K-contractions

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Contractions

Let $\mathcal{H}$ be a complex Hilbert space.

Lemma

Let $T \in B(\mathcal{H})$. The following assertions are equivalent:

1. *T is a contraction (i.e., $\|T\| \leq 1$),*
2. *$1/K(T, T^*) \geq 0$, where*

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, \ (z, w) \mapsto \frac{1}{1 - z\overline{w}} = \frac{1}{1 - \langle z, w \rangle}$$

is the reproducing kernel of the Hardy space on the unit disc $H^2(\mathbb{D})$.

Example

1. The shift operator $M_z \in B(H^2(\mathbb{D}))$ satisfies
$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \geq 0.$$
2. Every unitary $U \in B(\mathcal{H})$ fulfills
$$\frac{1}{K}(U, U^*) = 0.$$

Definition

We say that a contraction $T \in B(\mathcal{H})$ belongs to the class $C_{\cdot 0}$ or is *pure* if

$$T_\infty = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} T^N T^{*N} = 0.$$

Example

The shift operator $M_z \in B(H^2(\mathbb{D}))$ belongs to $C_{\cdot 0}$.

Theorem

Let $T \in B(\mathcal{H})$ be a contraction. Then

$$\pi_T : \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D}_T), \ h \mapsto \sum_{n=0}^{\infty} (D_T T^{*n} h) z^n,$$

where $D_T = (1 - TT^*)^{1/2} = (1/K(T, T^*))^{1/2}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle T_\infty h, h \rangle$$

for all $h \in \mathcal{H}$, and

$$\pi_T T^* = (M_z^{\mathcal{D}_T})^* \pi_T.$$

Corollary

A contraction $T \in B(\mathcal{H})$ is in $C_{\cdot 0}$ if and only if π_T is an isometry.

The $C_{\cdot 0}$ case

Corollary

Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:

1. *T is a contraction which belongs to $C_{\cdot 0}$,*
2. *there exist a Hilbert space $\mathcal{D}$, and an isometry $\pi\colon \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D})$ such that*

$$\pi T^* = (M_z^{\mathcal{D}})^* \pi.$$

A Beurling type theorem

Remark

If $T \in B(\mathcal{H})$ is a $C_{.0}$-contraction and $\mathcal{S} \in \mathrm{Lat}(T)$, then $T|_{\mathcal{S}}$ is also $C_{.0}$-contraction.

Theorem

Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H^2(\mathbb{D}, \mathcal{E})$, the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}(M_z^{\mathcal{E}})$,
2. *there exist a Hilbert space $\mathcal{D}$, and a bounded analytic function $\theta\colon \mathbb{D} \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta\colon H^2(\mathbb{D}, \mathcal{D}) \to H^2(\mathbb{D}, \mathcal{E}),\ f \mapsto \theta f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

The general case

Theorem

Let $T \in B(\mathcal{H})$ *be an operator and write* $H_K = H^2(\mathbb{D})$*. The following statements are equivalent:*

1. $1/K(T, T^*) \geq 0$,
2. *there exist Hilbert spaces* $\mathcal{D}$ *and* $\mathcal{K}$*, a unitary operator* $U \in B(\mathcal{K})$*, and an isometry* $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ *such that*
$$\Pi T^* = (M_z^{\mathcal{D}} \oplus U)^* \Pi.$$

Question

For which reproducing kernels K does an analogue theorem hold?
What happens if we look at commuting tuples $T = (T_1, \dots, T_d) \in B(\mathcal{H})^d$?

Unitarily invariant spaces on $\mathbb{B}_d$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with $a_0 = 1$ and such that

$$k(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{D})$$

defines a holomorphic function with radius of convergence at least 1 and no zeros in the unit disc $\mathbb{D}$. The map

$$K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z,w) \mapsto \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n$$

defines a semianalytic positive definite function and hence, there exists a reproducing kernel Hilbert space $H_K \subset \mathcal{O}(\mathbb{B}_d)$ with kernel K. The space H_K is a so called *unitarily invariant space on* $\mathbb{B}_d$.

