K-contractions

Dominik Schillo

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Contractions

Let $\mathcal{H}$ be a Hilbert space.

Definition

Let $T \in B(\mathcal{H})$. We call T a *contraction* if $\|T\| \leq 1$.

Lemma

Let $T \in B(\mathcal{H})$ and define

$$\sigma_T : B(\mathcal{H}) \to B(\mathcal{H}), \ X \mapsto TXT^*.$$

The following assertions are equivalent:

1. *T is a contraction,*
2. *T^* is a contraction,*
3. $1 - TT^* \geq 0$,
4. $1 - \sigma_T(1) \geq 0$.

Hardy space

The function

$$K\colon \mathbb{D}\times\mathbb{D}\to\mathbb{C},\ (z,w)\mapsto \frac{1}{1-z\overline{w}}=\frac{1}{1-\langle z,w\rangle}$$

is the reproducing kernel of the Hardy space on the unit disc, i.e., we have $K(\cdot,w)\in H^2(\mathbb{D})$ and

$$\langle f, K(\cdot,w)\rangle = f(w)$$

for all $w\in\mathbb{D}$ and $f\in H^2(\mathbb{D})$, where

$$H^2(\mathbb{D})=\left\{f=\sum_{k=0}^{\infty} f_k z^k\in\mathcal{O}(\mathbb{D})\ ;\ \|f\|^2=\sum_{k=0}^{\infty}|f_k|^2<\infty\right\}$$

and, for $f=\sum_{k=0}^{\infty} f_k z^k,\ g=\sum_{k=0}^{\infty} g_k z^k\in H^2(\mathbb{D})$,

$$\langle f,g\rangle=\sum_{k=0}^{\infty} f_k\overline{g_k}.$$

Furthermore, we have

$$K(z,w) = \sum_{k=0}^{\infty} \langle z, w \rangle^k$$

and

$$\frac{1}{K}(z,w) = 1 - \langle z, w \rangle$$

for all $z, w \in \mathbb{D}$.

Proposition

An operator $T \in B(\mathcal{H})$ is a contraction if and only if

$$\frac{1}{K}(T) = \frac{1}{K}(T,T) = 1 - \sigma_T(1) \geq 0.$$

Proposition

The map

$$\varphi\colon \ell^2(\mathbb{N}) \to H^2(\mathbb{D}),\ (a_k)_{k\in\mathbb{N}} \mapsto \sum_{k=0}^{\infty} a_k z^k$$

is an Hilbert space isomorphism.

Let $S \in B(\ell^2(\mathbb{N}))$ be the right shift on $\ell^2(\mathbb{N})$. The operator $M_z \in B(H^2(\mathbb{D}))$ defined by

$$M_z = \varphi S \varphi^*$$

satisfies

$$(M_z f)(z) = zf(z)$$

for $f \in H^2(\mathbb{D})$ and $z \in \mathbb{D}$, i.e., M_z is the multiplication operator on $H^2(\mathbb{D})$ with symbol z. We call M_z the *shift operator on* $H^2(\mathbb{D})$.

Lemma

The following statements hold:

1. *The shift operator* $M_z \in B(H^2(\mathbb{D}))$ *satisfies*
$$\frac{1}{K}(M_z) = P_{\mathbb{C}} \geq 0.$$
2. *Every coisometry* $V \in B(\mathcal{H})$ *satisfies*
$$\frac{1}{K}(V) = 0.$$
In particular, every unitary $U \in B(\mathcal{H})$ *fulfills*
$$\frac{1}{K}(U) = 0.$$

Lemma

Let $T \in B(\mathcal{H})$ be a contraction. Then

$$T_\infty = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \sigma_T^N(1) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} T^N T^{*N}$$

exists and defines a positive operator.

Definition

We say that a contraction $T \in B(\mathcal{H})$ belongs to the class $C_{\cdot 0}$ or is *pure* if

$$T_\infty = 0.$$

Example

The shift operator $M_z \in B(H^2(\mathbb{D}))$ belongs to $C_{\cdot 0}$.

