# $K$-contractions

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# Contractions

Let $\mathcal{H}$ be a complex Hilbert space.

**Lemma**

*Let $T \in B(\mathcal{H})$. The following assertions are equivalent:*

1. *$T$ is a contraction (i.e., $\|T\| \leq 1$),*
2. *$1/K(T, T^*) \geq 0$, where*

$$K \colon \mathbb{D} \times \mathbb{D} \to \mathbb{C}, \; (z, w) \mapsto \frac{1}{1 - z\overline{w}} = \frac{1}{1 - \langle z, w \rangle}$$

*is the reproducing kernel of the Hardy space on the unit disc $H^2(\mathbb{D})$.*

## Example

1. The shift operator $M_z \in B(H^2(\mathbb{D}))$ satisfies
$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \geq 0.$$
2. Every unitary $U \in B(\mathcal{H})$ fulfills
$$\frac{1}{K}(U, U^*) = 0.$$

## Lemma

1. *If $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are contractions, then $T \oplus S$ is a contraction.*
2. *If $T \in B(\mathcal{H})$ is a contraction and $\mathcal{M} \subset \mathcal{H}$ is a subspace, then $T|_{\mathcal{M}}$ is a contraction.*

**Definition**

We say that a contraction $T \in B(\mathcal{H})$ belongs to the class $C_{\cdot 0}$ or is *pure* if

$$T_\infty = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} T^N T^{*N} = 0.$$

**Example**

The shift operator $M_z \in B(H^2(\mathbb{D}))$ belongs to $C_{\cdot 0}$.

## Theorem

*Let $T \in B(\mathcal{H})$ be a contraction. Then*

$$\pi_T \colon \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D}_T), \; h \mapsto \sum_{n=0}^{\infty} (D_T T^{*n} h) z^n,$$

*where $D_T = (1 - TT^*)^{1/2} = (1/K(T, T^*))^{1/2}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have*

$$\|\pi_T h\|^2 = \|h\|^2 - \langle T_\infty h, h \rangle$$

*for all $h \in \mathcal{H}$, and*

$$\pi_T T^* = (M_z^{\mathcal{D}_T})^* \pi_T.$$

## Corollary

*A contraction $T \in B(\mathcal{H})$ is in $C_{\cdot 0}$ if and only if $\pi_T$ is an isometry.*

# The $C_{\cdot 0}$ case

## Corollary

*Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:*

1. *$T$ is a contraction which belongs to $C_{\cdot 0}$,*
2. *there exist a Hilbert space $\mathcal{D}$, and an isometry $\pi \colon \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D})$ such that*

$$\pi T^* = (M_z^{\mathcal{D}})^* \pi.$$

# A Beurling-type theorem

**Remark**

If $T \in B(\mathcal{H})$ is a $C_{\cdot 0}$-contraction and $\mathcal{S} \in \mathrm{Lat}(T)$, then $T|_{\mathcal{S}}$ is also $C_{\cdot 0}$-contraction.

**Theorem**

*Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H^2(\mathbb{D}, \mathcal{E})$, the following statements are equivalent:*

1. $\mathcal{S} \in \mathrm{Lat}(M_z^{\mathcal{E}})$,
2. *there exist a Hilbert space $\mathcal{D}$, and a bounded analytic function $\theta \colon \mathbb{D} \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta \colon H^2(\mathbb{D}, \mathcal{D}) \to H^2(\mathbb{D}, \mathcal{E}), \; f \mapsto \theta f$$

*is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.*

# The general case

## Theorem

*Let $T \in B(\mathcal{H})$ be an operator and write $H_K = H^2(\mathbb{D})$. The following statements are equivalent:*

1. $1/K(T, T^*) \geq 0$,
2. *there exist Hilbert spaces $\mathcal{D}$ and $\mathcal{K}$, a unitary operator $U \in B(\mathcal{K})$, and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T^* = (M_z^{\mathcal{D}} \oplus U)^* \Pi.$$

## Question

For which reproducing kernels $K$ does an analogue theorem hold?
What happens if we look at commuting tuples $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$?

# Unitarily invariant spaces on $\mathbb{B}_d$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with $a_0 = 1$ and such that

$$k(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{D})$$

defines a holomorphic function with radius of convergence at least 1 and no zeros in the unit disc $\mathbb{D}$. The map

$$K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z,w) \mapsto \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n$$

defines a semianalytic positive definite function and hence, there exists a reproducing kernel Hilbert space $H_K \subset \mathcal{O}(\mathbb{B}_d)$ with kernel $K$. The space $H_K$ is a so called *unitarily invariant space on* $\mathbb{B}_d$.