Furthermore, we assume $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$ such that the *K-shift* $M_z = (M_{z_1}, \ldots, M_{z_d}) \in B(H_K)^d$ is well-defined.
Since k has no zeros in $\mathbb{D}$, the function

$$\frac{1}{k} \colon \mathbb{D} \to \mathbb{C}, \;\; z \mapsto \frac{1}{k(z)}$$

is again holomorphic and hence admits a Taylor expansion

$$\frac{1}{k}(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (z \in \mathbb{D})$$

with a suitable sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

Example

1. If $a_n = 1$ for all $n \in \mathbb{N}$, then H_K is the Hardy space ($d = 1$) or the *Drury-Arveson space* ($d \geq 2$).
2. If $\nu > 0$ and $a_n = a_n^{(\nu)} = (-1)^n \binom{-\nu}{n}$ for all $n \in \mathbb{N}$, then
$$K(z, w) = K^{(\nu)}(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\nu} \qquad (z, w \in \mathbb{B}_d),$$
i.e., $H_{K^{(\nu)}}$ is a *generalized Bergman space*.
3. The space H_K is a *complete Nevanlinna-Pick space* if and only if
$$c_n \leq 0$$
for all $n \geq 1$.

Definition

Let $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ be a commuting tuple. Define

$$\left(\frac{1}{K}\right)_N (T, T^*) = \sum_{n=0}^{N} c_n \sigma_T^n(1)$$

for all $N \in \mathbb{N}$, where

$$\sigma_T \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \sum_{i=1}^{d} T_i X T_i^*.$$

Furthermore, we write

$$\frac{1}{K}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \left(\frac{1}{K}\right)_N (T, T^*)$$

if the latter exists.

Definition

Let $T \in B(\mathcal{H})^d$ be a commuting tuple.

1. We call T a *K-contraction* if $1/K(T, T^*) \geq 0$.
2. We call T a *spherical unitary* if T satisfies $\sigma_T(1) = 1$ and consists of normal operators.

Example

1. If $d = 1$, a $K^{(1)}$-contraction is a contraction.
2. If $d \geq 2$, a $K^{(1)}$-contraction is a row contraction.
3. Let $m \in \mathbb{N}^*$. We call a commuting tuple $T \in B(\mathcal{H})^d$ an *m-hypercontraction* if T is a $K^{(\ell)}$-contraction for all $\ell = 1, \ldots, m$.

Proposition (Chen)

If there exists a natural number $p \in \mathbb{N}$ such that

$$c_n \geq 0 \text{ for all } n \geq p \quad \text{or} \quad c_n \leq 0 \text{ for all } n \geq p$$

holds, then

$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

Example

The condition above is satisfied in the case when H_K is a

1. generalized Bergman space,
2. complete Nevanlinna-Pick space.

For the rest of this section, we suppose that the condition in the last proposition holds.

Definition

Let $T \in B(\mathcal{H})^d$ be a K-contraction. We define

$$\Sigma_N(T) = 1 - \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T, T^*) \right)$$

for $N \in \mathbb{N}$ and write

$$\Sigma(T) = \tau_{\mathrm{SOT}}\text{-} \lim_{N \to \infty} \Sigma_N(T)$$

if the latter exists. If $\Sigma(T) = 0$, we call T K-pure.

Remark

If $K = K^{(1)}$ and $T \in B(\mathcal{H})^d$ is a $K^{(1)}$-contraction, then

$$\Sigma_N(T) = \sigma_T^{N+1}(1)$$

for all $N \in \mathbb{N}$, and hence,

$$\Sigma(T) = T_\infty \geq 0.$$

Proposition

Let $T \in B(\mathcal{H})^d$ be a K-contraction such that $\Sigma(T)$ exists. The map

$$\pi_T \colon \mathcal{H} \to H_K(\mathcal{D}_T),\ h \mapsto \sum_{\alpha \in \mathbb{N}^d} \left(a_{|\alpha|} \frac{|\alpha|!}{\alpha!} D_T T^{*\alpha} h \right) z^\alpha,$$

where $D_T = (1/K(T, T^*))^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle \Sigma(T) h, h \rangle$$

for all $h \in \mathcal{H}$ and

$$\pi_T T_i^* = (M_{z_i}^{\mathcal{D}_T})^* \pi_T$$

for all $i = 1, \ldots, d$.