Theorem

Let $T \in B(\mathcal{H})$ be a contraction. Then

$$\pi_T : \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{D}_T, \ h \mapsto \sum_{k=0}^{\infty} z^k \otimes D_T T^{*k} h,$$

where $D_T = (1 - TT^*)^{1/2} = (1/K(T))^{1/2}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle T_\infty h, h \rangle = \|h\|^2 - \left\| T_\infty^{1/2} h \right\|^2$$

for all $h \in \mathcal{H}$, and

$$\pi_T T^* = (M_z \otimes 1_{\mathcal{D}_T})^* \pi_T .$$

The $C_{\cdot 0}$ case

Corollary

A contraction $T \in B(\mathcal{H})$ is in $C_{\cdot 0}$ if and only if π_T is an isometry.

Corollary

Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:

1. *T is a contraction which belongs to $C_{\cdot 0}$,*
2. *there exist a Hilbert space $\mathcal{D}$, and an isometry $\pi \colon \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{D}$ such that*

$$\pi T^* = (M_z \otimes 1_{\mathcal{D}})^* \pi.$$

Beurling's theorem

Remark

If $T \in B(\mathcal{H})$ is a $C_{\cdot 0}$ contraction and $\mathcal{S} \in \mathrm{Lat}(T)$, then $T|_{\mathcal{S}}$ is also $C_{\cdot 0}$ contraction.

Lemma

Let $T \in B(\mathcal{H})$ be a $C_{\cdot 0}$ contraction and $\mathcal{S} \subset \mathcal{H}$. The following assertions are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}(T)$,
2. *there exist a Hilbert space $\mathcal{D}$, and a partial isometry $\psi \colon H^2(\mathbb{D}) \otimes \mathcal{D} \to \mathcal{H}$ with*

$$T\psi = \psi(M_z \otimes 1_{\mathcal{D}})$$

and $\mathrm{Im}(\psi) = \mathcal{S}$.

Theorem (Beurling)

Let $\mathcal{S} \subset H^2(\mathbb{D})$. The following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}(M_z)$,
2. *there exist a Hilbert space $\mathcal{D}$, and an analytic function $\theta \colon \mathbb{D} \to B(\mathcal{D}, \mathbb{C})$ such that*

$$M_\theta \colon H^2(\mathbb{D}) \otimes \mathcal{D} \to H^2(\mathbb{D}),\; f \mapsto \theta f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

The general case

Lemma

Let $T \in B(\mathcal{H})$ be a contraction. Then there exist a Hilbert space $\mathcal{K}_T$ with $T_\infty^{1/2}\mathcal{H} \subset \mathcal{K}_T$ and an unitary operator $U_T \in B(\mathcal{K}_T)$ such that

$$T_\infty^{1/2} T^* = U_T^* T_\infty^{1/2}.$$

Furthermore, $\mathcal{K}_T$ and U_T can be chosen such that

$$\mathcal{K}_T = \bigvee \left\{ U_T^k T_\infty^{1/2} h \ ; \ k \in \mathbb{N} \text{ and } h \in \mathcal{H} \right\}.$$

Remark

Let $T \in B(\mathcal{H})$ be a contraction. The operator

$$\Pi_T \colon \mathcal{H} \to (H^2(\mathbb{D}) \otimes \mathcal{D}_T) \oplus \mathcal{K}_T, \; h \mapsto \pi_T h \oplus T_\infty^{1/2} h$$

is an isometry which satisfies

$$\Pi_T T^* = ((M_z \otimes 1_{\mathcal{D}_T}) \oplus U_T)^* \Pi_T.$$

Theorem

Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:

1. *T is a contraction,*
2. *there exist Hilbert spaces $\mathcal{D}$ and $\mathcal{K}$, an unitary operator $U \in B(\mathcal{K})$, and an isometry $\Pi \colon \mathcal{H} \to (H^2(\mathbb{D}) \otimes \mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T^* = ((M_z \otimes 1_{\mathcal{D}}) \oplus U)^* \Pi.$$

If we use the notation $H_K = H^2(\mathbb{D})$, we can reformulate the last theorem.

Theorem

Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent.

1. $1/K(T) \geq 0$.
2. *There exist Hilbert spaces $\mathcal{D}$ and $\mathcal{K}$, an unitary operator $U \in B(\mathcal{K})$, and an isometry $\Pi \colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T^* = ((M_z \otimes 1_{\mathcal{D}}) \oplus U)^* \Pi.$$

Question

For which reproducing kernels K does the theorem above hold?
What happens if we look at commuting tuples $T = (T_1, \ldots, T_n) \in B(\mathcal{H})^n$?

