0)$ such that $\overline{W_k} \subset W_{k+1}$ for all $k \in \mathbb{N}_0$. □

In what follows, by a segmented exhaustion of a segmented set W we shall always mean a representation as described in part (d) of the preceding lemma.

For any bounded open subset $U \subset \mathbb{C}$, we denote by $A(U, X)$ the natural generalization of the disc algebra, i.e. the space of all continuous functions $f : \overline{U} \to X$ such that $f|_U$ is analytic. Equipped with the supremum-norm on $\overline{U}$, $A(U, X)$ becomes a Banach space.

Given a segmented set $W = \dot\bigcup_{i=1}^{n} Q_\varepsilon(\alpha_i, \beta_i) \subset \mathbb{C}$, we define

$$\mathcal{E}(\mathbb{T}\cap W, X) = \{f : \mathbb{T}\cap W \to X \quad | \quad (t \mapsto f(e^{it})) \in \mathcal{E}(\dot\bigcup_{i=1}^{n}(\alpha_i,\beta_i), X)\},$$

equipped with the product topology induced by the canonical identification

$$\mathcal{E}(\mathbb{T}\cap W, X) \cong \prod_{i=1}^{n} \mathcal{E}((\alpha_i,\beta_i), X).$$

Hence, by definition, $\mathcal{E}(\mathbb{T}\cap W, X)$ is a Fréchet space, and $\mathcal{E}(\mathbb{T}\cap W) = \mathcal{E}(\mathbb{T}\cap W, \mathbb{C})$ is nuclear, because it is topologically isomorphic to a product of nuclear spaces.

If now $(W_k)_k$ denotes a segmented exhaustion for W, set

$$C^k_{W_k}(\mathbb{T}, X) = \{f : \mathbb{T}\cup \overline{W_k} \to X \mid f|_{\mathbb{T}} \in C^k(\mathbb{T}, X), f|_{\overline{W_k}} \in A(W_k, X)\} \quad (k \in \mathbb{N}_0),$$

which is easily seen to be a Banach space if it is equipped with the projective topology with respect to the inclusion map

$$C^k_{W_k}(\mathbb{T}, X) \longrightarrow C^k(\mathbb{T}, X) \oplus A(W_k, X), \quad f \mapsto f|_{\mathbb{T}} \oplus f|_{\overline{W_k}}.$$

Hence, a corresponding norm is given by $\|f\|_k = \|f|_{\mathbb{T}}\|_k + \|f|_{\overline{W_k}}\|_\infty$. As before, we use the notation $C^k_{W_k}(\mathbb{T})$ in the scalar-valued case.

Finally, fix any open set $U \subset \mathbb{C}$ with $U \cap \mathbb{T} \neq \emptyset$. Then the space

$$\mathcal{E}_U(\mathbb{T}, X) = \{f : \mathbb{T}\cup U \to X \quad | \quad f|_{\mathbb{T}} \in \mathcal{E}(\mathbb{T}, X), f|_U \in \mathcal{O}(U, X)\}$$

becomes a Fréchet space when equipped with the projective locally convex topology with respect to the inclusion map

$$\mathcal{E}_U(\mathbb{T}, X) \longrightarrow \mathcal{E}(\mathbb{T}, X) \oplus \mathcal{O}(U, X), \quad f \mapsto f|_{\mathbb{T}} \oplus f|_U,$$

which has obviously closed range. Moreover, note that $\mathcal{E}_U(\mathbb{T}) = \mathcal{E}_U(\mathbb{T}, \mathbb{C})$ is nuclear as a subspace of the sum $\mathcal{E}(\mathbb{T}) \oplus \mathcal{O}(U)$ of nuclear spaces. Furthermore, $\mathcal{E}_W(\mathbb{T})$ is actually a unital Fréchet algebra and the continuous bilinear map

$$\mathcal{E}_W(\mathbb{T}) \times C^k_{W_k}(\mathbb{T}, X) \to C^k_{W_k}(\mathbb{T}, X), \quad (\varphi, f) \mapsto \varphi \cdot f$$

turns $C^k_{W_k}(\mathbb{T}, X)$ into a Banach $\mathcal{E}_W(\mathbb{T})$-module. As a consequence, the mapping

$$\Phi : \mathcal{E}_W(\mathbb{T}) \to L(C^k_{W_k}(\mathbb{T}, X)), \quad \varphi \mapsto M_\varphi,$$

where

$$M_\varphi : C^k_{W_k}(\mathbb{T}, X) \to C^k_{W_k}(\mathbb{T}, X), \quad f \mapsto \varphi f$$

denotes the operator of the multiplication by φ, is a continuous $\mathcal{E}_W(\mathbb{T})$-functional calculus for the multiplication by the argument $M_z \in L(C^k_{W_k}(\mathbb{T}, X))$.

Now observe that the closed subspace $\overline{(z-T)\mathcal{E}_W(\mathbb{T},X)}^k \subset C^k_{W_k}(\mathbb{T},X)$ is invariant under $\Phi(\varphi)$ for any $\varphi \in \mathcal{E}_W(\mathbb{T})$, because M_φ and $(z-T)$ commute. (Here, $\overline{\{\cdot\}}^k$ refers to the closure with respect to the topology of $C^k_{W_k}(\mathbb{T},X)$.) Hence the quotient operator $[M_z]$ induced by M_z on the quotient space

$$C^k_{W_k}(\mathbb{T},X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T},X)}^k$$

possesses the $\mathcal{E}_W(\mathbb{T})$-functional calculus

$$\hat{\Phi} : \mathcal{E}_W(\mathbb{T}) \to L(C^k_{W_k}(\mathbb{T},X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T},X)}^k), \quad \varphi \mapsto [M_\varphi],$$

where $[M_\varphi]$ denotes the corresponding quotient of the multiplication by φ.

3.1.3 Definition. *Let $T \in L(X)$ be a bounded linear operator on a Banach space and let $U \subset \mathbb{C}$ be an open subset of the complex plane such that $\mathbb{T} \cap U \neq \emptyset$.*

(a) The operator T is said to be $\mathcal{E}_U(\mathbb{T})$-scalar if T possesses an $\mathcal{E}_U(\mathbb{T})$-functional calculus.

(b) If there is a Banach space $\hat{X}$ and a topological monomorphism $i : X \to \hat{X}$ intertwining T and an $\mathcal{E}_U(\mathbb{T})$-scalar operator $\hat{T} \in L(\hat{X})$, then T is said to be $\mathcal{E}_U(\mathbb{T})$-subscalar.

3.1.4 Theorem. *Suppose that $W \subset \mathbb{C}$ is a segmented subset. Then the unique continuous linear map*

$$\mathcal{E}_W(\mathbb{T})\widehat{\otimes}X \xrightarrow{H} \mathcal{E}_W(\mathbb{T},X)$$

with $H(f \otimes x) = f \cdot x$ for $f \in \mathcal{E}_W(\mathbb{T})$ and $x \in X$ is a topological isomorphism.

Proof. Given any Banach space X, the sequence

$$0 \longrightarrow \mathcal{E}_W(\mathbb{T},X) \xrightarrow{R_1} \mathcal{E}(\mathbb{T},X) \oplus \mathcal{O}(W,X) \xrightarrow{R_2} \mathcal{E}(\mathbb{T} \cap W, X),$$

where R_1 denotes the natural injection and R_2 is the continuous linear operator given by

$$R_2(\varphi \oplus f) = \varphi|_{\mathbb{T}\cap W} - f|_{\mathbb{T}\cap W},$$

is easily seen to be exact. In particular, the corresponding sequence

$$0 \longrightarrow \mathcal{E}_W(\mathbb{T}) \xrightarrow{L_1} \mathcal{E}(\mathbb{T}) \oplus \mathcal{O}(W) \xrightarrow{L_2} \mathcal{E}(\mathbb{T} \cap W)$$

is exact in the scalar-valued case. Using the fact that all the spaces contained in this latter sequence are nuclear, we obtain the exactness of

$$0 \longrightarrow \mathcal{E}_W(\mathbb{T})\widehat{\otimes}X \xrightarrow{L_1\otimes\mathbf{1}} (\mathcal{E}(\mathbb{T}) \oplus \mathcal{O}(W))\widehat{\otimes}X \xrightarrow{L_2\otimes\mathbf{1}} \mathcal{E}(\mathbb{T} \cap W)\widehat{\otimes}X$$

as an application of Theorem A1.6 in [7]. Consequently, the columns in the following diagram are exact:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{E}_W(\mathbb{T})\widehat{\otimes}X & \xrightarrow{H} & \mathcal{E}_W(\mathbb{T},X) \\
{\scriptstyle L_1\otimes\mathbf{1}}\downarrow & & {\scriptstyle R_1}\downarrow \\
(\mathcal{E}(\mathbb{T})\oplus\mathcal{O}(W))\widehat{\otimes}X & \xrightarrow{H_1} & \mathcal{E}(\mathbb{T},X)\oplus\mathcal{O}(W,X) \\
{\scriptstyle L_2\otimes\mathbf{1}}\downarrow & & {\scriptstyle R_2}\downarrow \\
\mathcal{E}(\mathbb{T}\cap W)\widehat{\otimes}X & \xrightarrow{H_2} & \mathcal{E}(\mathbb{T}\cap W,X).
\end{array}$$

