K-contractions

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Contractions

Let $\mathcal{H}$ be a complex Hilbert space.

Lemma

Let $T \in B(\mathcal{H})$. The following assertions are equivalent:

1. *T is a contraction (i.e., $\|T\| \leq 1$),*
2. *$1/K(T, T^*) \geq 0$, where*

$$K\colon \mathbb{D} \times \mathbb{D} \to \mathbb{C}, \ (z, w) \mapsto \frac{1}{1 - z\overline{w}} = \frac{1}{1 - \langle z, w \rangle}$$

is the reproducing kernel of the Hardy space on the unit disc $H^2(\mathbb{D})$.

Example

1. The shift operator $M_z \in B(H^2(\mathbb{D}))$ satisfies
$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \geq 0.$$
2. Every unitary $U \in B(\mathcal{H})$ fulfills
$$\frac{1}{K}(U, U^*) = 0.$$

Definition

We say that a contraction $T \in B(\mathcal{H})$ belongs to the class $C_{\cdot 0}$ or is *pure* if

$$T_\infty = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} T^N T^{*N} = 0.$$

Example

The shift operator $M_z \in B(H^2(\mathbb{D}))$ belongs to $C_{\cdot 0}$.

Theorem

Let $T \in B(\mathcal{H})$ be a contraction. Then

$$\pi_T \colon \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D}_T), \; h \mapsto \sum_{k=0}^{\infty} (D_T T^{*k} h) z^k,$$

where $D_T = (1 - TT^*)^{1/2} = (1/K(T, T^*))^{1/2}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle T_\infty h, h \rangle$$

for all $h \in \mathcal{H}$, and

$$\pi_T T^* = (M_z^{\mathcal{D}_T})^* \pi_T.$$

Corollary

A contraction $T \in B(\mathcal{H})$ is in $C_{\cdot 0}$ if and only if π_T is an isometry.

The $C_{\cdot 0}$ case

Corollary

Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:

1. *T is a contraction which belongs to $C_{\cdot 0}$,*
2. *there exist a Hilbert space $\mathcal{D}$, and an isometry $\pi \colon \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D})$ such that*
$$\pi T^* = (M_z^{\mathcal{D}})^* \pi.$$

A Beurling type theorem

Remark

If $T \in B(\mathcal{H})$ is a $C_{\cdot 0}$ contraction and $\mathcal{S} \in \mathrm{Lat}(T)$, then $T|_{\mathcal{S}}$ is also $C_{\cdot 0}$ contraction.

Theorem

Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H^2(\mathbb{D}, \mathcal{E})$, the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}(M_z^{\mathcal{E}})$,
2. *there exist a Hilbert space $\mathcal{D}$, and a bounded analytic function $\theta \colon \mathbb{D} \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta \colon H^2(\mathbb{D}, \mathcal{D}) \to H^2(\mathbb{D}, \mathcal{E}), \; f \mapsto \theta f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

The general case

Theorem

Let $T \in B(\mathcal{H})$ be an operator and write $H_K = H^2(\mathbb{D})$. The following statements are equivalent:

1. $1/K(T, T^*) \geq 0$,
2. *there exist Hilbert spaces $\mathcal{D}$ and $\mathcal{K}$, a unitary operator $U \in B(\mathcal{K})$, and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T^* = (M_z^{\mathcal{D}} \oplus U)^* \Pi.$$

Question

For which reproducing kernels K does an analogue theorem hold?
What happens if we look at commuting tuples $T = (T_1, \ldots, T_n) \in B(\mathcal{H})^n$?

Unitarily invariant spaces on $\mathbb{B}_n$

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers with $a_0 = 1$ and such that

$$k(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (z \in \mathbb{D})$$

defines a holomorphic function with radius of convergence at least 1 and no zeros in the unit disc $\mathbb{D}$. The map

$$K\colon \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{C},\ (z,w) \mapsto \sum_{k=0}^{\infty} a_k \langle z, w\rangle^k$$

defines a semianalytic positive definite function and hence, there exists a reproducing kernel Hilbert space $H_K \subset \mathcal{O}(\mathbb{B}_n)$ with kernel K. The space H_K is a so called *unitarily invariant space on* $\mathbb{B}_n$.

Furthermore, we assume $\sup_{k\in\mathbb{N}} a_k/a_{k+1} < \infty$ such that the K-shift $M_z = (M_{z_1}, \dots, M_{z_n}) \in B(H_K)^n$ is well-defined.
Since k has no zeros in $\mathbb{D}$, the function

$$\frac{1}{k} : \mathbb{D} \to \mathbb{C},\ z \mapsto \frac{1}{k(z)}$$

is again holomorphic and hence admits a Taylor expansion

$$\frac{1}{k}(z) = \sum_{k=0}^{\infty} c_k z^k \qquad (z \in \mathbb{D})$$

with a suitable sequence $(c_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$.

Example

1. If $a_k = 1$ for all $k \in \mathbb{N}$, then H_K is the Hardy space ($n = 1$) or the Drury-Arveson space ($n \geq 2$).
2. If $\nu > 0$ and $a_k = a_k^{(\nu)} = (-1)^k \binom{-\nu}{k}$ for all $k \in \mathbb{N}$, then

$$K(z, w) = K^{(\nu)}(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\nu} \qquad (z, w \in \mathbb{B}_n),$$

i.e., $H_{K^{(\nu)}}$ is a weighted Bergman space.