Unitarily invariant spaces on $\mathbb{B}_n$

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers with $a_0 = 1$ and such that

$$k(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (z \in \mathbb{D})$$

defines a holomorphic function with radius of convergence at least 1 and no zeros in the unit disc $\mathbb{D}$. The map

$$K\colon \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{C},\ (z,w) \mapsto \sum_{k=0}^{\infty} a_k \langle z, w\rangle^k$$

defines a semianalytic positive definite function and hence, there exists a repdroducing kernel Hilbert space $H_K \subset \mathcal{O}(\mathbb{B}_n)$ with kernel K. The space H_K is a so called *unitarily invariant space on* $\mathbb{B}_n$.

We have that

$$K(z,w) = \sum_{\alpha \in \mathbb{N}^n} a_{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha$$

for all $z, w \in \mathbb{B}_n$. Furthermore, one can show that

$$H_K = \left\{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}_n) \; ; \; \|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{a_{|\alpha|}} \frac{\alpha!}{|\alpha|!} |f_\alpha|^2 < \infty \right\}.$$

Since k has no zeros in $\mathbb{D}$, the function

$$\frac{1}{k} : \mathbb{D} \to \mathbb{C}, \; z \mapsto \frac{1}{k(z)}$$

is again holomorphic and hence admits a Taylor expansion

$$\frac{1}{k}(z) = \sum_{k=0}^{\infty} c_k z^k \qquad (z \in \mathbb{D})$$

with a suitable sequence $(c_k)_{k \in \mathbb{N}}$ in $\mathbb{R}$.

Example

1. If $a_k = 1$ for all $k \in \mathbb{N}$, then H_K is the Hardy space ($n = 1$) or the Drury-Arveson space ($n \geq 2$).
2. If $\nu > 0$ and $a_k = a_k^{(\nu)} = (-1)^k \binom{-\nu}{k}$ for all $k \in \mathbb{N}$, then

$$K(z, w) = K^{(\nu)}(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{\nu}} \qquad (z, w \in \mathbb{B}_n),$$

   i.e., $H_{K^{(\nu)}}$ is a weighted Bergman space.
3. The space H_K is an *irreducible complete Nevanlinna-Pick space* if and only if

$$c_k \leq 0$$

   for all $k \geq 1$.

Definition

Let $T = (T_1, \dots, T_n) \in B(\mathcal{H})^n$ be a commuting tuple. Define

$$\sigma_T \colon B(\mathcal{H}) \to B(\mathcal{H}),\ X \mapsto \sum_{i=1}^{n} T_i X T_i^*$$

and

$$\left(\frac{1}{K}\right)_N (T) = \sum_{k=0}^{N} c_k \sigma_T^k(1)$$

for all $N \in \mathbb{N}$. Furthermore, we write

$$\frac{1}{K}(T) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \left(\frac{1}{K}\right)_N (T)$$

if the latter exists.

Definition

Let $T \in B(\mathcal{H})^n$ be a commuting tuple.

1. We call T a *K-contraction* if $1/K(T) \geq 0$.
2. We call T a *row contraction* if T is $K^{(1)}$-contraction, i.e.,
$$\frac{1}{K^{(1)}}(T) = 1 - \sigma_T(1) \geq 0.$$
3. We call T a *row unitary or spherical unitary* if T is a *row isometry* (i.e. $\sigma_T(1) = 1$) and consists of normal operators.

Remark

If $\sum_{k=0}^{\infty} c_k$ is absolutely convergent (e.g. if $K = K^{(\nu)}$ for $\nu > 0$) and $T \in B(\mathcal{H})^n$ is a row contraction, then
$$\frac{1}{K}(T) = \tau_{\|\cdot\|}\text{-}\sum_{k=0}^{\infty} c_k \sigma_T^k(1).$$

Example

1. If $n = 1$, a row contraction is a contraction.
2. Let $m \in \mathbb{N}^*$. We call a commuting tuple $T \in B(\mathcal{H})^n$ an *m-hypercontraction* if and only if T is a row contraction as well as a $K^{(m)}$-contraction.