The pure case

Theorem (Eschmeier, S.)

Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:

1. *T is a K-contraction which is K-pure,*
2. *there exist a Hilbert space $\mathcal{D}$ and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D})$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}})^* \Pi$$

for all $i = 1, \ldots, d$.

A Beurling type theorem

Theorem (Eschmeier, S.)

Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H_K(\mathcal{E})$, the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$ *and* $M_z^{\mathcal{E}}|_{\mathcal{S}}$ *is K-pure,*
2. *there exist a Hilbert space $\mathcal{D}$ and a bounded analytic function $\theta\colon \mathbb{B}_d \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta\colon H_K(\mathcal{D}) \to H_K(\mathcal{E}),\ f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

The general case

Definition

We call a K-contraction $T \in B(\mathcal{H})^d$ *strong* if $\Sigma(T) \geq 0$ and $\Sigma(T) = \sigma_T(\Sigma(T))$ holds.

Remark

1. If a K-contraction is K-pure, then it is also strong. Hence, the K-shift $M_z \in B(H_K)^d$ is a strong K-contraction.
2. Every spherical unitary is a strong K-contraction.

Lemma

Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. Then there exist a Hilbert space $\mathcal{L}$ with $\Sigma(T)^{1/2}\mathcal{H} \subset \mathcal{L}$, and a spherical unitary $W \in B(\mathcal{L})^d$ such that

$$\Sigma(T)^{1/2} T_i^* h = W_i^* \Sigma(T)^{1/2} h$$

for all $h \in \mathcal{H}$ and $i = 1, \ldots, d$. Furthermore, $\mathcal{L}$ and W can be chosen such that

$$\mathcal{L} = \bigvee \left\{ W^\alpha \Sigma(T)^{1/2} h \;;\; \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\}$$

holds.

Remark

Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. Using the notations from the last lemma, we define

$$\Pi_T \colon \mathcal{H} \to H_K(\mathcal{D}_T) \oplus \mathcal{L}, \ h \mapsto \pi_T h \oplus \Sigma(T)^{1/2} h.$$

Then Π_T is an isometry with

$$\Pi_T T_i^* = (M_{z_i}^{\mathcal{D}_T} \oplus W_i)^* \Pi_T$$

for $i = 1, \ldots, d$. Furthermore, one can achieve that

$$\mathcal{L} = \bigvee \left\{ W^\alpha \Sigma(T)^{1/2} h \ ; \ \alpha \in \mathbb{N}^d \text{ and } h \in \mathcal{H} \right\},$$

which will be assumed further on.

Theorem (Eschmeier, S.)

Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:

- *T is a strong K-contraction,*
- *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi\colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \quad (i = 1, \ldots, d).$$

If we assume that H_K is regular, i.e., $\lim_{n\to\infty} a_n / a_{n+1} = 1$, then the above are also equivalent to

- *there is a unital completely contractive linear map $\rho\colon \operatorname{span}\left\{1, M_{z_i}, M_{z_i} M_{z_i}^* \;;\; i = 1, \ldots, d\right\} \to B(\mathcal{H})$ with*

$$\rho(M_{z_i}) = T_i, \; \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, d).$$

The Toeplitz algebra

Suppose that H_K is regular and that

$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \in \bigvee \left\{ M_z^{\alpha} M_z^{*\beta} \; ; \; \alpha, \beta \in \mathbb{N}^d \right\}.$$

Example

If $K = K^{(\nu)}$ for $\nu \geq 1$, then the above conditions are satisfied.

