A Beurling–Lax–Halmos Theorem for Spaces with a Complete Nevanlinna-Pick Factor

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30^{th} IWOTA, 2019-07-25

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Definition

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. We denote by

$$H^2(\mathbb{D}) = \left\{ f = \sum_{n=0}^{\infty} a_n z^n \;;\; a_n \in \mathbb{C},\ \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\} \subset \mathcal{O}(\mathbb{D})$$

the *(scalar-valued) Hardy space*.

Definition

We call a bounded analytic function $\Phi\colon \mathbb{D} \to \mathbb{C}$ *inner* if $\|\Phi\|_{\mathbb{D}} \leq 1$ and $\lim_{r\to 1}|\Phi(r\lambda)| = 1$ for almost all λ in the unit circle $\mathbb{T}$.

Theorem (Beurling, 1949)

Let $\mathcal{M} \subset H^2(\mathbb{D})$ be a non-zero closed subspace. The following are equivalent.

1. *$\mathcal{M}$ is invariant for z (i.e., $z\mathcal{M} \subset \mathcal{M}$).*
2. *There exists an inner function Φ such that*

$$\mathcal{M} = \Phi \cdot H^2(\mathbb{D}).$$

Let X be a non-empty set.

Definition

Let $k \colon X \times X \to \mathbb{C}$ be a function. We call k a *kernel* and write $k \succeq 0$ if

$$(k(x_i, x_j))_{i,j=1}^{n} \in \mathbb{C}^{n \times n}$$

is positive semi-definite for every finite subset $\{x_1, \ldots, x_n\} \subset X$. The corresponding *reproducing kernel Hilbert space* on X is denoted by $\mathcal{H}_k$. Thus,

$$k(\cdot, x) \in \mathcal{H}_k \quad \text{for all } x \in X$$

and

$$\langle f, k(\cdot, x) \rangle = f(x) \quad \text{for all } f \in \mathcal{H}_k, x \in X.$$

Example

1. If
$$s_1\colon \mathbb{D}\times\mathbb{D}\to\mathbb{C},\ (z,w)\mapsto \frac{1}{1-z\overline{w}},$$
then $\mathcal{H}_{s_1} = H^2(\mathbb{D})$.

Let $d \geq 1$ be a natural number.

2. If
$$s_d\colon \mathbb{B}_d\times\mathbb{B}_d\to\mathbb{C},\ (z,w)\mapsto \frac{1}{1-\langle z,w\rangle},$$
then $\mathcal{H}_{s_d} = H^2_d$, the Drury–Arveson space on the unit ball in $\mathbb{B}_d \subset \mathbb{C}^d$.

Example

3. Let $m \geq 2$ be a natural number. If

$$k^{(m)} \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, \ (z, w) \mapsto \frac{1}{(1 - \langle z, w \rangle)^m},$$

then $\mathcal{H}_{k^{(m)}} = A_m(\mathbb{B}_d)$, the weighted Bergman space on $\mathbb{B}_d$.

4. If

$$k \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, \ (z, w) \mapsto \sum_{n=0}^{\infty} \frac{1}{n+1} \langle z, w \rangle^n,$$

then $\mathcal{H}_k$ coincides with the Dirichlet space on $\mathbb{B}_d$.

Definition

Let $\mathcal{E}, \mathcal{F}$ be Hilbert spaces and let k, ℓ be kernels on X.
We write $\operatorname{Mult}(\mathcal{H}_\ell \otimes \mathcal{F}, \mathcal{H}_k \otimes \mathcal{E})$ for the space of $B(\mathcal{F}, \mathcal{E})$-valued functions on X that multiply $\mathcal{H}_\ell \otimes \mathcal{F}$ into $\mathcal{H}_k \otimes \mathcal{E}$.
Furthermore, we often identify $\Phi \in \operatorname{Mult}(\mathcal{H}_\ell \otimes \mathcal{F}, \mathcal{H}_k \otimes \mathcal{E})$ with its associated multiplication operator $M_\Phi \colon \mathcal{H}_\ell \otimes \mathcal{F} \to \mathcal{H}_k \otimes \mathcal{E}$.