Note that the first horizontal map H is the one considered in the assertion of the theorem, while H_1 is given by the composition of the well-known topological identifications

$$(\mathcal{E}(\mathbb{T})\oplus\mathcal{O}(W))\widehat{\otimes}X=(\mathcal{E}(\mathbb{T})\widehat{\otimes}X)\oplus(\mathcal{O}(W)\widehat{\otimes}X)=\mathcal{E}(\mathbb{T},X)\oplus\mathcal{O}(W,X),$$

(for the first one, see Theorem 15.5.3 in [8]). Finally, the map H_2 is by definition the topological isomorphism making the diagram

$$\begin{array}{ccc}
\mathcal{E}(\mathbb{T}\cap W)\widehat{\otimes}X & \xrightarrow{H_2} & \mathcal{E}(\mathbb{T}\cap W,X) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\left(\prod_{i=1}^n\mathcal{E}(\alpha_i,\beta_i)\right)\widehat{\otimes}X & \xrightarrow{\cong} & \prod_{i=1}^n\mathcal{E}((\alpha_i,\beta_i),X)
\end{array}$$

commutative, where $W=\dot{\bigcup}_{i=1}^n Q_\varepsilon(\alpha_i,\beta_i)$.

Now, it is easy to check that the diagram containing the map H is commutative. By diagram chasing one obtains that the continuous linear map H is bijective. Hence H is a topological isomorphism between Fréchet spaces. □

3.1.5 Lemma. *Let W be a segmented set with a given segmented exhaustion $(W_k)_k$. Then, given any Banach space X, the following topological identification holds:*

$$\mathcal{E}_W(\mathbb{T},X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T},X)}=\varprojlim C^k_{W_k}(\mathbb{T},X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T},X)}^k.$$

Proof. Step (1). First observe that there are canonical topological identifications

$$\mathcal{E}_W(\mathbb{T},X)=\varprojlim C^k_{W_k}(\mathbb{T},X)\quad\text{and}\quad\mathcal{E}_W(\mathbb{T},X)=\varprojlim\overline{\mathcal{E}_W(\mathbb{T},X)}^k,$$

where $\overline{\{\cdots\}}^k$ denotes the closure in $C^k_{W_k}(\mathbb{T}, X)$ relative to the $C^k_{W_k}(\mathbb{T}, X)$-topology. Note that, for trivial reasons, the projective spectrum corresponding to the representation on the right-hand side is reduced. So as an application of Lemma 2.2.1, we obtain that the left upper horizontal map in the following diagram has dense range:

$$\begin{array}{ccccc} \varprojlim \overline{\mathcal{E}_W(\mathbb{T}, X)}^k & \xrightarrow{z-T} & \varprojlim \overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)}^k & \xrightarrow{inj} & \varprojlim \overline{\mathcal{E}_W(\mathbb{T}, X)}^k \\ \imath \uparrow & & \imath_0 \uparrow & & \imath \uparrow \\ \mathcal{E}_W(\mathbb{T}, X) & \xrightarrow{z-T} & \overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)} & \xrightarrow{inj} & \mathcal{E}_W(\mathbb{T}, X). \end{array}$$

Here, $\imath$ stands for the second of the two identifications established at the beginning of the proof, and $\imath_0$ denotes the corresponding restriction. Since the diagram commutes, we therefore have shown that $\imath_0$ has dense range. But the right half of the diagram tells us that $\imath_0$ is in addition a topological monomorphism, and hence is actually a topological isomorphism.

Step (2). We apply Theorem 3.2.4 of [7] to obtain that the sequence

$$\begin{aligned} 0 \longrightarrow\ & \varprojlim \overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)}^k \longrightarrow \varprojlim C^k_{W_k}(\mathbb{T}, X) \\ \longrightarrow\ & \varprojlim C^k_{W_k}(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)}^k \longrightarrow 0 \end{aligned}$$

is exact. Using the results of Step (1), we obtain the exactness of

$$\begin{aligned} 0 \longrightarrow\ & \overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)} \longrightarrow \mathcal{E}_W(\mathbb{T}, X) \\ \longrightarrow\ & \varprojlim C^k_{W_k}(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)}^k \longrightarrow 0, \end{aligned}$$

which completes the proof. □

3.1.6 Theorem. *Given $T \in L(X)$ and $W \subset \mathbb{C}$ segmented, the following conditions are equivalent:*

(a) T is $\mathcal{E}_W(\mathbb{T})$-subscalar;

(b) the map $j : X \longrightarrow \mathcal{E}_W(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)}$, $\quad x \mapsto [x]$ is a topological monomorphism;

(c) for each $x' \in X'$ there is a distribution $u \in L(\mathcal{E}_W(\mathbb{T}), X')$ satisfying $u(1) = x'$ and $u(z\varphi) = T'u(\varphi)$ whenever $\varphi \in \mathcal{E}_W(\mathbb{T})$.

Proof. (b) ⇒ (a). Using the fact that j is a topological monomorphism as well as the inverse limit representation established in the preceding lemma, we obtain the existence of an integer $l \in \mathbb{N}_0$ such that the mapping

$$X \xrightarrow{j_l} C^l_{W_l}(\mathbb{T}, X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T}, X)}^l, \quad x \mapsto [x]$$

becomes a topological monomorphism. It is easy to check that j_l intertwines T and the quotient of the operator M_z on $C^l_{W_l}(\mathbb{T},X)/\overline{(z-T)\mathcal{E}_W(\mathbb{T},X)}^l$, which is $\mathcal{E}_W(\mathbb{T})$-scalar, as we have seen in the remarks preceding Theorem 3.1.4.

(a) $\Rightarrow$ (c). Let $\hat{T} \in L(\hat{X})$ be an $\mathcal{E}_W(\mathbb{T})$-scalar extension of T on a Banach space $\hat{X}$. Let $i : X \to \hat{X}$ be the corresponding intertwining map and $\Phi : \mathcal{E}_W(\mathbb{T}) \to L(\hat{X})$ the functional calculus for $\hat{T}$. Given any vector $x' \in X'$, the surjectivity of the adjoint i' allows us to choose an $\hat{x}' \in \hat{X}'$ with the property that $i'\hat{x}' = x'$. As can easily be verified, the distribution

$$u : \mathcal{E}_W(\mathbb{T}) \to X', \quad \varphi \mapsto i'(\Phi(\varphi))'\hat{x}'$$

satisfies all the required properties.

(c) $\Rightarrow$ (b). To see that (c) and (b) are equivalent, it is enough to make use of the fact that a mapping between Fréchet spaces is a topological monomorphism if and only if its adjoint is onto. Since the argument is essentially the same as in the proof of the implication (c) $\Rightarrow$ (b) in Theorem 2.2.3, we only indicate how the dual space of $\mathcal{E}_W(\mathbb{T},X)$ can be identified with $L(\mathcal{E}_W(\mathbb{T}),X')$.