Definition

Let $i \in \{1, \dots, n\}$. We define

$$(M_{z_i} f)(z) = z_i f(z) \qquad (f \in H_K,\ z \in \mathbb{B}_n).$$

Then $M_{z_i} : H_K \to H_K$ is a well-defined bounded operator on H_K if and only if $\sup_{k \in \mathbb{N}} \frac{a_k}{a_{k+1}} < \infty$. From now on, we shall assume that this condition holds.

We call the commuting tuple $M_z = (M_{z_1}, \dots, M_{z_n}) \in B(H_K)^n$ the *K-shift* on H_K.

Remark

The K-shift $M_z \in B(H_K)^n$ is a row contraction if and only if $a_k \leq a_{k+1}$ for all $k \in \mathbb{N}$.

Example

1. If $\nu \geq 1$, then $M_z \in B(H_{K^{(\nu)}})^n$ is a row contraction.
2. If $0 < \nu < 1$, then $M_z \in B(H_{K^{(\nu)}})^n$ is *not* a row contraction.

Definition

Let $T \in B(\mathcal{H})^n$ be a K-contraction. We define

$$\Sigma_N(T) = 1 - \sum_{k=0}^{N} a_k \sigma_T^k \left(\frac{1}{K}(T) \right)$$

for $N \in \mathbb{N}$ and write

$$\Sigma(T) = \tau_{\mathrm{SOT}}\text{-} \lim_{N\to\infty} \Sigma_N(T)$$

if the latter exists. If $\Sigma(T) = 0$, we call T pure.

Remark

If $K = K^{(1)}$ and $T \in B(\mathcal{H})^n$ is a row contraction, then

$$\Sigma_N(T) = \sigma_T^{N+1}(1)$$

for all $N \in \mathbb{N}$, and hence,

$$\Sigma(T) = T_\infty \geq 0.$$

Proposition

Let $T \in B(\mathcal{H})^n$ be a K-contraction such that the sequence $(\Sigma_N(T))_{N\in\mathbb{N}}$ is norm-bounded. The map

$$\pi_T : \mathcal{H} \to H_K \otimes \mathcal{D}_T,\ h \mapsto \sum_{\alpha\in\mathbb{N}^n} a_{|\alpha|} \frac{|\alpha|!}{\alpha!}(z^\alpha \otimes D_T T^{*\alpha} h),$$

where $D_T = (1/K(T))^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{D_T\mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle \Sigma(T)h, h\rangle$$

for all $h \in \mathcal{H}$ and

$$\pi_T T_i^* = (M_{z_i} \otimes 1_{\mathcal{D}_T})^* \pi_T$$

for all $i = 1, \ldots, n$.

Remark

In the setting of the last proposition, if T is pure, then π_T is an isometry. Conversely, if π_T is a well-defined isometry, then the proof of the last proposition shows that T is pure.

In many cases (e.g. if $K = K^{(\nu)}$ for $\nu > 0$) we have that

$$\frac{1}{K}(M_z) = P_{\mathbb{C}}.$$

Proposition

Assume that $1/K(M_z) = P_{\mathbb{C}}$*. The* K*-shift* $M_z \in B(H_K)^n$ *satisfies*

$$\Sigma_N(M_z) \geq 0$$

for all $N \in \mathbb{N}$ *and is pure.*

The pure case

Theorem (Eschmeier, S.)

Assume that $1/K(M_z) = P_{\mathbb{C}}$. *Let* $T \in B(\mathcal{H})^n$ *be a commuting tuple. The following statements are equivalent:*

1. T *is pure,*
2. *there exist a Hilbert space* $\mathcal{D}$ *and an isometry*

$$\Pi \colon \mathcal{H} \to H_K \otimes \mathcal{D}$$

such that

$$\Pi T_i^* = (M_{z_i} \otimes 1_{\mathcal{D}})^* \Pi$$

for all $i = 1, \ldots, n$.

Beurling's theorem

Proposition (Eschmeier, S.)

Assume that $1/K(M_z) = P_{\mathbb{C}}$. Let $T \in B(\mathcal{H})^n$ be pure and $\mathcal{S} \subset \mathcal{H}$. The following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}(T)$ *and* $T|_{\mathcal{S}}$ *is pure,*
2. *there exist a Hilbert space* $\mathcal{D}$, *and a partial isometry* $\pi \colon H_K \otimes \mathcal{D} \to \mathcal{H}$ *with*

$$T_i \pi = \pi(M_{z_i} \otimes 1_{\mathcal{D}})$$

for all $i = 1, \ldots, n$ *and* $\mathrm{Im}(\pi) = \mathcal{S}$.