3. The space H_K is a *complete Nevanlinna-Pick space* if and only if

$$c_k \leq 0$$

for all $k \geq 1$.

Definition

Let $T = (T_1, \ldots, T_n) \in B(\mathcal{H})^n$ be a commuting tuple. Define

$$\left(\frac{1}{K}\right)_N (T, T^*) = \sum_{k=0}^{N} c_k \sigma_T^k(1)$$

for all $N \in \mathbb{N}$, where

$$\sigma_T \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \sum_{i=1}^{n} T_i X T_i^*.$$

Furthermore, we write

$$\frac{1}{K}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \left(\frac{1}{K}\right)_N (T, T^*)$$

if the latter exists.

Definition

Let $T \in B(\mathcal{H})^n$ be a commuting tuple.

1. We call T a *K-contraction* if $1/K(T, T^*) \geq 0$.
2. We call T a *spherical unitary* if T satisfies $\sigma_T(1) = 1$ and consists of normal operators.

Example

1. If $n = 1$, a $K^{(1)}$-contraction is a contraction.
2. If $n \geq 2$, a $K^{(1)}$-contraction is a row contraction.
3. Let $m \in \mathbb{N}^*$. We call a commuting tuple $T \in B(\mathcal{H})^n$ an *m-hypercontraction* if T is a $K^{(\ell)}$-contraction for all $\ell = 1, \ldots, m$.

Proposition (Chen, 2012)

If there exists a $p \in \mathbb{N}$ such that

$$c_k \geq 0 \text{ for all } k \geq p \quad \text{or} \quad c_k \leq 0 \text{ for all } k \geq p$$

holds, then

$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}}.$$

Example

The condition above is satfisfied in the case when H_K is a

1. weighted Bergman space,
2. complete Nevanlinna-Pick space.

From now on, we assume that the condition in the last proposition holds.

Definition

Let $T \in B(\mathcal{H})^n$ be a K-contraction. We define

$$\Sigma_N(T) = 1 - \sum_{k=0}^{N} a_k \sigma_T^k \left(\frac{1}{K}(T, T^*) \right)$$

for $N \in \mathbb{N}$ and write

$$\Sigma(T) = \tau_{\mathrm{SOT}}\text{-} \lim_{N \to \infty} \Sigma_N(T)$$

if the latter exists. If $\Sigma(T) = 0$, we call T K-pure.

Remark

If $K = K^{(1)}$ and $T \in B(\mathcal{H})^n$ is a $K^{(1)}$-contraction, then

$$\Sigma_N(T) = \sigma_T^{N+1}(1)$$

for all $N \in \mathbb{N}$, and hence,

$$\Sigma(T) = T_\infty \geq 0.$$

Proposition

Let $T \in B(\mathcal{H})^n$ be a K-contraction such that $\Sigma(T)$ exists. The map

$$\pi_T \colon \mathcal{H} \to H_K(\mathcal{D}_T),\ h \mapsto \sum_{\alpha \in \mathbb{N}^n} \left(a_{|\alpha|} \frac{|\alpha|!}{\alpha!} D_T T^{*\alpha} h \right) z^\alpha,$$

where $D_T = (1/K(T,T^*))^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle \Sigma(T)h, h \rangle$$

for all $h \in \mathcal{H}$ and

$$\pi_T T_i^* = (M_{z_i}^{\mathcal{D}_T})^* \pi_T$$

for all $i = 1, \ldots, n$.

The pure case

Theorem (Eschmeier, S.)

Let $T \in B(\mathcal{H})^n$ be a commuting tuple. The following statements are equivalent:

1. *T is K-pure,*
2. *there exist a Hilbert space $\mathcal{D}$ and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D})$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}})^* \Pi$$

for all $i = 1, \ldots, n$.

A Beurling type theorem

Theorem (Eschmeier, S.)

Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H_K(\mathcal{E})$, the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$ *and* $M_z^{\mathcal{E}}|_{\mathcal{S}}$ *is* K*-pure,*
2. *there exist a Hilbert space* $\mathcal{D}$ *and a bounded analytic function* $\theta\colon \mathbb{B}_n \to B(\mathcal{D}, \mathcal{E})$ *such that*

$$M_\theta\colon H_K(\mathcal{D}) \to H_K(\mathcal{E}),\ f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

The general case

Definition

We call a K-contraction $T \in B(\mathcal{H})^n$ *strong* if $\Sigma(T) \geq 0$ and $\Sigma(T) = \sigma_T(\Sigma(T))$ holds.

Remark

1. Every K-pure K-contraction is a strong K-contraction. Hence, the K-shift $M_z \in B(H_K)^n$ is a strong K-contraction.
2. Every spherical unitary is a strong K-contraction.

Theorem (Eschmeier, S.)

Let $T \in B(\mathcal{H})^n$ be a commuting tuple. The following statements are equivalent:

- *T is a strong K-contraction,*
- *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^n$, and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \quad (i = 1, \ldots, n).$$

If we assume that H_K is regular, i.e., $\lim_{k\to\infty} a_k/a_{k+1} = 1$, then the above are also equivalent to

- *there is a unital completely contractive linear map $\rho\colon \operatorname{span}\left\{1, M_{z_i}, M_{z_i}M_{z_i}^* \;;\; i = 1, \ldots, n\right\} \to B(\mathcal{H})$ with*

$$\rho(M_{z_i}) = T_i, \; \rho(M_{z_i}M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, n).$$