Using the canonical identification $L(\mathcal{E}_W(\mathbb{T}),X') = (\mathcal{E}_W(\mathbb{T})\widehat{\otimes}X)'$ (see, for instance, Jarchow [8], Theorem 21.5.9) as well as the tensor product representation $\mathcal{E}_W(\mathbb{T})\widehat{\otimes}X = \mathcal{E}_W(\mathbb{T},X)$ established in Theorem 3.1.4, we obtain a topological isomorphism

$$\psi : L(\mathcal{E}_W(\mathbb{T}),X') \to \mathcal{E}_W(\mathbb{T},X)', \quad u \mapsto \psi(u),$$

where $\psi(u)$ denotes the unique continuous linear form on $\mathcal{E}_W(\mathbb{T},X)$ mapping $\varphi \cdot x$ to $\langle x, u(\varphi)\rangle$, whenever $\varphi \in \mathcal{E}_W(\mathbb{T})$, $x \in X$. □

3.2 A sufficient condition for $(\beta)_\mathcal{E}$ modulo S

Given an open set $U \subset \mathbb{C}$ and a Banach space X, let $\mathcal{E}(U,X)$ denote the space of all X-valued C^∞-functions on U carrying its natural Fréchet space topology. By a result due to Eschmeier and Putinar, an operator $T \in L(X)$ is subscalar (i.e. possesses an extension to a generalized scalar operator in the sense of Colojoară and Foiaş) if and only if the multiplication map

$$\mathcal{E}(\mathbb{C},X) \xrightarrow{z-T} \mathcal{E}(\mathbb{C},X)$$

is a topological monomorphism. In this case T is said to satisfy property $(\beta)_\mathcal{E}$.

To localize this concept, we have to consider for any open subset $U \subset \mathbb{C}$ the space

$$\mathcal{E}_U(\mathbb{C},X) = \{f \in \mathcal{E}(\mathbb{C},X) : f|_U \in \mathcal{O}(U,X)\},$$

which clearly is a Fréchet space with respect to the relative topology induced by $\mathcal{E}(\mathbb{C}, X)$.

In Becker [2] the following natural localization of the results cited above is shown to be true:

3.2.1 Theorem. *Let X be a Banach space. Given an operator $T \in L(X)$ and a compact subset $S \subset \sigma(T)$, the following conditions are equivalent:*

(a) The map $\mathcal{E}(\mathbb{C} \setminus S, X) \stackrel{z-T}{\longrightarrow} \mathcal{E}(\mathbb{C} \setminus S, X)$ is a topological monomorphism;

(b) $\mathcal{E}_U(\mathbb{C}, X) \stackrel{z-T}{\longrightarrow} \mathcal{E}_U(\mathbb{C}, X)$ is topologcal monomorphism for any choice of a bounded open set $U \supset S$;

(c) for each bounded open set $U \supset S$ there exists an extension of T possessing an $\mathcal{E}_U(\mathbb{C})$-functional calculus.

If one of the above conditions is fulfilled, then T is said to satisfy property $(\beta)_{\mathcal{E}}$ modulo S.

The next result will be used in the final chapter to show that the Cesàro operator on H^p $(1 < p < \infty)$ satisfies property $(\beta)_{\mathcal{E}}$ modulo $S = \{0\}$.

3.2.2 Theorem. *Suppose that $T \in L(X)$ has the following properties:*

(a) $\sigma(T) \subset \overline{\mathbb{D}}$

(b) there exists an analytic left resolvent for T on $\mathbb{D}$, i.e. a function

$$L \in \mathcal{O}(\mathbb{D}, L(X)) \text{ with } L(z)(z - T) = \mathbf{1} \quad (z \in \mathbb{D}),$$

and there exists a compact set $\emptyset \neq S \subsetneq \mathbb{T}$, such that the left resolvent L has the following growth-behaviour near $\mathbb{T}$: For each open set U in the plane that contains S, there are constants $c_U > 0$, $k_U \in \mathbb{N}_0$ such that

$$\|L(z)\| \leq \frac{c_U}{|1 - |z||^{k_U}} \quad (z \in \mathbb{D} \cap (\mathbb{C} \setminus U)).$$

(c) There are $c > 0, k \in \mathbb{N}_0$ such that

$$\|R(z, T)\| \leq \frac{c}{|1 - |z||^k} \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}).$$

Then, for each open set $W \subset \mathbb{C}$ containing S, and each vector $x' \in X'$, there is a distribution $u \in L(\mathcal{E}_W(\mathbb{T}), X')$ satisfying

$$u(z\varphi) = T'u(\varphi) \quad (\varphi \in \mathcal{E}_W(\mathbb{T})) \quad \text{and} \quad u(1) = x'.$$

Before proving this result, let us state the following consequences:

3.2.3 Corollary. *Under the hypotheses of Theorem 3.2.2 we have:*

(a) *For each open set $U \subset \mathbb{C}$, $U \supset S$, the operator T is $\mathcal{E}_U(\mathbb{T})$-subscalar.*

(b) *T satisfies property $(\beta)_{\mathcal{E}}$ modulo S.*

Proof.

(a) By Lemma 3.1.2 there is a segmented set W satisfying $S \subset W \subset U$. Then apply Theorem 3.1.6 and the result of the above theorem to conclude that T is $\mathcal{E}_W(\mathbb{T})$-subscalar. Finally, use the continuous restriction map $\mathcal{E}_U(\mathbb{T}) \to \mathcal{E}_W(\mathbb{T})$ to see that an $\mathcal{E}_W(\mathbb{T})$-scalar extension of T is also $\mathcal{E}_U(\mathbb{T})$-scalar.

(b) Note that, via the continuous restriction map $\mathcal{E}_U(\mathbb{C}) \to \mathcal{E}_U(\mathbb{T})$, each $\mathcal{E}_U(\mathbb{T})$-functional calculus yields an $\mathcal{E}_U(\mathbb{C})$-functinal calculus. □

During the proof of Theorem 3.2.2, we will make use of the following "integration by parts"-formulas. Since they can be established by a simple induction, we leave the proof to the reader.

3.2.4 Lemma. *Let E be a Banach space, $0 < r < 1$ and $\varphi \in \mathcal{E}(\mathbb{T})$.*

(a) *Suppose that we are given a function $f \in \mathcal{O}(\mathbb{D}, E)$, together with a sequence $(f_n)_{n\geq 0}$ of primitives in $\mathcal{O}(\mathbb{D}, E)$, i.e. $f_0 = f$ and $f'_{n+1} = f_n$ for $n \in \mathbb{N}_0$. Then, for each $n \in \mathbb{N}$, there are polynomials $p_j^n \in \mathbb{C}[z]$ $(j = 1, \cdots, n)$ (not depending on φ and r) such that the following formula holds:*

$$\int_{\partial\mathbb{D}} \varphi(\zeta) f(\zeta r) d\zeta = \int_0^{2\pi} \frac{1}{r^n} \Big(\sum_{j=1}^{n} p_j^n(e^{-it}) \frac{d^j}{dt^j} \varphi(e^{it}) \Big) f_n(re^{it}) dt.$$

(b) *Suppose that $f \in \mathcal{O}(\mathbb{C} \setminus \overline{\mathbb{D}}, E)$ vanishes at infinity, and that $(\tilde{f}_n)_{n\geq 0}$ is a sequence of primitives in $\mathcal{O}(\mathbb{D}, E)$ for the holomorphic continuation $\tilde{f} : \mathbb{D} \to E$ of the function $\mathbb{D} \setminus \{0\} \to E, z \mapsto f(\frac{1}{z})$. Then, for each $n \in \mathbb{N}_0$, one can find polynomials $q_j^n \in \mathbb{C}[z]$ $(j = 0, \cdots, n)$ (not depending on φ, r) such that*

$$\int_{\partial\mathbb{D}} \varphi(\zeta) f(\frac{\zeta}{r}) d\zeta = \int_0^{2\pi} \frac{1}{r^n} \Big(\sum_{j=0}^{n} q_j^n(e^{it}) \frac{d^j}{dt^j} \varphi(e^{it}) \Big) \tilde{f}_n(re^{-it}) dt.$$

□

3.2.5 Lemma. *Suppose that $f \in \mathcal{O}(\mathbb{D}, E)$ is an analytic function satisfying the following growth restriction:*

There is a star-shaped set $V \subset \mathbb{D}$ with center 0, and there are constants $c > 0$ and $k \in \mathbb{N}_0$ such that

$$\|f(z)\| \leq \frac{c}{|1 - |z||^k} \quad (z \in V).$$

Then there exists a sequence $(f_n)_{n \geq 0}$ of primitives for f in $\mathcal{O}(\mathbb{D}, E)$ such that $f_{k+2}|_V$ has a continuous extension to $\overline{V}$.