Theorem (Eschmeier, S.)

Assume that $1/K(M_z) = P_{\mathbb{C}}$. *Let* $\mathcal{E}$ *be a Hilbert space,* $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_n, \mathcal{E})$ *a reproducing kernel Hilbert space, and let* $M_z \in B(H(\mathcal{E}))^n$ *be pure. For* $\mathcal{S} \subset H(\mathcal{E})$, *the following statements are equivalent:*

1. $\mathcal{S} \in \mathrm{Lat}\,(M_z)$ *and* $M_z|_{\mathcal{S}}$ *is pure,*
2. *there exist a Hilbert space* $\mathcal{D}$ *and an analytic function* $\theta\colon \mathbb{B}_n \to B(\mathcal{D}, \mathcal{E})$ *such that*

$$M_\theta\colon H_K(\mathcal{D}) \to H(\mathcal{E}),\; f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

Theorem (Sarkar, 2016)

Let $\mathcal{E}$ be a Hilbert space and let $H(\mathcal{E}) \subset \mathcal{O}(\mathbb{B}_n, \mathcal{E})$ be a reproducing kernel Hilbert space of analytic functions such that $M_z \in B(H(\mathcal{E}))^n$ is a row contraction as well as $\mathcal{S} \subset H(\mathcal{E})$. Then the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}\,(M_z)$,
2. *there exist a Hilbert space $\mathcal{D}$ and an analytic function $\theta \colon \mathbb{B}_n \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta \colon H_{K^{(1)}}(\mathcal{D}) \to H(\mathcal{E}), \; f \mapsto \theta \cdot f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

The general case

Definition

We call a K-contraction $T \in B(\mathcal{H})^n$ *strong* if $\Sigma(T) \geq 0$ and $\Sigma(T) = \sigma_T(\Sigma(T))$ holds.

Remark

Every pure K-contraction is a strong K-contraction. Hence, the K-shift $M_z \in B(H_K)^n$ is a strong K-contraction if we assume that $1/K(M_z) \geq 0$.

Proposition

Let $V \in B(\mathcal{H})^n$ be a row isometry.

1. *The limit*
$$\frac{1}{K}(V) = \tau_{\mathrm{SOT}}\text{-}\lim_{N\to\infty} \sum_{k=0}^{N} c_k \sigma_V^k(1)$$
exists if and only if the series $\sum_{k=0}^{\infty} c_k$ converges.
2. *The tuple V is a K-contraction if and only if $\sum_{k=0}^{\infty} c_k \geq 0$.*
3. *Assume that $\sum_{k=0}^{\infty} c_k = 1/\sum_{k=0}^{\infty} a_k \in [0,\infty)$. Then V is a strong K-contraction.*

Remark

If $\sum_{k=0}^{\infty} c_k$ is abolutely convergent, then $\sum_{k=0}^{\infty} c_k = 1/\sum_{k=0}^{\infty} a_k$.

Lemma

Let $T \in B(\mathcal{H})^n$ be a strong K-contraction. Then there exist a Hilbert space $\mathcal{K}_T$ with $\Sigma(T)^{\frac{1}{2}}\mathcal{H} \subset \mathcal{K}_T$, and a spherical unitary $U_T \in B(\mathcal{K}_T)^n$ such that

$$\Sigma(T)^{\frac{1}{2}} T_i{}^* = U_T{}_i^* \Sigma(T)^{\frac{1}{2}}$$

for all $i = 1, \ldots, n$. Furthermore, $\mathcal{K}_T$ and U_T can be chosen such that

$$\mathcal{K}_T = \bigvee \left\{ U_T^{\alpha} \Sigma(T)^{\frac{1}{2}} h \; ; \; \alpha \in \mathbb{N}^n \text{ and } h \in \mathcal{H} \right\}$$

holds.

Theorem (Eschmeier, S.)