Theorem (Beurling, 1949; Lax, 1959; Halmos, 1961)

Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{M} \subset H^2(\mathbb{D}) \otimes \mathcal{E}$ be a non-zero closed subspace. The following are equivalent.

1. *$\mathcal{M}$ is invariant for z.*
2. *There exist a Hilbert space $\mathcal{F} \subset \mathcal{E}$ and an isometric multiplier $\Phi \in \mathrm{Mult}(H^2(\mathbb{D}) \otimes \mathcal{F}, H^2(\mathbb{D}) \otimes \mathcal{E})$ such that*
$$\mathcal{M} = \Phi(H^2(\mathbb{D}) \otimes \mathcal{F}).$$

Theorem (Ball–Bolotnikov, 2013)

Let $m \geq 2$ be a natural number. Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{M} \subset A_m(\mathbb{D}) \otimes \mathcal{E}$ be a non-zero closed subspace. The following are equivalent.

1. *$\mathcal{M}$ is invariant for z.*
2. *There exist an auxiliary Hilbert space $\mathcal{F}$ and a partially isometric multiplier $\Phi \in \mathrm{Mult}(H^2(\mathbb{D}) \otimes \mathcal{F}, A_m(\mathbb{D}) \otimes \mathcal{E})$ such that*

$$\mathcal{M} = \Phi \cdot (H^2(\mathbb{D}) \otimes \mathcal{F}).$$

Theorem (Sarkar, 2015 & 2016)

Let $\mathcal{E}$ be a Hilbert space, let $\mathcal{H}_k \subset \mathcal{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space of analytic functions such that the coordinate functions $(z_1, \dots, z_d)$ form a row contraction on $\mathcal{H}_k$ (i.e., $\sum_{n=1}^d M_{z_n} M_{z_n}^* \leq \mathrm{id}_{\mathcal{H}_k}$) and let $\mathcal{M} \subset \mathcal{H}_k \otimes \mathcal{E}$ be a non-zero closed subspace. The following are equivalent.

1. *$\mathcal{M}$ is invariant for $z_1, \dots, z_d$.*
2. *There exist an auxiliary Hilbert space $\mathcal{F}$ and a partially isometric multiplier $\Phi \in \mathrm{Mult}(H_d^2 \otimes \mathcal{F}, \mathcal{H}_k \otimes \mathcal{E})$ such that*
$$\mathcal{M} = \Phi \cdot (H_d^2 \otimes \mathcal{F}).$$

Remark

Let $\mathcal{E}$ be a Hilbert space, and let $k\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ be a kernel. The following are equivalent.

1. coordinate functions $(z_1, \dots, z_d)$ form a row contraction on $\mathcal{H}_k$.
2. $k/s_d \succeq 0$.

In this case, for a closed subspace $\mathcal{M} \subset \mathcal{H}_k \otimes \mathcal{E}$, the following are equivalent.

1. $\mathcal{M}$ is invariant under multiplication by $z_1, \dots, z_d$.
2. $\mathcal{M}$ is $\mathrm{Mult}(\mathcal{H}_{s_d})$-invariant (i.e., invariant for all $\varphi \in \mathrm{Mult}(\mathcal{H}_{s_d})$).

Definition

A kernel $k \colon X \times X \to \mathbb{C}$ is said to be *normalized* if there exists a point $x_0 \in X$ with $k(x, x_0) = 1$ for all $x \in X$.

Definition/Theorem (Agler–McCarthy, 2000)

A normalized kernel s is a *complete Nevanlinna–Pick* kernel if and only if s is non-vanishing and $1 - 1/s \succeq 0$.

Example

1. The Drury–Arveson space and the Dirichlet space are complete Nevanlinna–Pick spaces.
2. The weighted Bergman spaces are *not* complete Nevanlinna–Pick spaces.

Theorem (McCullough–Trent, 2000)

Let X be a set, let s be a normalized complete Nevanlinna–Pick kernel on X. Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{M} \subset \mathcal{H}_s \otimes \mathcal{E}$ be a non-zero closed subspace. The following are equivalent.

