Schatten-class Perturbations of Toeplitz Operators

Dominik Schillo

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Definition

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with the canonical probability measure m. The *Hardy space with respect to m* will be denoted by

$$H^2(m) = \left\{ f \in L^2(m) \; ; \; \hat{f}(n) = 0 \text{ for all } n < 0 \right\} \subset L^2(m).$$

Let $f \in L^\infty(m)$. We call

$$T_f : H^2(m) \to H^2(m), \; g \mapsto P_{H^2(m)}(fg),$$

the *Toeplitz operator with symbol f*.

Theorem (Brown-Halmos, 1964)

For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exists $f \in L^\infty(m)$ such that $X = T_f$,*
2. *$T_z^* X T_z - X = 0$, where $z \in L^\infty(m)$ is the identity map.*

Definition

We define

$$H^\infty(m) = L^\infty(m) \cap H^2(m) \subset L^\infty(m)$$

and we call

$$I_m = \{f \in H^\infty(m) \; ; \; |f| = 1 \; m\text{-a.e.}\}$$

the set of *inner functions with respect to m*.

Theorem (Brown-Halmos, 1964)

For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exists $f \in L^\infty(m)$ such that $X = T_f$,*
2. *$T_u^* X T_u - X \in \{0\}$ for all $u \in I_m$.*

Theorem (Gu, 2004)

For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exist $f \in L^\infty(m)$ and $F \in \mathcal{F}(H^2(m))$ such that $X = T_f + F$,*
2. *$T_u^* X T_u - X \in \mathcal{F}(H^2(m))$ for all $u \in I_m$.*

Theorem (Xia, 2009)

For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exist $f \in L^\infty(m)$ and $K \in \mathcal{K}(H^2(m))$ such that $X = T_f + K$,*
2. *$T_u^* X T_u - X \in \mathcal{K}(H^2(m))$ for all $u \in I_m$.*

Definition

Let $p \in [1, \infty)$, and let H be a Hilbert space. An operator $S \in \mathcal{B}(H)$ is a *Schatten-p-class operator* if

$$\|S\|_p^p = \operatorname{tr}(|S|^p) < \infty.$$

Denote by

$$\mathcal{S}_p(H) = \left\{ S \in \mathcal{B}(H) \; ; \; \|S\|_p < \infty \right\}$$

the set of all Schatten-p-class operators.

Remark

We have

$$\mathcal{F}(H) \subset \mathcal{S}_p(H) \subset \mathcal{S}_q(H) \subset \mathcal{K}(H) \subset \mathcal{B}(H)$$

for all $1 \leq p \leq q < \infty$.

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with the canonical probability measure $m \in M_1^+(\mathbb{T})$.

Theorem (Didas-Eschmeier-S., 2017)

Let $p \in [1, \infty)$. For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exist $f \in L^\infty(m)$ and $S \in \mathcal{S}_p(H^2(m))$ such that $X = T_f + S$,*
2. *$T_u^* X T_u - X \in \mathcal{S}_p(H^2(m))$ for all $u \in I_m$.*

Definition

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc, and let $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on $\mathbb{D}$. We denote by

1. $A(\mathbb{D}) = \left\{ f \in C(\overline{\mathbb{D}}) \; ; \; f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}) \right\}$ the *disc algebra*,
2. $\partial_{A(\mathbb{D})}$ the *Shilov boundary* of $A(\mathbb{D})$ (i.e., the smallest closed subset of $\overline{\mathbb{D}}$ such that

$$\sup_{z \in \overline{\mathbb{D}}} |f(z)| = \sup_{z \in \partial_{A(\mathbb{D})}} |f(z)|$$

for all $f \in A(\mathbb{D})$).

Proposition

We have

$$\partial_{A(\mathbb{D})} = \partial \mathbb{D} = \mathbb{T}.$$

Definition

Let $D \subset \mathbb{C}^d$ be a bounded domain, and let $\mathcal{O}(D)$ be the set of all scalar-valued analytic functions on D. We denote by

1. $A(D) = \{f \in C(\overline{D})\ ;\ f|_D \in \mathcal{O}(D)\} \subset C(\overline{D})$ the *domain algebra* of D,
2. $\partial_{A(D)}$ the *Shilov boundary* of $A(D)$ (i.e., the smallest closed subset of $\overline{D}$ such that

$$\sup_{z\in\overline{D}} |f(z)| = \sup_{z\in\partial_{A(D)}} |f(z)|$$

for all $f \in A(D)$).

Let $D \subset \mathbb{C}^d$ be a bounded strictly pseudoconvex or a bounded symmetric and circled domain. We denote by $\mu \in M_1^+(\partial_{A(D)})$ the canonical probability measure on $\partial_{A(D)}$.

Example

1. $D = \mathbb{B}_d$: $\partial_{A(\mathbb{B}_d)} = \partial\mathbb{B}_d = \mathbb{S}_d$ and $\mu = \sigma$.
2. $D = \mathbb{D}^d$: $\partial_{A(\mathbb{D}^d)} = \mathbb{T}^d$ and $\mu = \otimes_d m$.