Proof. Define $f_0 = f$. The inductive definition

$$f_{j+1}(z) = \int_{[0,z]} f_j(\zeta) d\zeta \quad (z \in \mathbb{D}, j \in \mathbb{N}_0),$$

where $[0, z]$ denotes the line-segment joining 0 and z, yields a sequence $(f_j)_{j \geq 0}$ of primitives for f.

Inductively, one can show that

$$\|f_j(z)\| \leq \frac{c}{(1 - |z|)^{k-j}} \quad (z \in V, j = 0, \cdots, k - 1).$$

Hence, for f_k we get the estimate

$$\begin{aligned} \|f_k(z)\| &\leq \int_0^{|z|} \|f_{k-1}(t\frac{z}{|z|})\frac{z}{|z|}\|dt \\ &\leq \int_0^{|z|} \frac{c}{1-t} dt \\ &= -c \log(1 - |z|) \quad (z \in V). \end{aligned}$$

Integrating once again we see that the function f_{k+1} is bounded on V:

$$\|f_{k+1}(z)\| \leq -c \int_0^1 \log u \, du = c \quad (z \in V).$$

Hence the function $f_{k+2}|_V$, which is holomorphic on V, has a bounded derivative, and can therefore be continuously extended to $\overline{V}$. □

Now we are ready for the proof of Theorem 3.2.2.

Proof of 3.2.2. Fix an open set $W \supset S$ and an $x' \in X'$. Our aim is to construct a distribution $u \in L(\mathcal{E}_W(\mathbb{T}), X')$ satisfying $u(z\varphi) = T'u(\varphi)$ for all $\varphi \in \mathcal{E}_W(\mathbb{T})$ and $u(1) = x'$.

In view of Lemma 3.1.2 we are allowed to assume that W is segmented. Otherwise we can choose a segmented set $\tilde{W}$ satisfying $S \subset \tilde{W} \subset W$. If $\tilde{u}$ is a distribution as desired for $\tilde{W}$, then the composition

$$\mathcal{E}_W(\mathbb{T}) \xrightarrow{rest} \mathcal{E}_{\tilde{W}}(\mathbb{T}) \xrightarrow{\tilde{u}} X'$$

is a possible choice for u.

Now choose a sequence (r_m) in $(0,1)$ with the property that $r_m \uparrow 1$ as $m \to \infty$. Then try to define u via the formula

$$u(\varphi) = \lim_{m\to\infty} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \varphi(\zeta)\Big(R(\frac{\zeta}{r_m}, T') - L'(r_m\zeta)\Big) d\zeta x' \quad (\varphi \in \mathcal{E}_W(\mathbb{T})).$$

Here $L' \in \mathcal{O}(\mathbb{D}, L(X'))$ is defined by $L'(z) = (L(z))'$ for each $z \in \mathbb{D}$. Note that $(z - T')L'(z) = \mathbf{1}$ $(z \in \mathbb{D})$ and that L' satisfies the same norm estimates as L.

The proof of the fact that the above definition yields an element of $L(\mathcal{E}_W(\mathbb{T}), X')$ having the claimed properties will be divided into several steps.

(1) The existence of $\lim_{m\to\infty} \int_{\partial\mathbb{D}} \varphi(\zeta)L'(r_m\zeta)d\zeta$.
Since W is assumed to be segmented, we have a representation of the form $W = \dot\bigcup_{i=1}^n Q_\varepsilon(\alpha_i, \beta_i)$. Remember that $W \supset S$ and $S \subsetneq \mathbb{T}$ is compact. Hence we can choose a real number $\delta > 0$ such that the segmented set $W_0 = \dot\bigcup_{i=1}^n Q_{\frac{\varepsilon}{2}}(\alpha_i + \delta, \beta_i - \delta) \subset W$ still satisfies $W_0 \supset S$.

By Lemma 3.1.2 the set $V = \mathbb{D} \cap (\mathbb{C} \setminus \overline{W_0})$ is star-shaped with center 0. By hypothesis we can choose constants $c > 0$ and $k \in \mathbb{N}_0$ such that

$$\|L'(z)\| \leq \frac{c}{|1 - |z||^k} \quad (z \in V).$$

As an application of Lemma 3.2.5 we obtain a sequence of primitives $(f_n)_{n\geq 0}$ in $\mathcal{O}(\mathbb{D}, L(X'))$ for $f_0 = L'$, such that $f_{k+2} \in A(V, L(X'))$. Now Lemma 3.2.4 ensures that there are polynomials p_j^{k+2} such that the formula

$$\int_{\partial\mathbb{D}} \varphi(\zeta)L'(r_m\zeta)d\zeta = \int_0^{2\pi} \frac{1}{r_m^{k+2}} \Big(\sum_{j=1}^{k+2} p_j^{k+2}(e^{-it}) \frac{d^j}{dt^j}\varphi(e^{it})\Big) f_{k+2}(r_m e^{it})dt$$

holds. The fundamental observation about this formula is that the expression in brackets can be written in the form $g(e^{it})$ with a function $g \in \mathcal{E}_W(\mathbb{T})$ (which is in particular holomorphic on W). To see this, first note that $\mathcal{E}_W(\mathbb{T})$ is an algebra containing all functions of the type $\mathbb{T} \cup W \to \mathbb{C}, z \mapsto p(\frac{1}{z})$, where p is a polynomial. Therefore it suffices to check that for any given $\varphi \in \mathcal{E}_W(\mathbb{T})$ we can find an element $g \in \mathcal{E}_W(\mathbb{T})$ satisfying $\frac{d}{dt}\varphi(e^{it}) = g(e^{it})$ for all $t \in \mathbb{R}$.

The formula $\frac{d}{dz}\varphi(e^{iz}) = \varphi'(e^{iz})ie^{iz}$, which is valid for all $z \in \mathbb{C}$ with $e^{iz} \in W$ and $\varphi \in \mathcal{E}_W(\mathbb{T})$, shows that such a function g simply can be defined as follows

$$g : \mathbb{T} \cup W \to \mathbb{C}, \quad z \mapsto \begin{cases} \frac{d}{dt}\varphi(e^{it}) & (z = e^{it} \in \mathbb{T}) \\ \varphi'(z)iz & (z \in W). \end{cases}$$

In view of these remarks it is clear that we can choose a function $g \in \mathcal{E}_W(\mathbb{T})$ that solves the equation

$$\int_{\partial\mathbb{D}} \varphi(\zeta)L'(r_m\zeta)d\zeta = \int_0^{2\pi} \frac{1}{r_m^{k+2}} g(e^{it})f_{k+2}(r_m e^{it})dt.$$

The integral on the right-hand side can be decomposed into the sum

$$\begin{aligned} & \frac{1}{r_m^{k+2}} \sum_{\nu=1}^{n} \int_{\alpha_\nu+\frac{\delta}{2}}^{\beta_\nu-\frac{\delta}{2}} g(e^{it})f_{k+2}(r_m e^{it})dt \\ + \quad & \frac{1}{r_m^{k+2}} \sum_{\nu=1}^{n-1} \int_{\beta_\nu-\frac{\delta}{2}}^{\alpha_{\nu+1}+\frac{\delta}{2}} g(e^{it})f_{k+2}(r_m e^{it})dt \\ + \quad & \frac{1}{r_m^{k+2}} \int_{\beta_n-\frac{\delta}{2}}^{\alpha_1+\frac{\delta}{2}+2\pi} g(e^{it})f_{k+2}(r_m e^{it})dt. \end{aligned}$$

Remembering the definiton of W, W_0 and V, we can state that except for the integrals in the first sum, passing to the limit $m \to \infty$ causes no damage, since the argument of f_{k+2} varies only in V. Because of $f_{k+2} \in A(V, L(X'))$, we can pass to the limit under the integral sign.

Note that the integrals in the first sum can not be handled in this way, because the integration path is not contained in the set V. But using the holomorphy of g on W, it is possible to deform the path in an appropriate way without changing the value of the integrals.