Assume that $1/K(M_z) = P_{\mathbb{C}}$ *and* $\sum_{k=0}^{\infty} c_k = 1/\sum_{k=0}^{\infty} a_k \in [0, \infty)$. *Let* $T \in B(\mathcal{H})^n$ *be a commuting tuple. The following statements are equivalent:*

1. T *is a strong* K*-contraction,*
2. *there exist Hilbert spaces* $\mathcal{D}, \mathcal{K}$*, a spherical unitary* $U \in B(\mathcal{K})^n$*, and an isometry*

$$\Pi \colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$$

such that

$$\Pi T_i^* = ((M_{z_i} \otimes 1_{\mathcal{D}}) \oplus U_i)^* \Pi$$

for all $i = 1, \ldots, n$.

Radial K-hypercontractions

Lemma

Let $m \in \mathbb{N}^*$ and let $T \in B(\mathcal{H})^n$ be a $K^{(m)}$-contraction. Then, for $0 < r < 1$, the tuple $rT \in B(\mathcal{H})^n$ is a pure $K^{(m)}$-contraction.

Definition

We call a commuting tuple $T \in B(\mathcal{H})^n$ with $\sigma(T) \subset \overline{\mathbb{B}}_n$ a *radial K-hypercontraction* if, for all $0 < r < 1$, $rT \in B(\mathcal{H})^n$ is a K-contraction.

Example

Every row isometry is a radial K-hypercontraction.

Remark

We define

$$k_r : \overline{\mathbb{D}} \to \mathbb{C},\ z \mapsto k(rz) \qquad (r \in [0,1)).$$

For $r, s \in [0,1]$, the function k_s/k_r has a Taylor expansion on $\mathbb{D}$

$$\sum_{k=0}^{\infty} a_k(s,r) z^k.$$

Note that

$$a_k(1,0) = a_k \quad \text{and} \quad a_k(0,1) = c_k \qquad (k \in \mathbb{N}).$$

Remark (Guo, Hu, Xu)

If $\lim_{k\to\infty} a_k/a_{k+1} = 1$, then $\sigma(M_z) \subset \overline{\mathbb{B}}_n$. From now on, we shall assume that this condition holds.

Proposition

The K-shift $M_z \in B(H_K)^n$ is a radial K-hypercontraction if and only if

$$a_k(1, r) \geq 0$$

for all $k \in \mathbb{N}$ and $0 < r < 1$.

Remark (Olofsson)

All irreducible complete Nevanlinna-Pick spaces and all weighted Bergman spaces fulfill the property above.

Proposition

Assume that M_z is a radial K-hypercontraction. Let $T \in B(\mathcal{H})^n$ be a radial K-hypercontraction. Then

$$\frac{1}{K}_{\mathrm{rad}}(T) = \tau_{\mathrm{SOT}}\text{-}\lim_{r \to 1} \frac{1}{K}(rT)$$

exists and defines a positive operator.

Corollary

If M_z is a radial K-hypercontraction, then

$$\frac{1}{K}_{\mathrm{rad}}(M_z) = P_{\mathbb{C}}.$$

Theorem (Olofsson; Eschmeier-S.)

Assume that M_z is a row contraction and a radial K-hypercontraction (and another technical assumption). Let $T \in B(\mathcal{H})^n$ be a commuting tuple. The following are equivalent:

1. *T is a row contraction as well as radial K-hypercontraction,*
2. *T is a strong K-contraction,*
3. *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^n$, and an isometry $\Pi \colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T_i^* = ((M_{z_i} \otimes 1_{\mathcal{D}}) \oplus U_i)^* \Pi \qquad (i = 1, \ldots, n).$$

In this case, we have

$$\Sigma(T) = T_\infty \quad \text{and} \quad \tau_{\mathrm{SOT}}\text{-}\sum_{k=0}^{\infty} a_k \sigma_T^k \left(\frac{1}{K}(T) \right) + T_\infty = 1.$$

ν-hypercontractions

Remark (Agler; Müller, Vasilescu)

Let $m \in \mathbb{N}^*$. A commuting tuple $T \in B(\mathcal{H})^n$ is a m-hypercontraction if and only if T is $K^{(k)}$-contraction for $1 \leq k \leq m$.

Definition

Let $\nu \geq 1$ be a real number. We call a commuting tuple $T \in B(\mathcal{H})^n$ an *ν-hypercontraction* if

$$\frac{1}{K^{(\mu)}}(T) = \tau_{\|\cdot\|}\text{-}\sum_{k=0}^{\infty} c_k^{(\mu)} \sigma_T^k(1) \geq 0$$

for all $1 \leq \mu \leq \nu$.

