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Schatten-*p*-class perturbations of Toeplitz operators

Dominik Schillo

09.05.2016

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- Regular triples
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Let $\mathbb T$ be the unit circle in $\mathbb C$ with the canonical probability measure m.

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Let $\mathbb T$ be the unit circle in $\mathbb C$ with the canonical probability measure m.

Definition

We call

$$H^2(m)=\left\{f\in L^2(m)\;;\;\hat{f}(n)=0\; ext{for all}\;n<0
ight\}\subset L^2(m)$$

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the Hardy space with respect to m.

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Definition

Let $f \in L^{\infty}(m)$. We call the compression of the multiplication operator

$$M_f: L^2(m) \to L^2(m), g \mapsto fg$$

to $H^2(m)$ the Toeplitz operator with symbol f and denote it by T_f , i.e.

$$T_f = P_{H^2(m)} M_f |_{H^2(m)},$$

where $P_{H^2(m)}$: $L^2(m) \to H^2(m)$ is the orthogonal projection onto $H^2(m)$.

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Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^{\infty}(m)$ such that $X = T_f$) if and only if

$$T_z^*XT_z-X=0,$$

where $z \in L^{\infty}(m)$ is the identity map.

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Definition

Define

$$H^{\infty}(m) = L^{\infty}(m) \cap H^{2}(m) \subset L^{\infty}(m)$$

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Definition

Define

$$H^{\infty}(m) = L^{\infty}(m) \cap H^{2}(m) \subset L^{\infty}(m)$$

and we call

$$I_m=\{f\in H^\infty(m)\;;\; |f|=1\;m$$
-a.e. $\}$

the set of inner functions with respect to m.

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Definition

Define

$$H^{\infty}(m) = L^{\infty}(m) \cap H^{2}(m) \subset L^{\infty}(m)$$

and we call

$$I_m=\{f\in H^\infty(m)\;;\;|f|=1\;m ext{-a.e.}\}$$

the set of inner functions with respect to m.

Theorem (Brown-Halmos condition II)

An operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_m$.

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Theorem (Gu, 2004)

An operator $X \in B(H^2(m))$ is a finite rank Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(m)$ and $F \in F(H^2(m))$ such that $X = T_f + F$) if and only if

$$T_u^*XT_u - X \in F(H^2(m))$$

for all $u \in I_m$.

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Theorem (Xia, 2009)

An operator $X \in B(H^2(m))$ is a compact Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(m)$ and $K \in K(H^2(m))$ such that $X = T_f + K$) if and only if

$$T_u^*XT_u - X \in K(H^2(m))$$

for all $u \in I_m$.

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Definition

Let $p \in [1, \infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a *Schatten-p-class* operator if

$$\|S\|_p^p = \operatorname{tr}(|S|^p) = \sum_{e \in \mathcal{E}} \left\langle |S|^p \, e, e
ight
angle = \sum_{e \in \mathcal{E}} \left\langle (S^*S)^{rac{p}{2}} e, e
ight
angle < \infty$$

for some orthonormal basis $\mathcal E$ of H.

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Definition

Let $p \in [1,\infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a *Schatten-p-class* operator if

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ight
angle < \infty$$

for some orthonormal basis $\mathcal E$ of H. Furthermore, we set

$$\mathcal{S}_p(H) = \left\{ S \in B(H) \; ; \; \left\|S\right\|_p < \infty
ight\}$$

equipped with $\|\cdot\|_p$

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Definition

Let $p \in [1,\infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a Schatten-p-class operator if

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for some orthonormal basis \mathcal{E} of H. Furthermore, we set

$$\mathcal{S}_p(H) = \left\{ S \in B(H) \; ; \; \left\|S
ight\|_p < \infty
ight\}$$

equipped with $\left\|\cdot\right\|_{p}$ and

$$\mathcal{S}_0(H)=\mathcal{F}(H)$$
 as well as $\mathcal{S}_\infty(H)=\mathcal{K}(H)$

both equipped with the operator norm.

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Theorem

Let $p \in \{0\} \cup [1, \infty]$. An operator $X \in B(H^2(m))$ is a $S_p(H^2(m))$ Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(m)$ and $S \in S_p(H^2(m))$ such that $X = T_f + S$) if and only if

$$T_u^*XT_u - X \in \mathcal{S}_p(H^2(m))$$

for all $u \in I_m$.