Furthermore, we suppose that $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$ such that the $K$-shift $M_z = (M_{z_1}, \dots, M_{z_d}) \in B(H_K)^d$ is well-defined.
Since $k$ has no zeros in $\mathbb{D}$, the function

$$\frac{1}{k} : \mathbb{D} \to \mathbb{C}, \ z \mapsto \frac{1}{k(z)}$$

is again holomorphic and hence admits a Taylor expansion

$$\frac{1}{k}(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (z \in \mathbb{D})$$

with a suitable sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

## Example

1. If $a_n = 1$ for all $n \in \mathbb{N}$, then $H_K$ is the Hardy space $(d = 1)$ or the *Drury-Arveson space* $(d \geq 2)$.
2. If $\nu > 0$ and $a_n = a_n^{(\nu)} = (-1)^n \binom{-\nu}{n}$ for all $n \in \mathbb{N}$, then
$$K(z, w) = K^{(\nu)}(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{\nu}} \qquad (z, w \in \mathbb{B}_d),$$
i.e., $H_{K^{(\nu)}}$ is a *weighted Bergman space*.
3. The space $H_K$ is a *complete Nevanlinna-Pick space* if and only if
$$c_n \leq 0$$
for all $n \geq 1$.

## Definition

Let $T = (T_1, \dots, T_d) \in B(\mathcal{H})^d$ be a commuting tuple. Define

$$\left(\frac{1}{K}\right)_N (T, T^*) = \sum_{n=0}^{N} c_n \sigma_T^k(1)$$

for all $N \in \mathbb{N}$, where

$$\sigma_T \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \sum_{i=1}^{d} T_i X T_i^*.$$

Furthermore, we write

$$\frac{1}{K}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \left(\frac{1}{K}\right)_N (T, T^*)$$

if the latter exists.

## Definition

Let $T \in B(\mathcal{H})^d$ be a commuting tuple.

1. We call $T$ a *$K$-contraction* if $1/K(T, T^*) \geq 0$.
2. We call $T$ a *spherical unitary* if $T$ satisfies $\sigma_T(1) = 1$ and consists of normal operators.

## Example

1. If $d = 1$, a $K^{(1)}$-contraction is a contraction.
2. If $d \geq 2$, a $K^{(1)}$-contraction is a row contraction.
3. Let $m \in \mathbb{N}^*$. We call a commuting tuple $T \in B(\mathcal{H})^d$ an *$m$-hypercontraction* if $T$ is a $K^{(\ell)}$-contraction for all $\ell = 1, \ldots, m$.

## Proposition (Chen, 2012)

*If there exists a natural number $p \in \mathbb{N}$ such that*

$$c_n \geq 0 \text{ for all } n \geq p \quad \text{or} \quad c_n \leq 0 \text{ for all } n \geq p$$

*holds, then*

$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

## Example

The condition above is satisfied in the case when $H_K$ is a

1. weighted Bergman space,
2. complete Nevanlinna-Pick space.

For the rest of this section, we suppose that the condition in the last proposition holds.

## Definition

Let $T \in B(\mathcal{H})^d$ be a $K$-contraction. We define

$$\Sigma_N(T) = 1 - \sum_{n=0}^{N} a_n \sigma_T^n \left( \frac{1}{K}(T, T^*) \right)$$

for $N \in \mathbb{N}$ and write

$$\Sigma(T) = \tau_{\mathrm{SOT}}\text{-} \lim_{N \to \infty} \Sigma_N(T)$$

if the latter exists. If $\Sigma(T) = 0$, we call $T$ $K$-pure.

## Remark

If $K = K^{(1)}$ and $T \in B(\mathcal{H})^d$ is a $K^{(1)}$-contraction, then

$$\Sigma_N(T) = \sigma_T^{N+1}(1)$$

for all $N \in \mathbb{N}$, and hence,

$$\Sigma(T) = T_\infty \geq 0.$$

## Proposition

*Let $T \in B(\mathcal{H})^d$ be a $K$-contraction such that $\Sigma(T)$ exists. The map*

$$\pi_T \colon \mathcal{H} \to H_K(\mathcal{D}_T), \ h \mapsto \sum_{\alpha \in \mathbb{N}^d} \left( a_{|\alpha|} \frac{|\alpha|!}{\alpha!} D_T T^{*\alpha} h \right) z^\alpha,$$

*where $D_T = (1/K(T, T^*))^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have*

$$\|\pi_T h\|^2 = \|h\|^2 - \langle \Sigma(T) h, h \rangle$$

*for all $h \in \mathcal{H}$ and*

$$\pi_T T_i^* = (M_{z_i}^{\mathcal{D}_T})^* \pi_T$$

*for all $i = 1, \ldots, d$.*

# The pure case

**Theorem (Eschmeier, S.)**

*Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

1. *$T$ is a $K$-contraction which is $K$-pure,*
2. *there exist a Hilbert space $\mathcal{D}$ and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D})$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}})^* \Pi$$

*for all $i = 1, \ldots, d$.*

# A Beurling-type theorem

**Theorem (Eschmeier, S.)**

*Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H_K(\mathcal{E})$, the following statements are equivalent:*

1. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$ *and* $M_z^{\mathcal{E}}|_{\mathcal{S}}$ *is $K$-pure,*
2. *there exist a Hilbert space $\mathcal{D}$ and a bounded analytic function $\theta\colon \mathbb{B}_d \to B(\mathcal{D}, \mathcal{E})$ such that*
$$M_\theta\colon H_K(\mathcal{D}) \to H_K(\mathcal{E}),\ f \mapsto \theta \cdot f$$
*is a partial isometry with* $\mathrm{Im}(M_\theta) = \mathcal{S}$.