Theorem

The Toeplitz algebra $C^*(M_z)$ coincides with the closed linear span of $\left\{ M_z^{\alpha} M_z^{*\beta} \; ; \; \alpha, \beta \in \mathbb{N}^d \right\}$, i.e.,

$$C^*(M_z) = \bigvee \left\{ M_z^{\alpha} M_z^{*\beta} \; ; \; \alpha, \beta \in \mathbb{N}^d \right\}.$$

Factorizations of coextensions

Theorem (Eschmeier, S.)

Let $T \in B(\mathcal{H})^d$ be a strong K-contraction. Furthermore, let $\mathcal{D}, \mathcal{K}$ be Hilbert spaces, $U \in B(\mathcal{K})^d$ a spherical unitary, and let $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ be an isometry such that

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \quad (i = 1, \ldots, d).$$

Then there exist isometries $V_s \in B(\mathcal{D}_T, \mathcal{D})$ and $V_u \in B(\mathcal{L}, \mathcal{K})$ such that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad \Pi \quad} & H_K(\mathcal{D}) \oplus \mathcal{K} \\ {\scriptstyle \Pi_T} \downarrow & \nearrow {\scriptstyle (1_{H_K} \otimes V_s) \oplus V_u} & \\ H_K(\mathcal{D}_T) \oplus \mathcal{L} & & \end{array}$$

commutes.

Radial K-hypercontractions

Lemma

Let $m \in \mathbb{N}^*$ and let $T \in B(\mathcal{H})^d$ be a $K^{(m)}$-contraction. Then, for $0 < r < 1$, the tuple $rT \in B(\mathcal{H})^d$ is a $K^{(m)}$-pure $K^{(m)}$-contraction.

Definition

We call a commuting tuple $T \in B(\mathcal{H})^d$ with $\sigma(T) \subset \overline{\mathbb{B}}_d$ a *radial K-hypercontraction* if, for all $0 < r < 1$, $rT \in B(\mathcal{H})^d$ is a K-contraction.

Example

Spherical unitaries are radial K-hypercontractions.

Remark

We define

$$k_r : \mathbb{D} \to \mathbb{C}, \; z \mapsto k(rz) \qquad (r \in [0,1]).$$

For $r, s \in [0,1]$, the function k_s/k_r has a Taylor expansion on $\mathbb{D}$

$$\sum_{n=0}^{\infty} a_n(s,r) z^n.$$

Note that

$$a_n(1,0) = a_n \quad \text{and} \quad a_n(0,1) = c_n \qquad (n \in \mathbb{N}).$$

Remark (Guo, Hu, Xu)

If H_K is regular, then $\sigma(M_z) \subset \overline{\mathbb{B}}_d$.

From now on, we suppose that H_K is regular.

Proposition (Olofsson)

The K-shift $M_z \in B(H_K)^d$ is a radial K-hypercontraction if and only if

$$a_n(1, r) \geq 0$$

for all $n \in \mathbb{N}$ and $0 < r < 1$.

Remark (Olofsson)

If H_K is a complete Nevanlinna-Pick space or a generalized Bergman space, then M_z is a radial K-hypercontraction.

Proposition (Olofsson)

Suppose that M_z is a radial K-hypercontraction. Let $T \in B(\mathcal{H})^d$ be a radial K-hypercontraction. Then

$$\frac{1}{K}_{\mathrm{rad}}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{r \to 1} \frac{1}{K}(rT, rT^*)$$

exists and defines a positive operator.

Corollary

If M_z is a radial K-hypercontraction, then

$$\frac{1}{K}_{\mathrm{rad}}(M_z, M_z^*) = P_{\mathbb{C}}.$$

Theorem (Eschmeier, S.)

Suppose that $M_z \in B(H_K)^d$ *is a radial* K*-hypercontraction. Let* $T \in B(\mathcal{H})^d$ *be a commuting tuple. The following assertions are equivalent:*

1. T *is a radial* K*-hypercontraction,*
2. *there exist Hilbert spaces* $\mathcal{D}, \mathcal{K}$*, a spherical unitary* $U \in B(\mathcal{K})^d$*, and an isometry* $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ *such that*
$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \qquad (i = 1, \ldots, d),$$
3. *there is a unital completely contractive linear map* $\rho \colon S \to B(\mathcal{H})$ *on the operator space* $S = \operatorname{span}\left\{1, M_{z_i}, M_{z_i} M_{z_i}^* \; ; \; i = 1, \ldots, d\right\}$ *with*
$$\rho(M_{z_i}) = T_i, \;\; \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, d).$$

Theorem (Eschmeier, S.)