Theorem (Olofsson; Eschmeier, S.)

Let $\nu \geq 1$ and let $T \in B(\mathcal{H})^n$ be a commuting tuple. The following assertions are equivalent:

1. *T is a row contraction and a radial $K^{(\nu)}$-hypercontraction.*
2. *T is an ν-hypercontraction.*
3. *T is row contraction and a $K^{(\nu)}$-contraction.*
4. *T is a strong $K^{(\nu)}$-contraction.*
5. *There exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^n$, and an isometry*

$$\Pi \colon \mathcal{H} \to (H_{K^{(\nu)}} \otimes \mathcal{D}) \oplus \mathcal{K}$$

   such that

$$\Pi T_i^* = ((M_{z_i} \otimes 1_{\mathcal{D}}) \oplus U_i)^* \Pi$$

   for all $i = 1, \ldots, n$.

Beurling's theorem for ν-hypercontractions

Theorem (Eschmeier, Klauk, S.)

Let $\mathcal{E}$ be a Hilbert space and $\mathcal{S} \subset H_{K^{(\nu)}}(\mathcal{E})$ be a subspace. For $M_z \in B(H_{K^{(\nu)}})^n$ and $1 \leq \mu \leq \nu$, the following statements are equivalent:

1. *We have $\mathcal{S} \in \mathrm{Lat}\,(M_z)$ and $M_z|_{\mathcal{S}}$ is a μ-hypercontraction,*
2. *there exist a Hilbert space $\mathcal{D}$ and an analytic function $\theta \colon \mathbb{B}_n \to B(\mathcal{D}, \mathcal{E})$ such that*
   $$M_\theta \colon H_{K^{(\mu)}}(\mathcal{D}) \to H_{K^{(\nu)}}(\mathcal{E}), \; f \mapsto \theta \cdot f$$
   is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

Current work

Theorem (Eschmeier, S.)

Assume that $\lim_{k\to\infty} a_k/a_{k+1} = 1$ *and* $M_z \in B(H_K)^n$ *is a radial* K*-hypercontraction. Let* $T \in B(\mathcal{H})^n$ *be a commuting tuple. The following assertions are equivalent:*

1. T *is a radial* K*-hypercontraction,*
2. *there exist Hilbert spaces* $\mathcal{D}, \mathcal{K}$*, a spherical unitary* $U \in B(\mathcal{K})^n$*, and an isometry* $\Pi\colon \mathcal{H} \to (H_K \otimes \mathcal{D}) \oplus \mathcal{K}$ *such that*
$$\Pi T_i^* = ((M_{z_i} \otimes 1_{\mathcal{D}}) \oplus U_i)^* \Pi \qquad (i = 1, \ldots, n),$$
3. *there is a unital completely contractive linear map* $\rho\colon S \to B(\mathcal{H})$ *on the operator space* $S = \operatorname{span}\left\{1, M_{z_i}, M_{z_i}M_{z_i}^* \;;\; i = 1, \ldots, n\right\}$ *with*
$$\rho(M_{z_i}) = T_i, \;\; \rho(M_{z_i}M_{z_i}^*) = T_iT_i^* \qquad (i = 1, \ldots, n).$$

Theorem (Eschmeier, S.)

Let $\nu > 0$ and let $T \in B(\mathcal{H})^n$ be a commuting tuple. The following assertions are equivalent:

1. *T is a radial $K^{(\nu)}$-hypercontraction,*
2. *T is a strong $K^{(\nu)}$-contraction,*
3. *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^n$, and an isometry $\Pi \colon \mathcal{H} \to (H_{K^{(\nu)}} \otimes \mathcal{D}) \oplus \mathcal{K}$ such that*
$$\Pi T_i^* = ((M_{z_i} \otimes 1_{\mathcal{D}}) \oplus U_i)^* \Pi \qquad (i = 1, \dots, n),$$
4. *there is a unital completely contractive linear map $\rho \colon S \to B(\mathcal{H})$ on the operator space $S = \operatorname{span}\left\{1, M_{z_i}, M_{z_i} M_{z_i}^* \; ; \; i = 1, \dots, n\right\}$ with*
$$\rho(M_{z_i}) = T_i, \;\; \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \dots, n).$$