Schatten-class perturbations of Toeplitz operators

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Joint work with
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Definition

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ with the canonical probability measure m. The *Hardy space with respect to m* will be denoted by

$$H^2(m) = \left\{ f \in L^2(m) \; ; \; \hat{f}(n) = 0 \text{ for all } n < 0 \right\} \subset L^2(m).$$

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$$H^\infty(m) = L^\infty(m) \cap H^2(m) \subset L^\infty(m)$$

and we call

$$I_m = \{ f \in H^\infty(m) \; ; \; |f| = 1 \;\; m\text{-a.e.} \}$$

the set of *inner functions with respect to* m.

Definition

Let $f \in L^\infty(m)$. We call

$$T_f \colon H^2(m) \to H^2(m),\ g \mapsto P_{H^2(m)}(fg),$$

where $P_{H^2(m)} \colon L^2(m) \to H^2(m)$ is the orthogonal projection onto $H^2(m)$, the *Toeplitz operator with symbol* f.

Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^\infty(m)$ such that $X = T_f$) if and only if

$$T_z^* X T_z - X = 0,$$

where $z \in L^\infty(m)$ is the identity map.

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Theorem (Brown-Halmos condition II)

An operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_m$.

Theorem (Gu, 2004)

An operator $X \in B(H^2(m))$ is a finite rank Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $F \in F(H^2(m))$ such that $X = T_f + F$) if and only if

$$T_u^* X T_u - X \in F(H^2(m))$$

for all $u \in I_m$.

Theorem (Xia, 2009)

An operator $X \in B(H^2(m))$ is a compact Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $K \in K(H^2(m))$ such that $X = T_f + K$) if and only if

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for all $u \in I_m$.

Definition

Let $p \in [1, \infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a *Schatten-p-class* operator if

$$\|S\|_p^p = \mathrm{tr}(|S|^p) < \infty.$$

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equipped with $\|\cdot\|_p$ and

$$\mathcal{S}_0(H) = F(H) \quad \text{as well as} \quad \mathcal{S}_\infty(H) = K(H)$$

both equipped with the operator norm.

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with the canonical probability measure $m \in M_1^+(\mathbb{T})$.

Theorem (Gu, Xia, Didas-Eschmeier-S.)

Let $p \in \{0\} \cup [1, \infty]$. An operator $X \in B(H^2(m))$ is a $\mathcal{S}_p(H^2(m))$ Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $S \in \mathcal{S}_p(H^2(m))$ such that $X = T_f + S$) if and only if

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the *disc algebra* and denote by $\partial_{A(\mathbb{D})}$ the *Shilov boundary* of $A(\mathbb{D})$ (i.e. the smallest closed subset of $\overline{\mathbb{D}}$ such that

$$\sup_{z \in \overline{\mathbb{D}}} |f(z)| = \sup_{z \in \partial_{A(\mathbb{D})}} |f(z)| \quad \text{for all } f \in A(\mathbb{D})).$$

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Proposition

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$$\partial_{A(\mathbb{D})} = \partial \mathbb{D} = \mathbb{T}.$$

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(ii) $S = \partial_{A(D)}$ the *Shilov boundary* of $A(D)$ (i.e. the smallest closed subset of $\overline{D}$ such that

$$\sup_{z \in \overline{D}} |f(z)| = \sup_{z \in S} |f(z)|$$

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Examples

(i) $D = \mathbb{B}_n$: $S = \partial\mathbb{B}_n$ and $\mu = \sigma$.

(ii) $D = \mathbb{D}^n$: $S = \mathbb{T}^n$ and $\mu = \otimes_n m$.

We have

$$H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}} \quad \text{and} \quad H^\infty(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}.$$

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$$I_\mu = \{f \in H^\infty(\mu)\ ;\ |f| = 1\ \mu\text{-a.e.}\}$$

the set of inner functions with respect to μ.

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Let $f \in L^\infty(\mu)$. We call

$$T_f \colon H^2(\mu) \to H^2(\mu),\ g \mapsto P_{H^2(\mu)}(fg),$$

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Theorem (Didas-Eschmeier-Everard, 2011)

An operator $X \in B(H^2(\mu))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_\mu$.

Definition

We denote by $H^\infty(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ the set of all bounded analytic functions on $\mathbb{D}$.

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Theorem

The map

$$r_m\colon H^\infty(\mathbb{D}) \to H^\infty(m),\ \theta \mapsto \tau_{w^*}\text{-}\lim_{r\to 1}[\theta(r\cdot)] =: \theta^*$$

is an isometric algebra isomorphism and weak homeomorphism with* $r_m(\theta|_{\mathbb{D}}) = [\theta|_{\mathbb{T}}]$ *for all* $\theta \in A(\mathbb{D})$.

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Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex or bounded symmetric and circled domain and $\mu \in M_1^+(S)$ be the probability measure on the Shilov boundary $S = \partial_{A(D)}$ obtained before.