For this aim, fix an arbitrary $\nu \in \{1, \cdots, n\}$. As the following calculation shows, each of the critical integrals can be interpreted as a contour integral:

$$\begin{aligned} \int_{\alpha_\nu+\frac{\delta}{2}}^{\beta_\nu-\frac{\delta}{2}} g(e^{it})f_{k+2}(r_m e^{it})dt &= \int_{\alpha_\nu+\frac{\delta}{2}}^{\beta_\nu-\frac{\delta}{2}} \frac{g(e^{it})}{ie^{it}} f_{k+2}(r_m e^{it})ie^{it}dt \\ &= \int_{\gamma_\nu} \frac{g}{iz}(\zeta)f_{k+2}(r_m\zeta)d\zeta, \end{aligned}$$

where the contour γ_ν is given by the parametrization

$$\gamma_\nu : [\alpha_\nu + \frac{\delta}{2}, \beta_\nu - \frac{\delta}{2}] \to \mathbb{T}, \quad t \mapsto e^{it}.$$

Note that the image of γ_ν is contained in $\mathbb{T}$, and therefore in $W \cap \frac{1}{r_m}\mathbb{D}$, where the integrand is holomorphic.

Cauchy's theorem guarantees that substituting γ_ν by the path $\omega_\nu = \omega_1^\nu \oplus \omega_2^\nu \oplus \omega_3^\nu$, defined by

$$\begin{aligned}
\omega_\nu^1 &: [0, \frac{2}{3}\varepsilon] \to \mathbb{C}, \quad t \mapsto e^{i(\alpha_\nu + \frac{\delta}{2})}(1-t) \\
\omega_\nu^2 &: [\alpha_\nu + \frac{\delta}{2}, \beta_\nu - \frac{\delta}{2}] \to \mathbb{C}, \quad t \mapsto (1 - \frac{2}{3}\varepsilon)e^{it} \\
\omega_\nu^3 &: [0, \frac{2}{3}\varepsilon] \to \mathbb{C}, \quad t \mapsto e^{i(\beta_\nu - \frac{\delta}{2})}(1 - \frac{2}{3}\varepsilon + t),
\end{aligned}$$

does not change the value of the integral. (Note that the image of ω_ν is contained in $W \setminus \overline{W_0}$.) But the integral

$$\int_{\omega_\nu} \frac{g}{iz}(\zeta) f_{k+2}(r_m\zeta)d\zeta$$

can be computed as

$$\begin{aligned}
&\int_{\alpha_\nu + \frac{\delta}{2}}^{\beta_\nu - \frac{\delta}{2}} \frac{g}{iz}\Big((1 - \frac{2}{3}\varepsilon)e^{it}\Big) f_{k+2}\Big(r_m(1 - \frac{2}{3}\varepsilon)e^{it}\Big) i(1 - \frac{2}{3}\varepsilon)e^{it}dt \\
- &\int_0^{\frac{2}{3}\varepsilon} \frac{g}{iz}\Big(e^{i(\alpha_\nu + \frac{\delta}{2})}(1-t)\Big) f_{k+2}\Big(r_m e^{i(\alpha_\nu + \frac{\delta}{2})}(1-t)\Big)dt \\
+ &\int_0^{\frac{2}{3}\varepsilon} \frac{g}{iz}\Big(e^{i(\beta_\nu - \frac{\delta}{2})}(1 - \frac{2}{3}\varepsilon + t)\Big) f_{k+2}\Big(r_m e^{i(\beta_\nu - \frac{\delta}{2})}(1 - \frac{2}{3}\varepsilon + t)\Big)dt.
\end{aligned}$$

Now looking at the argument of f_{k+2}, we can state that for $m \to \infty$ we do not leave the set $V = \mathbb{D} \cap (\mathbb{C} \setminus \overline{W_0})$. Again using the fact that $f_{k+2} \in A(V, L(X'))$ we conclude that the limit for $m \to \infty$ exists. This finishes the first step of the proof.

(2) The existence of $\lim_{m\to\infty} \int_{\partial\mathbb{D}} \varphi(\zeta)R(\frac{\zeta}{r_m}, T')d\zeta$.
Since the resolvent of a bounded operator vanishes at infinity, the definition $\tilde{R}(z) = R(\frac{1}{z}, T')$ for $z \in \mathbb{D} \setminus \{0\}$ induces a holomorphic function $\tilde{R} : \mathbb{D} \to L(X')$, which satisfies the same norm estimates as the resolvent of T'

$$\|\tilde{R}(z)\| = \|R(\frac{1}{z}, T')\| \leq \frac{c}{|1 - |\frac{1}{z}||^k} = c\frac{|z|^k}{||z| - 1|^k} \leq \frac{c}{|1 - |z||^k} \quad (z \in \mathbb{D}).$$

Again, an application of Lemma 3.2.5 yields a sequence $(f_n)_{n\geq 0}$ of primitives for $f_0 = \tilde{R}$ in $\mathcal{O}(\mathbb{D}, L(X'))$ such that $f_{k+2} \in A(\mathbb{D}, L(X'))$. By Lemma 3.2.4 there are polynomials q_j^{k+2} such that

$$\int_{\partial\mathbb{D}} \varphi(\zeta)R(\frac{\zeta}{r_m}, T')d\zeta = \int_0^{2\pi} \frac{1}{r_m^{k+2}}\Big(\sum_{j=0}^{k+2} q_j^{k+2}(e^{it})\frac{d^j}{dt^j}\varphi(e^{it})\Big) f_{k+2}(r_m e^{-it})dt.$$

Because of the continuity of f_{k+2} on the whole of $\overline{\mathbb{D}}$, we are allowed to pass to the limit $m \to \infty$ under the integral sign.

(3) We check that u satisfies the required algebraic conditions.
Using the fact that the limits considered in Step (1) and (2) exist, we easily deduce that

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\partial\mathbb{D}} \zeta\varphi(\zeta)\Big(R(\frac{\zeta}{r_m}, T') - L'(r_m\zeta)\Big)d\zeta \\
= &\lim_{m\to\infty} \int_{\partial\mathbb{D}} \varphi(\zeta)\Big(\frac{\zeta}{r_m}R(\frac{\zeta}{r_m}, T') - \zeta r_m L'(\zeta r_m)\Big)d\zeta \\
= &\lim_{m\to\infty} T' \int_{\partial\mathbb{D}} \varphi(\zeta)\Big(R(\frac{\zeta}{r_m}, T') - L'(r_m\zeta)\Big)d\zeta.
\end{aligned}$$

Since all the limits even exist in the norm topology of $L(X')$, the above equality still holds after evaluating at the given vector $x' \in X'$. Thus we have shown that $u(z\varphi) = T'u(\varphi)$ for all $\varphi \in \mathcal{E}_W(\mathbb{T})$.

To determine the value of u at the constant function $1 \in \mathcal{E}_W(\mathbb{T})$, we use the fact that $L'(r_m\cdot)$ is holomorphic on an open neighborhood of $\overline{\mathbb{D}}$ and apply the holomorphic functional calculus of T':

$$\begin{aligned}
u(1) &= \lim_{m\to\infty} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \Big(R(\frac{\zeta}{r_m}, T') - L'(r_m\zeta)\Big)d\zeta x' \\
&= \lim_{m\to\infty} \frac{1}{2\pi i} r_m \int_{\partial D_{\frac{1}{r_m}}(0)} R(\zeta, T')d\zeta x' \\
&= x'.
\end{aligned}$$

(4) It remains to show that u is an element of $L(\mathcal{E}_W(\mathbb{T}), X')$.

For any $m \in \mathbb{N}$, we define

$$u_m(\varphi) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \varphi(\zeta)\Big(R(\frac{\zeta}{r_m}, T') - L'(r_m\zeta)\Big)d\zeta x' \quad (\varphi \in \mathcal{E}_W(\mathbb{T})).$$

Obviously, for each fixed $m \in \mathbb{N}$, this definition gives a well defined linear operator $u_m : \mathcal{E}_W(\mathbb{T}) \to X'$, which is easily seen to be continuous by applying the standard estimate for path integrals. Since we have shown in (1) and (2) that the pointwise limit u of the sequence (u_m) exists, the Banach-Steinhaus theorem shows that u is continuous, and hence is an element of $L(\mathcal{E}_W(\mathbb{T}), X')$. □

4 Examples

4.1 Commuting tuples of shift operators on Hilbert spaces

Let H be a separable, infinite-dimensional Hilbert space. For a fixed integer $n \in \mathbb{N}$, an orthonormal basis of H may be given in the form $(h_\nu)_{\nu \in \mathbb{N}_0^n}$. Suppose that, for each $i \in \{1, \cdots, n\}$, a bounded sequence $(\alpha_\nu^{(i)})_{\nu \in \mathbb{N}_0^n}$ in $\mathbb{C} \setminus \{0\}$ is given. Then we denote by S_i the unique continuous linear operator on H satisfying

$$S_i h_\nu = \alpha_\nu^{(i)} h_{\nu+e_i} \quad (\nu \in \mathbb{N}_0^n),$$

where e_i stands for the canonical i-th unit vector in $\mathbb{R}^n$.