# The general case

## Definition

We call a $K$-contraction $T \in B(\mathcal{H})^d$ *strong* if $\Sigma(T) \geq 0$ and $\Sigma(T) = \sigma_T(\Sigma(T))$ holds.

## Remark

1. If a $K$-contraction is $K$-pure, then it is also strong. Hence, the $K$-shift $M_z \in B(H_K)^d$ is a strong $K$-contraction.
2. Every spherical unitary is a strong $K$-contraction.

## Theorem (Eschmeier, S.)

*Let $T \in B(\mathcal{H})^d$ be a commuting tuple. The following statements are equivalent:*

- *$T$ is a strong $K$-contraction,*
- *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \quad (i = 1, \dots, d).$$

*If we assume that $H_K$ is regular, i.e., $\lim_{n\to\infty} a_n/a_{n+1} = 1$, then the above are also equivalent to*

- *there is a unital completely contractive linear map $\rho \colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i}M_{z_i}^* \; ; \; i = 1, \dots, d\right\} \to B(\mathcal{H})$ with*

$$\rho(M_{z_i}) = T_i, \; \rho(M_{z_i}M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \dots, d).$$

# Radial $K$-hypercontractions

## Definition

We call a commuting tuple $T \in B(\mathcal{H})^d$ with $\sigma(T) \subset \overline{\mathbb{B}}_d$ a *radial $K$-hypercontraction* if, for all $0 < r < 1$, $rT \in B(\mathcal{H})^d$ is a $K$-contraction.

## Example

Spherical unitaries are radial $K$-hypercontractions.

## Remark

We define

$$k_r : \mathbb{D} \to \mathbb{C}, \ z \mapsto k(rz) \qquad (r \in [0,1]).$$

For $r, s \in [0,1]$, the function $k_s / k_r$ has a Taylor expansion on $\mathbb{D}$

$$\sum_{n=0}^{\infty} a_n(s,r) z^n.$$

Note that

$$a_n(1,0) = a_n \quad \text{and} \quad a_n(0,1) = c_n \qquad (n \in \mathbb{N}).$$

## Remark (Guo-Hu-Xu, 2004)

If $H_K$ is regular, then $\sigma(M_z) \subset \overline{\mathbb{B}}_d$.

From now on, we suppose that $H_K$ is regular.

## Proposition (Olofsson, 2015)

*The $K$-shift $M_z \in B(H_K)^d$ is a radial $K$-hypercontraction if and only if*

$$a_n(1, r) \geq 0$$

*for all $n \in \mathbb{N}$ and $0 < r < 1$.*

## Remark (Olofsson, 2015)

If $H_K$ is a complete Nevanlinna-Pick space or a generalized Bergman space, then $M_z$ is a radial $K$-hypercontraction.

## Proposition (Olofsson, 2015)

*Suppose that $M_z$ is a radial $K$-hypercontraction. Let $T \in B(\mathcal{H})^d$ be a radial $K$-hypercontraction. Then*

$$\frac{1}{K}_{\mathrm{rad}}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{r \to 1} \frac{1}{K}(rT, rT^*)$$

*exists and defines a positive operator.*

## Corollary

*If $M_z$ is a radial $K$-hypercontraction, then*

$$\frac{1}{K}_{\mathrm{rad}}(M_z, M_z^*) = P_{\mathbb{C}}.$$

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## Theorem (Eschmeier, S.)

*Suppose that* $M_z \in B(H_K)^d$ *is a radial* $K$*-hypercontraction. Let* $T \in B(\mathcal{H})^d$ *be a commuting tuple. The following assertions are equivalent:*

1. $T$ *is a radial* $K$*-hypercontraction,*
2. *there exist Hilbert spaces* $\mathcal{D}, \mathcal{K}$*, a spherical unitary* $U \in B(\mathcal{K})^d$*, and an isometry* $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ *such that*
$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \qquad (i = 1, \ldots, d),$$
3. *there is a unital completely contractive linear map* $\rho \colon S \to B(\mathcal{H})$ *on the operator space* $S = \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i} M_{z_i}^* \ ; \ i = 1, \ldots, d\right\}$ *with*
$$\rho(M_{z_i}) = T_i, \ \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, d).$$