Theorem (Didas-Eschmeier-S., 2017)

Let $p \in [1, \infty)$. An operator $X \in B(H^2(\mu))$ is a $\mathcal{S}_p(H^2(\mu))$ Toeplitz perturbation (i.e. there exist a $f \in L^\infty(\mu)$ and $S \in \mathcal{S}_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

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and $X \in B(H^2(\mu))$ *an operator such that*

$$Y = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{\alpha_k}^* X T_{\alpha_k} \in B(H^2(\mu))$$

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exists. If $T^*_u X T_u - X \in \mathcal{S}_\infty(H^2(\mu))$ *for all* $u \in I_\mu$*, then there exists a function* $f \in L^\infty(\mu)$ *such that*

$$X = T_f + \frac{1}{1-\alpha^2}(X - Y).$$

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Proposition (Hiai, 1997)

The map

$$\|\cdot\|_p : (B(H^2(\mu)), \tau_{\mathrm{WOT}}) \to [0,\infty], \; S \mapsto \|S\|_p$$

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$$\left\| \tau_{\text{WOT}}\text{-} \lim_{k\to\infty} X - T_{\alpha_k}^* X T_{\alpha_k} \right\|_p \leq \liminf_{k\to\infty} \left\| X - T_{\alpha_k}^* X T_{\alpha_k} \right\|_p$$

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Proposition (Aleksandrov, 1984)

Let $\alpha \in [0,1)$. Then there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in I_D such that

$$\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha.$$

Proposition (Xia, 2009)

Let $p \in [1, \infty]$. *Suppose that* $X \in B(H^2(\mu))$ *is an operator such that*

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

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Proposition (Xia, 2009)

Let $p \in [1, \infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_\mu$. Then, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\| T_{\theta^*}^* X T_{\theta^*} - X \|_p \leq \varepsilon$$

for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$.

Proof.

There exists $0 < \delta < 1$ such that $\left\| T_{\theta^*}^* X T_{\theta^*} - X \right\|_p \leq 1$ for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$.

Proof.

There exists $0 < \delta < 1$ such that $\left\| T_{\theta^*}^* X T_{\theta^*} - X \right\|_p \leq 1$ for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$.
$\implies$ There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with $\tau_{w^*}\text{-}\lim_{k \to \infty} \alpha_k^* = \alpha$.

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$\implies$ There exists $(\alpha_k)_{k\in\mathbb{N}}$ in I_D with $\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha$.
By passing to a subsequence we can achieve that $\left|\int_D 1 - \alpha_k^* \, \mathrm{d}\mu\right| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\mathrm{WOT}}\text{-}\lim_{k\to\infty} T_{\alpha_k^*}^* X T_{\alpha_k^*} \in B(H^2(\mu))$$

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exists. Hence

$$\left\| \tau_{\mathrm{WOT}}\text{-} \lim_{k\to\infty} X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \leq \liminf_{k\to\infty} \left\| X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \leq 1.$$

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$\implies$ There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with $\tau_{w^*}\text{-}\lim_{k\to\infty} \alpha_k^* = \alpha$.

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$\implies X - Y \in \mathcal{S}_p(H^2(\mu))$. $\square$

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The following ingredients are essential for the proof.

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(ii) The measure μ is a *faithful Henkin measure*.

Question

What about $p = 0, \infty$?

Theorem

An operator $X \in B(H^2(m))$ *is an analytic Toeplitz operator (i.e. there exists a* $f \in H^\infty(m)$ *such that* $X = T_f$*) if and only if*

$$[X, T_g] = XT_g - T_gX = 0.$$

for all $g \in H^\infty(m)$*.*

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Theorem (Davidson, 1977)

An operator $X \in B(H^2(m))$ *has the form* $X = T_f + K$ *with* $f \in H^\infty(m) + C(\mathbb{T})$ *and* $K \in K(H^2(m))$ *if and only if*

$$XT_g - T_gX \in K(H^2(m)).$$

for all $g \in H^\infty(m)$.

Theorem (Hartman, 1958)

We have

$$H^\infty(m) + C(\mathbb{T}) = \{f \in L^\infty(m)\ ;\ H_f \text{ is compact}\}$$

with $H_f = (1 - P_{H^2(m)})M_f|_{H^2(m)}$ *for* $f \in L^\infty(m)$.

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An operator $X \in B(H^2(m))$ *has the form* $X = T_f + K$ *with* $f \in L^\infty(m)$ *such that* H_f *is compact and* $K \in K(H^2(m))$ *if and only if*

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Theorem (Ding-Sun, 1997)

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + K$ for some $f \in L^\infty(\mu)$ such that H_f is compact and $K \in K(H^2(\mu))$ if and only if
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Theorem (Guo-Wang, 2006)

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + F$ for some $f \in L^\infty(\mu)$ such that H_f is of finite rank and $F \in F(H^2(\mu))$ if and only if

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for all $g \in H^\infty(\mu)$.

Let $n \geq 1$ and

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Theorem (Didas-Eschmeier-S., 2017)

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + S$ for some $f \in L^\infty(\mu)$ such that H_f is in the Schatten-p-class and $S \in \mathcal{S}_p(H^2(\mu))$ if and only if

$$XT_g - T_g X \in \mathcal{S}_p(H^2(\mu))$$

for all $g \in H^\infty(\mu)$.