For the rest of this section, we suppose that the S_i's defined in this way commute, i.e. the wight sequences $(\alpha_\nu^{(i)})_{\nu \in \mathbb{N}_0^n}$ satisfy the condition

$$\alpha_\nu^{(j)} \alpha_{\nu+e_j}^{(i)} = \alpha_\nu^{(i)} \alpha_{\nu+e_i}^{(j)} \quad (1 \leq i, j \leq n, \nu \in \mathbb{N}_0^n).$$

In this case we call $S = (S_1, \cdots, S_n) \in L(H)^n$ a commuting n-tuple of weighted shifts.

Observe that the product $A = S_1 \cdots S_n \in L(H)$ satisfies the relation

$$A^m \mathbb{C} h_\nu \; = \mathbb{C} h_{\nu + m \cdot e} \quad (m \in \mathbb{N}_0, \nu \in \mathbb{N}_0^n),$$

where $e = (1, \cdots, 1)$. In particular, the case $m = 1$ shows that $\langle A^* A h_\nu, h_\mu \rangle = 0$ whenever $\nu, \mu \in \mathbb{N}_0^n$ are different, and hence

$$A^* A h_\nu \in \mathbb{C} h_\nu \quad (\nu \in \mathbb{N}_0^n).$$

Now it is an obvious fact that

$$A^* A^{m+1} H = (A^* A) A^m H \subset A^m H \quad (m \in \mathbb{N}_0),$$

and as an application of Corollary 2.2.12, we obtain the next result.

4.1.1 Theorem. *A commuting n-tuple of weighted Hilbert-space shift operators $S \in L(H)^n$ as defined above is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if each component of S is an $\mathcal{E}(\mathbb{T})$-subscalar operator on H.* □

Now fix an integer $i \in \{1, \cdots, n\}$. By the same arguments as above we obtain that

$$S_i^* S_i^{m+1} H \subset S_i^m H \quad (m \in \mathbb{N}_0).$$

Therefore, an application of Theorem 2.2.11 shows that, for each $i \in \{1, \cdots, n\}$, the operator S_i is $\mathcal{E}(\mathbb{T})$-subscalar if and only if there are constants $b > 0$, $k \in \mathbb{N}_0$ such that

$$\frac{\|x\|}{b(1+m)^k} \le \|S_i^m x\| \le b(1+m)^k \|x\| \quad (m \in \mathbb{N}_0, x \in H).$$

4.1.2 Theorem. *A commuting n-tuple of weighted shift operators $S \in L(H)^n$ as defined above is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if there are constants $c > 0$ and $\kappa \in \mathbb{N}_0$ such that*

$$\frac{\|x\|}{c(1+|\alpha|)^\kappa} \le \|S^\alpha x\| \le c(1+|\alpha|)^\kappa \|x\| \quad (\alpha \in \mathbb{N}_0^n, x \in H).$$

Proof. Applying Lemma 2.2.5 to the components of S shows that the above growth condition is necessarily satisfied by any $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuple S on a Banach space. Indeed, if $b > 0$ and $k \in \mathbb{N}_0$ are constants such that the estimates

$$\frac{\|x\|}{b(1+m)^k} \le \|S_i^m x\| \le b(1+m)^k \|x\| \quad (m \in \mathbb{N}, x \in H, i = 1, \cdots, n)$$

hold, we deduce that

$$\frac{\|x\|}{b^n(1+|\alpha|)^{nk}} \le \|S^\alpha x\| \le b^n (1+|\alpha|)^{nk} \|x\| \quad (x \in H, \alpha \in \mathbb{N}_0^n),$$

as desired.

Conversely, suppose that a commuting n-tuple of shift operators $S \in L(H)^n$ satisfies the above growth condition. By the observation preceding the theorem we know that each component of S is $\mathcal{E}(\mathbb{T})$-subscalar. Hence an application of Theorem 4.1.1 finishes the proof. □

4.1.3 Corollary. *A commuting system of weighted shifts $S \in L(H)^n$ as defined above is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if there are constants $c > 0, \kappa \in \mathbb{N}_0$ such that the weight sequences $(\alpha_\nu^{(i)})_{\nu \in \mathbb{N}_0^n}$ $(i = 1, \cdots, n)$ satisfy the estimates*

$$\begin{aligned} \inf_{\nu \in \mathbb{N}_0^n} |\alpha_\nu^{(i)} \cdots \alpha_{\nu+(m-1)e_i}^{(i)}| &\ge \frac{1}{c(1+m)^\kappa} \\ \sup_{\nu \in \mathbb{N}_0^n} |\alpha_\nu^{(i)} \cdots \alpha_{\nu+(m-1)e_i}^{(i)}| &\le c(1+m)^\kappa \end{aligned}$$

for all $m \in \mathbb{N}_0$, $i \in \{1, \cdots, n\}$.

Proof. If $S \in L(H)^n$ is an n-tuple of commuting shifts, fix constants $c > 0, \kappa \in \mathbb{N}_0$ as in the statement of the preceding theorem. Because of

$$S_i^m h_\nu = \alpha_\nu^{(i)} \cdots \alpha_{\nu+(m-1)e_i}^{(i)} h_{\nu+me_i} \quad (m \in \mathbb{N}_0, \nu \in \mathbb{N}_0^n, i = 1, \cdots, n),$$

where $(h_\nu)_{\nu \in \mathbb{N}_0^n}$ denotes the orthonormal basis of H fixed at the beginning of this section, we deduce that

$$\frac{1}{c(1+m)^\kappa} \leq |\alpha_\nu^{(i)} \cdots \alpha_{\nu+(m-1)e_i}^{(i)}| \leq c(1+m)^\kappa \qquad (m \in \mathbb{N}_0, \nu \in \mathbb{N}_0^n, i = 1, \cdots, n),$$

as desired.

To see the reverse implication note that, for an arbitrary vector $x = \sum_{\nu \in \mathbb{N}_0^n} \lambda_\nu h_\nu \in H$, we have

$$\|S_i^m x\|^2 = \sum_{\nu \in \mathbb{N}_0^n} |\lambda_\nu|^2 |\alpha_\nu^{(i)} \cdots \alpha_{\nu+(m-1)e_i}^{(i)}|^2 \quad (m \in \mathbb{N}_0, i = 1, \cdots, n).$$

Supposed that the growth conditions from the statement of the theorem are valid, it follows that

$$\frac{\|x\|^2}{c^2(1+m)^{2\kappa}} \leq \|S_i^m x\|^2 \leq c^2(1+m)^{2\kappa}\|x\|^2 \quad (m \in \mathbb{N}_0, i = 1, \cdots, n),$$

for all $x \in H$. Hence an application of the preceding theorem finishes the proof.

□

4.2 The Cesàro operator on the Hardy space

4.2.1 H^p-spaces

Recall that, for $1 \leq p < \infty$, the Hardy space H^p over the unit disc is defined to be the vector space of all holomorphic functions $f \in \mathcal{O}(\mathbb{D})$ for which the expression

$$\|f\|_{H^p} = \Big(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \Big)^{\frac{1}{p}}$$

is finite, and that $\| \cdot \|_{H^p}$ defines a norm on H^p turning this space into a Banach space. Furthermore, it is is well known that, for a function $f \in H^p$, the non-tangential boundary values $\lim_{r \uparrow 1} f(re^{it})$ exist for a.e. $t \in [0, 2\pi]$ and form an L^p function on $\mathbb{T}$ (with respect to the Lesbesgue measure) which will be denoted by f^*.

Now assume that $1 < p < \infty$, and let p' denote the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have a natural sesquilinear dual pairing between H^p and $H^{p'}$ given by

$$\langle f, g\rangle = \int_{\mathbb{T}} f^* \overline{g^*} d\lambda \quad (f \in H^p, g \in H^{p'}),$$

where λ denotes the normalized Lebesgue measure on $\mathbb{T}$.

It turns out that the map $H^{p'} \to (H^p)'$, $\quad g \mapsto \langle \cdot, g\rangle$ is an anti-linear topological isomorphism, but not an isometry.