We have

$$H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}} \quad \text{and} \quad H^\infty(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}.$$

Definition

We define

$$H^2(\mu) = \overline{A(D)|_{\partial_{A(D)}}}^{\tau_{\|\cdot\|_{L^2(\mu)}}} \subset L^2(\mu)$$

and

$$H^\infty(\mu) = \overline{A(D)|_{\partial_{A(D)}}}^{\tau_{w^*}} \subset L^\infty(\mu).$$

Furthermore, we denote by

$$I_\mu = \{f \in H^\infty(\mu)\ ;\ |f| = 1\ \mu\text{-a.e.}\}$$

the set of *inner functions with respect to* μ.

Definition

Let $f \in L^\infty(\mu)$. We call

$$T_f \colon H^2(\mu) \to H^2(\mu),\ g \mapsto P_{H^2(\mu)}(fg),$$

the *Toeplitz operator with symbol* f.

Theorem (Didas-Eschmeier-Everard, 2011)

For $X \in \mathcal{B}(H^2(\mu))$, the following are equivalent:

1. *there exists $f \in L^\infty(\mu)$ such that $X = T_f$,*
2. *$T_u^* X T_u - X = 0$ for all $u \in I_\mu$.*

Definition

We denote by $H^\infty(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ the set of all bounded analytic functions on $\mathbb{D}$.

Theorem

The map

$$r_m\colon H^\infty(\mathbb{D}) \to H^\infty(m),\ \theta \mapsto \tau_{w^*}\text{-}\lim_{r\to 1}[\theta(r\cdot)] =: \theta^*$$

is an isometric algebra isomorphism and a weak homeomorphism with* $r_m(f|_{\mathbb{D}}) = [f|_{\mathbb{T}}]$ *for all* $f \in A(\mathbb{D})$.

Definition

We denote by $H^\infty(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Theorem

There exists a map

$$r_\mu \colon H^\infty(D) \to H^\infty(\mu)$$

which is an isometric algebra isomorphism and a weak homeomorphism with* $r_\mu(f|_D) = [f|_{\partial_{A(D)}}]$ *for all* $f \in A(D)$. *We write* $\theta^* = r_\mu(\theta)$ *for* $\theta \in H^\infty(D)$.

Let $D \subset \mathbb{C}^d$ be a bounded strictly pseudoconvex or bounded symmetric and circled domain and $\mu \in M_1^+(\partial_{A(D)})$ be the probability measure on the Shilov boundary $\partial_{A(D)}$.

Theorem (Didas-Eschmeier-S., 2017)

Let $p \in [1, \infty)$. For $X \in \mathcal{B}(H^2(\mu))$, the following are equivalent:

1. *there exist $f \in L^\infty(\mu)$ and $S \in \mathcal{S}_p(H^2(\mu))$ such that $X = T_f + S$,*
2. *$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$ for all $u \in I_\mu$.*

Proposition

Let $(\alpha_k)_{k\in\mathbb{N}}$ *be a sequence in* $H^\infty(\mu)$ *with*

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k = \alpha \in [0,1),$$

and let $X \in \mathcal{B}(H^2(\mu))$ *be an operator such that*

$$Y = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{\alpha_k}^* X T_{\alpha_k} \in \mathcal{B}(H^2(\mu))$$

exists. If $T_u^* X T_u - X \in \mathcal{K}(H^2(\mu))$ *for all* $u \in I_\mu$*, then there exists a function* $f \in L^\infty(\mu)$ *such that*

$$X = T_f + \frac{1}{1-\alpha^2}(X - Y).$$

Proposition (Hiai, 1997)

Let $p \in [1, \infty)$. The map

$$\|\cdot\|_p : (\mathcal{B}(H^2(\mu)), \tau_{\mathrm{WOT}}) \to [0, \infty], \; S \mapsto \|S\|_p$$

is lower semi-continuous.

In the setting of the proposition on the last slide, we obtain

$$\left\| \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} X - T_{\alpha_k}^* X T_{\alpha_k} \right\|_p \leq \liminf_{k\to\infty} \left\| X - T_{\alpha_k}^* X T_{\alpha_k} \right\|_p .$$

We denote by

$$I_D = r_\mu^{-1}(I_\mu) = \{\theta \in H^\infty(D) \; ; \; \theta^* \in I_\mu\}$$

the set of *inner functions with respect to D and μ*.

Proposition (Aleksandrov, 1984)

Let $\alpha \in [0,1)$. Then there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in I_D such that

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha.$$

Proposition (Xia, 2009)

Let $p \in [1, \infty)$. *Suppose that* $X \in \mathcal{B}(H^2(\mu))$ *is an operator such that*

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$. *Then, for all* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\| T_{\theta^*}^* X T_{\theta^*} - X \|_p \leq \varepsilon$$

for all $\theta \in I_D$ *with* $\left| \int_{\partial_{A(D)}} 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$.