1. $\mathcal{M}$ *is* $\mathrm{Mult}(\mathcal{H}_s)$*-invariant.*
2. *There exist an auxiliary Hilbert space $\mathcal{F}$ and a partially isometric multiplier $\Phi \in \mathrm{Mult}(\mathcal{H}_s \otimes \mathcal{F}, \mathcal{H}_s \otimes \mathcal{E})$ such that*

$$\mathcal{M} = \Phi \cdot (\mathcal{H}_s \otimes \mathcal{F})$$

Theorem (Clouâtre–Hartz–S., 2019)

Let X be a set, let k be a kernel on X and let s be a normalized complete Nevanlinna–Pick kernel on X such that $k/s \succeq 0$. Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{M} \subset \mathcal{H}_k \otimes \mathcal{E}$ be a non-zero closed subspace. The following are equivalent.

1. *$\mathcal{M}$ is $\operatorname{Mult}(\mathcal{H}_s)$-invariant.*
2. *There exist an auxiliary Hilbert space $\mathcal{F}$ and a partially isometric multiplier $\Phi \in \operatorname{Mult}(\mathcal{H}_s \otimes \mathcal{F}, \mathcal{H}_k \otimes \mathcal{E})$ such that*

$$\mathcal{M} = \Phi \cdot (\mathcal{H}_s \otimes \mathcal{F}).$$

Idea of the proof.

(1) $\implies$ (2): Show that $k^{\mathcal{M}}/s \succeq 0$, where $k^{\mathcal{M}}$ is the reproducing kernel of $\mathcal{M}$. (Here we use the fact that s is a normalized complete Nevanlinna–Pick space.) Then use Kolmogorov's factorization theorem to obtain Φ. $\square$

Theorem (Clouâtre–Hartz–S., 2019)

Assume the setting of the main theorem.
Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{H}_k \otimes \mathcal{E}$ be two non-zero closed subspaces and let $\mathcal{F}, \mathcal{G}$ be Hilbert spaces.
If $\Phi \in \operatorname{Mult}(\mathcal{H}_s \otimes \mathcal{F}, \mathcal{H}_k \otimes \mathcal{E})$ and $\Psi \in \operatorname{Mult}(\mathcal{H}_s \otimes \mathcal{G}, \mathcal{H}_\ell \otimes \mathcal{E})$ are partially isometric multipliers with

$$\mathcal{M} = \Phi \cdot (\mathcal{H}_s \otimes \mathcal{F}) \quad \text{and} \quad \mathcal{N} = \Psi \cdot (\mathcal{H}_s \otimes \mathcal{G}),$$

then there exists a contractive multiplier
$\Gamma \in \operatorname{Mult}(\mathcal{H}_s \otimes \mathcal{G}, \mathcal{H}_s \otimes \mathcal{F})$ with

$$\Psi = \Phi\Gamma.$$

Theorem (Clouâtre–Hartz–S., 2019)

Assume the setting of main theorem.
Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{H}_k \otimes \mathcal{E}$ be two non-zero closed subspaces and let $\mathcal{F}$ be a Hilbert space.
If $\Phi \in \mathrm{Mult}(\mathcal{H}_s \otimes \mathcal{F}, \mathcal{H}_k \otimes \mathcal{E})$ is a partially isometric multiplier with

$$\mathcal{M} = \Phi \cdot (\mathcal{H}_s \otimes \mathcal{F})$$

and if $\mathcal{N}$ is $\mathrm{Mult}(\mathcal{H}_s)$-invariant, then there exist a Hilbert space $\mathcal{G}$ and a partially isometric multiplier $\Gamma \in \mathrm{Mult}(\mathcal{H}_s \otimes \mathcal{G}, \mathcal{H}_s \otimes \mathcal{F})$ such that $\Phi\Gamma \in \mathrm{Mult}(\mathcal{H}_s \otimes \mathcal{G}, \mathcal{H}_k \otimes \mathcal{E})$ is a partially isometric multiplier and

$$\mathcal{N} = (\Phi\Gamma) \cdot (\mathcal{H}_s \otimes \mathcal{G}).$$

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