The adjoint of $T \in L(H^p)$ with respect to the above duality is an operator in $L(H^{p'})$, which will be denoted by T^*. Note that the mapping

$$L(H^p) \to L(H^{p'}), \quad T \mapsto T^*$$

is conjugate-linear and continuous. Moreover, there is a constant $c_p^* > 1$ such that $\|T'\| \leq c_p^* \|T^*\|$ and $\|T^*\| \leq c_p^* \|T'\|$, where T' is the usual adjoint of T on $(H^p)'$.

4.2.2 The Cesàro operator
For an H^p-function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ we define

$$(C_p f)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \Big(\sum_{k=0}^{n} a_k \Big) z^n.$$

The proof of the fact that $C_p : H^p \to H^p$ is a bounded linear operator for $1 \leq p < \infty$ is due to Siskakis (see [12]).

Following [10], we will work with the operator $T_p = 1 - C_p \in L(H^p)$ instead of C_p. The next theorem is the key result about T_p.

4.2.3 Theorem. (A. Siskakis, T.L. Miller, V.G. Miller, R.C. Smith)
Let $1 < p < \infty$. Then the following assertions hold:

(a) $\sigma(T_p) = \overline{D}_{\frac{p}{2}}(1 - \frac{p}{2})$;

(b) $\exists k > 0 : \quad \|(\lambda - T_p)^{-1}\| \leq \frac{k}{dist(\lambda, \partial\sigma(T_p))} \quad (\lambda \in \rho(T_p))$;

(c) *there is a function $L : Int\ \sigma(T_p) \to L(H^p)$ satisfying*

$$L(\lambda)(\lambda - T_p) = \mathbf{1} \quad (\lambda \in Int\ \sigma(T_p) = D_{\frac{p}{2}}(1 - \frac{p}{2})),$$

and there are constants $a, b > 0$ such that for $\lambda \in Int\ \sigma(T_p)$ we have:

$$\|L(\lambda)\| \leq \frac{a \exp(b|1-\lambda|^{-2})}{dist(\lambda, \partial\sigma(T_p))}.$$

For a proof of these facts containing a complete description of how to construct such a left resolvent L see [10]. Analyzing this construction in detail, we will show that L is in fact holomorphic. To realize this claim, we have to gather some facts implicitly contained in [10].

For $z \in \overline{\mathbb{D}}$ and $t \geq 0$, define:

$$\begin{aligned} k_t(z) &= (e^{-t} - 1)z + 1 \\ \psi_t(z) &= e^{-t} z (k_t(z))^{-1}. \end{aligned}$$

Remark that $k_t(\overline{\mathbb{D}}) = \overline{D}_{(1-e^{-t})}(1) \subset \{z \in \mathbb{C} : \text{Re } z > 0\}$, so the definition of ψ_t makes sense. Furthermore, the powers $(e^t k_t(z))^\alpha = \exp(\alpha \log e^t k_t(z))$ are well defined whenever $\alpha \in \mathbb{C}, z \in \overline{\mathbb{D}}$. Since the mapping $\mathbb{D} \ni z \mapsto (e^t k_t(z))^\alpha \in \mathbb{C}$ belongs to $A(\mathbb{D})$, the corresponding multiplication operator on H^p,

$$M_{(e^t k_t)^\alpha} : H^p \longrightarrow H^p,$$

is bounded. By Littlewood's subordination theorem (see [6]), the composition operator

$$Q_t : H^p \longrightarrow H^p, \quad f \mapsto f \circ \psi_t$$

is a contraction on H^p whenever $t \geq 0$. The following definition is due to Siskakis. Set

$$\Gamma_{t,\alpha} = M_{(e^t k_t)^{-\alpha}} \circ Q_t \quad (\alpha \in \mathbb{C})$$

to obtain for each fixed $\alpha \in \mathbb{C}$ a strongly continuous semigroup $(\Gamma_{t,\alpha})_{t \geq 0}$ of operators on H^p $(1 \leq p < \infty)$, see Lemma 2.4 (i) in [10]. In the sequel we shall write $\Lambda_{p,\alpha}$ for the infinitesimal generator of $(\Gamma_{t,\alpha})_{t \geq 0}$ and $\omega_{p,\alpha}$ for the type of the corresponding semigroup.

It is a well-known fact that, for any $z \in \mathbb{C}$ satisfying $\text{Re } z > \omega_{p,\alpha}$, the formula

$$R(z, \Lambda_{p,\alpha})x = \int_0^\infty e^{-zt} \Gamma_{t,\alpha}(x) dt \quad (x \in H^p)$$

defines a bounded linear inverse for the operator $(z - \Lambda_{p,\alpha})$ (see [5], Theorem VIII.1.11).

For $\lambda \in \text{Int } \sigma(T_p)$, it follows that (see [10])

$$\text{Re } \frac{\lambda}{1-\lambda} > \omega_{p', 1 - \frac{\lambda}{1-\lambda}},$$

where p' is the conjugate exponent of p. This implies that

$$R\Big(\frac{\lambda}{1-\lambda}, \Lambda_{p', 1-\frac{\lambda}{1-\lambda}}\Big)$$

is a bounded linear operator on $H^{p'}$, and hence the same is true of

$$U_\lambda = \frac{1}{\lambda - 1}\Big(\mathbf{1} - \frac{\lambda}{1-\lambda} S \circ R\Big(\frac{\lambda}{1-\lambda}, \Lambda_{p',1-\frac{\lambda}{1-\lambda}}\Big)\Big),$$

where $S : H^{p'} \to H^{p'}$ denotes the multiplication by the argument. Then the explicit formula for the left resolvent L from [10] is the following

$$L : \text{Int}\, \sigma(T_p) \to L(H^p), \quad \lambda \mapsto (U_{\overline{\lambda}})^*(\mathbf{1} - A_p S^*),$$

where $A_p = (C_{p'})^*$.

4.2.4 Theorem. *The left resolvent L : Int $\sigma(T_p) \to L(H^p)$ defined above is analytic.*

Proof. For abbreviation, set $D = \text{Int}\, \sigma(T_p) = D_{\frac{p}{2}}(1 - \frac{p}{2})$. The proof is divided into several steps:

(1) Since $L(\lambda) = (U_{\overline{\lambda}})^* \circ (\mathbf{1} - A_p S^*)$, it suffices to check that $(U_{\overline{\lambda}})^*$ depends analytically on $\lambda \in D$, or equivalently, that

$$D \to L(H^{p'}), \quad \lambda \mapsto U_\lambda$$

is analytic. Because of $1 \notin D = D_{\frac{p}{2}}(1 - \frac{p}{2})$, it suffices to show the analyticity of the map

$$D \to L(H^{p'}), \quad \lambda \mapsto R\Big(\frac{\lambda}{1-\lambda}, \Lambda_{p',1-\frac{\lambda}{1-\lambda}}\Big).$$

Note that $\eta : D \to \mathbb{C}, \quad \lambda \mapsto \frac{\lambda}{1-\lambda}$ is holomorphic, and its range $W = \eta(D)$ satisfies

$$W \subset \{z \in \mathbb{C} : \text{Re}\, z > \omega_{p',1-z}\},$$

see [10]. Thus it suffices to show that the family of bounded linear operators $R(\alpha, \Lambda_{p',1-\alpha})$ on $H^{p'}$ depends analytically on $\alpha \in W$. But this is equivalent to the assertion that, for each fixed vector $x \in H^{p'}$, the mapping

$$W \to H^{p'}, \alpha \mapsto R(\alpha, \Lambda_{p',1-\alpha})x = \int_0^\infty e^{-\alpha t}\Gamma_{t,1-\alpha}(x)dt$$

is analytic. So our aim is to show that the last integral depends analytically on the parameter α in W.

(2) Note that, by definition, $\Gamma_{t,\alpha} = M_{(e^t k_t)^{-\alpha}} \circ Q_t$ for any $\alpha \in \mathbb{C}$. Hence, to show that the integrand under consideration is holomorphic, it suffices to ckeck that

$$\mathbb{C} \to L(H^{p'}), \quad \alpha \mapsto M_{(e^t k_t)^{-\alpha}}$$

is analytic. To ensure this, consider the family of functions

$$f_t : \overline{\mathbb{D}} \to \mathbb{C}, \quad z \mapsto \log\,(e^t k_t(z)) \quad (t \in [0,\infty))$$

belonging to $A(\mathbb{D})$. Then

$$G_t : \mathbb{C} \to A(\mathbb{D}), \quad \alpha \mapsto \exp(\alpha f_t) = \sum_{n=0}^{\infty} \frac{(f_t)^n}{n!}\alpha^n,$$

is an $A(\mathbb{D})$-valued holomorphic function for each fixed $t \in [0,\infty)$. Hence the desired holomorphy of $M_{(e^t k_t)^{-\alpha}} = M_{G_t(\alpha)}$ follows from the fact that the assignment $A(\mathbb{D}) \to L(H^{p'}), \quad f \mapsto M_f$ is a continuous linear operator.