Let $\nu > 0$ and let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following assertions are equivalent:

1. *T is a radial $K^{(\nu)}$-hypercontraction,*
2. *T is a strong $K^{(\nu)}$-contraction,*
3. *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi\colon \mathcal{H} \to H_{K^{(\nu)}}(\mathcal{D}) \oplus \mathcal{K}$ such that*
$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \qquad (i = 1, \ldots, d),$$
4. *there is a unital completely contractive linear map $\rho\colon S \to B(\mathcal{H})$ on the operator space $S = \operatorname{span}\left\{1, M_{z_i}, M_{z_i} M_{z_i}^* \; ; \; i = 1, \ldots, d\right\}$ with*
$$\rho(M_{z_i}) = T_i, \; \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, d).$$

ν-hypercontractions

Definition

Let $\nu \geq 1$ be a real number. We call a commuting tuple $T \in B(\mathcal{H})^d$ an *ν-hypercontraction* if

$$\frac{1}{K^{(\mu)}}(T, T^*) = \tau_{\|\cdot\|}\text{-}\sum_{n=0}^{\infty} c_n^{(\mu)} \sigma_T^n(1) \geq 0$$

for all $1 \leq \mu \leq \nu$.

Remark (Agler; Müller, Vasilescu)

Let $m \in \mathbb{N}^*$. A commuting tuple $T \in B(\mathcal{H})^d$ is a m-hypercontraction if and only if T is a row contraction and a $K^{(m)}$-contraction.

Theorem (Olofsson; Eschmeier, S.)

Let $\nu \geq 1$ and let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following assertions are equivalent:

1. *T is a row contraction and a radial $K^{(\nu)}$-hypercontraction,*
2. *T is an ν-hypercontraction,*
3. *T is row contraction and a $K^{(\nu)}$-contraction,*
4. *T is a strong $K^{(\nu)}$-contraction.*

In this case, we have $\Sigma(T) = T_\infty$.

A Beurling type theorem for ν-hypercontractions

Theorem (Eschmeier, Klauk, S.)

Let $\mathcal{E}$ be a Hilbert space and $\mathcal{S} \subset H_{K^{(\nu)}}(\mathcal{E})$ be a subspace. For $M_z^{\mathcal{E}} \in B(H_{K^{(\nu)}}(\mathcal{E}))^d$ and $1 \leq \mu \leq \nu$, the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$ *and* $M_z^{\mathcal{E}}|_{\mathcal{S}}$ *is a* $K^{(\mu)}$*-contraction,*
2. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$ *and* $M_z^{\mathcal{E}}|_{\mathcal{S}}$ *is a* μ*-hypercontraction,*
3. *there exist a Hilbert space* $\mathcal{D}$ *and a bounded analytic function* $\theta\colon \mathbb{B}_d \to B(\mathcal{D}, \mathcal{E})$ *such that*

$$M_\theta\colon H_{K^{(\mu)}}(\mathcal{D}) \to H_{K^{(\nu)}}(\mathcal{E}),\ f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

Theorem (Sarkar)

Let $\mathcal{E}$ be a Hilbert space and let $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_n, \mathcal{E})$ be a reproducing kernel Hilbert space of analytic functions such that $M_z^{\mathcal{E}} \in B(H(\mathcal{E}))^n$ is a row contraction as well as $\mathcal{S} \subset H(\mathcal{E})$. Then the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$,
2. *there exist a Hilbert space $\mathcal{D}$ and an analytic function $\theta \colon \mathbb{B}_n \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta \colon H_{K^{(1)}}(\mathcal{D}) \to H(\mathcal{E}), \; f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.