Proof of the main theorem.

There exists $0 < \delta < 1$ such that $\|T^*_{\theta^*} X T_{\theta^*} - X\|_p \leq 1$ for all $\theta \in I_D$ with $\left|\int_{\partial_{A(D)}} 1 - \theta^* \, \mathrm{d}\mu\right| \leq \delta$. Set $\alpha = 1 - \delta/2$.
$\implies$ There exists $(\alpha_k)_{k\in\mathbb{N}}$ in I_D with $\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha$.
By passing to a subsequence, we can achieve that $\left|\int_{\partial_{A(D)}} 1 - \alpha_k^* \, \mathrm{d}\mu\right| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T^*_{\alpha_k^*} X T_{\alpha_k^*} \in \mathcal{B}(H^2(\mu))$$

exists. Hence,

$$\left\|\tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} X - T^*_{\alpha_k^*} X T_{\alpha_k^*}\right\|_p \leq \liminf_{k\to\infty} \left\|X - T^*_{\alpha_k^*} X T_{\alpha_k^*}\right\|_p \leq 1.$$

$\implies X - Y \in \mathcal{S}_p(H^2(\mu))$. $\square$

Remark

The following ingredients are essential for the proof:

1. the triple $(A(D)|_{\partial_{A(D)}}, \partial_{A(D)}, \mu)$ is *regular (in the sense of Aleksandrov)*,
2. the measure μ is a *faithful Henkin measure*.

Theorem (Brown-Halmos, 1964)

For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exists $f \in H^\infty(m)$ such that $X = T_f$,*
2. *$XT_g - T_gX \in \{0\}$ for all $g \in H^\infty(m)$.*

Theorem (Davidson, 1977)

For $X \in \mathcal{B}(H^2(m))$, the following are equivalent:

1. *there exist $f \in H^\infty(m) + C(\mathbb{T})$ and $K \in \mathcal{K}(H^2(m))$ such that $X = T_f + K$,*
2. *$XT_g - T_gX \in \mathcal{K}(H^2(m))$ for all $g \in H^\infty(m)$.*

Theorem (Hartman, 1958)

We have

$$H^\infty(m) + C(\mathbb{T}) = \{f \in L^\infty(m) \; ; \; H_f \text{ is compact}\}$$

with $H_f = \left(\mathrm{id}_{L^2(m)} - P_{H^2(m)}\right) M_f|_{H^2(m)}$ *for* $f \in L^\infty(m)$.

Theorem (Davidson, Hartman)

For $X \in \mathcal{B}(H^2(m))$, *the following are equivalent:*

1. *there exist* $f \in L^\infty(m)$ *such that* H_f *is compact and* $K \in \mathcal{K}(H^2(m))$ *such that* $X = T_f + K$,
2. $XT_g - T_gX \in \mathcal{K}(H^2(m))$ *for all* $g \in H^\infty(m)$.

Let $d \geq 1$ and

$$D = \mathbb{B}_d.$$

Theorem (Ding-Sun, 1997)

For $X \in \mathcal{B}(H^2(\mu))$, the following are equivalent:

1. *there exist $f \in L^\infty(\mu)$ such that H_f is compact and $K \in \mathcal{K}(H^2(\mu))$ such that $X = T_f + K$,*
2. *$XT_g - T_gX \in \mathcal{K}(H^2(\mu))$ for all $g \in H^\infty(\mu)$.*

Let $d \geq 1$ and

$$D = \mathbb{B}_d \quad \text{or} \quad D = \mathbb{D}^d.$$

Theorem (Guo-Wang, 2006)

For $X \in \mathcal{B}(H^2(\mu))$, the following are equivalent:

1. *there exist $f \in L^\infty(\mu)$ such that H_f is of finite rank and $F \in \mathcal{F}(H^2(\mu))$ such that $X = T_f + F$,*
2. *$XT_g - T_g X \in \mathcal{F}(H^2(\mu))$ for all $g \in H^\infty(\mu)$.*

Let $d \geq 1$, and let $D \subset \mathbb{C}^d$ be a bounded strictly pseudoconvex or bounded symmetric and circled domain and $\mu \in M_1^+(\partial_{A(D)})$ be the probability measure on the Shilov boundary $\partial_{A(D)}$.

Theorem (Didas-Eschmeier-S., 2017)

Let $p \in [1, \infty)$. For $X \in \mathcal{B}(H^2(\mu))$, the following are equivalent:

1. *there exist $f \in L^\infty(\mu)$ such that H_f is in the Schatten-p-class and $S \in \mathcal{S}_p(H^2(\mu))$ such that $X = T_f + S$,*
2. *$XT_g - T_gX \in \mathcal{S}_p(H^2(\mu))$ for all $g \in H^\infty(\mu)$.*