(3) To apply a standard result about integrals depending holomorphically on a parameter, it suffices to majorize the norm of the integrand locally uniformly in α by an integrable function of the variable $t \in [0,\infty)$.

Starting with the estimate $e^{-t} \leq |k_t(z)| \leq 2 - e^{-t} \leq 2$, where $z \in \overline{\mathbb{D}}, t \in [0,\infty)$, we obtain that $1 \leq |e^t k_t(z)| \leq 2e^t$, and therefore

$$|\log\,(e^t k_t(z))| \leq \log|e^t k_t(z)| + \frac{\pi}{2} \leq \log 2e^t + 2 \leq t + 3 \quad (t \in [0,\infty), z \in \overline{\mathbb{D}}).$$

It follows that $\|f_t\|_\infty \leq t + 3$, implying

$$\|G_t(\alpha)\|_\infty \leq \exp(|\alpha|\|f_t\|_\infty) \leq e^{|\alpha|(t+3)} \quad (t \in [0,\infty), \alpha \in \mathbb{C}).$$

From this we deduce that, for any choice of $\alpha, \alpha_0 \in W$, $t \in [0,\infty)$, the estimate

$$\|M_{(e^t k_t)^{(\alpha-\alpha_0)}}\| \leq \|G_t(\alpha - \alpha_0)\|_\infty \leq e^{3|\alpha-\alpha_0|} e^{|\alpha-\alpha_0|t}.$$

holds. A simple calulation shows that

$$e^{-\alpha t}\Gamma_{t,1-\alpha} = e^{-(\alpha-\alpha_0)t} M_{(e^t k_t)^{(\alpha-\alpha_0)}} e^{-\alpha_0 t}\Gamma_{t,1-\alpha_0},$$

and we finally obtain

$$\begin{aligned} \|e^{-\alpha t}\Gamma_{t,1-\alpha}\| &\leq e^{|\alpha_0-\alpha|t} \quad e^{3|\alpha-\alpha_0|} e^{|\alpha-\alpha_0|t} \quad \exp(-\mathrm{Re}\ \alpha_0 t) \quad \|\Gamma_{t,1-\alpha_0}\| \\ &\leq e^{3|\alpha-\alpha_0|} \quad \exp\left(t(2|\alpha-\alpha_0| - \mathrm{Re}\ \alpha_0)\right) \quad \|\Gamma_{t,1-\alpha_0}\|. \end{aligned}$$

Because of $\alpha_0 \in W$ we can choose real numbers $\tau \in (\omega_{p',1-\alpha_0}, \mathrm{Re}\ \alpha_0)$ and $\varepsilon > 0$ in such a way that

$$\omega_{p',1-\alpha_0} < \tau + 2\varepsilon < \mathrm{Re}\ \alpha_0.$$

Furthermore, $\varepsilon > 0$ can be assumed to be so small that $D_\varepsilon(\alpha_0) \subset W$. Now by the general theory of operator semigroups (see [5], Corollary VIII.1.5), we know that our choice of τ guarantees the existence of a constant $M_\tau > 0$ satisfying

$$\|\Gamma_{t,1-\alpha_0}\| \leq M_\tau e^{\tau t} \quad (t \geq 0).$$

Combining this with the estimates from above, we obtain

$$\|e^{-\alpha t}\Gamma_{t,1-\alpha}\| \leq e^{3\varepsilon} M_\tau \exp\left(t(2\varepsilon - \text{Re }\alpha_0)\right)e^{\tau t},$$

whenever $\alpha \in D_\varepsilon(\alpha_0)$. Now observe that the expression at the right-hand side is clearly integrable in $t \in [0, \infty)$ because of $\tau + 2\varepsilon - \text{Re }\alpha_0 < 0$. Therefore the proof is complete. □

The next step is to reformulate all the facts about T_p in such a way that Theorem 3.2.2 can be applied. The proof of the following lemma is straight-forward; therefore, it will be omitted.

4.2.5 Lemma. *For $1 < p < \infty$, the operator $S_p = \frac{2}{p}(T_p - (1 - \frac{p}{2})) \in L(H^p)$ satisfies the following conditions:*

(a) $S_p = 1 - \frac{2}{p}C_p$;

(b) $\sigma(S_p) = \overline{\mathbb{D}}$;

(c) $\exists k > 0 : \quad \|R(\lambda, S_p)\| \leq \frac{k}{|1-|\lambda||} \quad (\lambda \notin \overline{\mathbb{D}})$;

(d) the function $L_p : \mathbb{D} \to L(H^p), \quad \lambda \mapsto \frac{p}{2}L\left(\frac{p}{2}\lambda + 1 - \frac{p}{2}\right)$ *is analytic and satisfies the relation* $L_p(\lambda)(\lambda - S_p) = \mathbf{1}$ $(\lambda \in \mathbb{D})$. *Furthermore, there are constants* $\tilde{a}, \tilde{b} > 0$, *such that*

$$\|L_p(\lambda)\| \leq \frac{\tilde{a}\exp(\tilde{b}|1-\lambda|^{-2})}{|1-|\lambda||} \quad (\lambda \in \mathbb{D}).$$

□

As a consequence, we immediatly obtain:

4.2.6 Theorem. *Let $1 < p < \infty$. Then the operator $S_p \in L(H^p)$ has property $(\beta)_{\mathcal{E}}$ modulo $\{1\}$.*

Proof. In view of the above lemma, this follows as an application of Corollary 3.2.3, taking $S = \{1\}$. To check the growth condition required for an application of Theorem 3.2.2, assume that an open set $U \supset S$ is given. Then fix a $\delta > 0$ such that $D_\delta(1) \subset U$ and note that, for all $\delta \in \mathbb{D} \cap (\mathbb{C} \setminus D_\delta(1))$, we have $|1 - \lambda| \geq \delta$. Consequently

$$\|L_p(\lambda)\| \leq \frac{\tilde{a}\exp(\tilde{b}\delta^{-2})}{|1-|\lambda||} \quad (\lambda \in \mathbb{D} \cap (\mathbb{C} \setminus D_\delta(1))),$$

and this estimate remains true for $\lambda \in \mathbb{D} \cap (\mathbb{C} \setminus U)$. □

Suppose that $T \in L(X)$ is an operator on a Banach space X, $S \subset \mathbb{C}$ is a compact subset and $a, b \in \mathbb{C}$ are given numbers. Making use of the topological isomorphism

$$\mathcal{E}(\mathbb{C} \setminus S, X) \longrightarrow \mathcal{E}(\mathbb{C} \setminus (aS+b), X), \quad f \mapsto \hat{f},$$

where $\hat{f}$ is given by $\hat{f}(z) = f(\frac{z-b}{a})$ for $z \in \mathbb{C} \setminus (aS+b)$, one easily deduces that T satisfies property $(\beta)_\mathcal{E}$ modulo S if and only if $aT + b$ has property $(\beta)_\mathcal{E}$ modulo $aS + b$. So, because of the identity $C_p = -\frac{p}{2}S_p + \frac{p}{2}$, we can state:

4.2.7 Corollary. *For $1 < p < \infty$, the Cesàro operator C_p on H^p has property $(\beta)_\mathcal{E}$ modulo $\{0\}$.* □

Since the implications

$$(\beta)_\mathcal{E} \text{ modulo } \{0\} \quad \Rightarrow \quad (\beta) \text{ modulo } \{0\} \quad \Rightarrow \quad (\beta)$$

hold, we have implicitly proven one of the main results from [10], namely that, for $1 < p < \infty$, C_p is subdecomposable. Furthermore, Corollary 4.2.7 may be helpful to decide the following question arised in [10]:

4.2.8 Problem. Let $1 < p < \infty$. Does the Cesàro operator C_p on the Hardy space H^p have property $(\beta)_\mathcal{E}$?

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