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The classical problem	Constraints and formulation of the problem ●0000 000 0000	Main result 000000	Open questions
Toeplitz operators			

Let \mathbb{D} be the unit disc in \mathbb{C} and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on \mathbb{D} . We call

$${\mathcal A}({\mathbb D})=ig\{f\in {\mathcal C}(\overline{\mathbb D})\;;\; f|_{\mathbb D}\in {\mathcal O}({\mathbb D})ig\}$$

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the disc algebra.

The classical problem	Constraints and formulation of the problem ●0000 ○000 0000	Main result 000000	Open question
Toeplitz operators			

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the disc algebra.

Proposition

The following statements hold.

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Toeplitz operators			

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Proposition

The following statements hold. (i) $H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}}$.

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Toeplitz operators			

Let \mathbb{D} be the unit disc in \mathbb{C} and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on \mathbb{D} . We call

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Proposition

The following statements hold. (i) $H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}}$. (ii) $H^{\infty}(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}$.

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Toeplitz operators			

Let \mathbb{D} be the unit disc in \mathbb{C} and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on \mathbb{D} . We call

$${\mathcal A}({\mathbb D})=ig\{f\in {\mathcal C}(\overline{\mathbb D})\;;\; f|_{\mathbb D}\in {\mathcal O}({\mathbb D})ig\}$$

the disc algebra.

Proposition

The following statements hold. (i) $H^{2}(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|}} L^{2(m)}$. (ii) $H^{\infty}(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^{*}}}$. (iii) $H^{\infty}(m)H^{2}(m) \subset H^{2}(m)$.

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Toeplitz operators			

Let D be a bounded domain in \mathbb{C}^n .

The classical problem	Constraints and formulation of the problem ○●○○○ ○○○○	Main result 000000	Open questions
Toeplitz operators			
Let <i>D</i> be a	bounded domain in \mathbb{C}^n .		
Definition			
We denote l	ру		
(i) A(D) = algebra	$= \left\{ f \in C(\overline{D}) \; ; \; f _D \in \mathcal{O}(D) ight\} \subset C$ of D ,	(\overline{D}) the doma	ain

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The classical problem	Constraints and formulation of the problem ○●○○○ ○○○○	Main result 000000	Open questions
Toeplitz operators			
let Dbe a b	bounded domain in \mathbb{C}^n		

We denote by

(i)
$$A(D) = \{ f \in C(\overline{D}) ; f|_D \in \mathcal{O}(D) \} \subset C(\overline{D})$$
 the domain algebra of D ,

(ii) $S = \partial_{A(D)}$ the Shilov boundary of A(D) (i.e. the smallest closed subset of \overline{D} such that

$$\sup_{z\in\overline{D}}|f(z)|=\sup_{z\in S}|f(z)|$$

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for all $f \in A(D)$).

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Toeplitz operators			

(i)
$$D = \mathbb{B}_n$$
: $S = \partial \mathbb{B}_n$

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The classical problem	Constraints and formulation of the problem 00●00 000 0000	Main result 000000	Open questions
Toeplitz operators			

(i)
$$D = \mathbb{B}_n$$
: $S = \partial \mathbb{B}_n$
(ii) $D = \mathbb{D}^n$: $S = \mathbb{T}^n$

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The classical problem	Constraints and formulation of the problem 00●00 000 0000	Main result 000000	Open questions
Toeplitz operators			

(i)
$$D = \mathbb{B}_n$$
: $S = \partial \mathbb{B}_n$
(ii) $D = \mathbb{D}^n$: $S = \mathbb{T}^n$
(iii) D strictly pseudoconvex: $S = \partial D$.

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Toeplitz operators

Let $\mu \in M^+(S)$ be a positive Borel measure on S.

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The classical problem	Constraints and formulation of the problem 000€0 000 0000	Main result 000000	Open questior
Toeplitz operators			
Let $\mu\in \mathbf{\Lambda}$	$\mathcal{I}^+(S)$ be a positive Borel measure of	on <i>S</i> .	
Definition			
We define	$H^2(\mu) = \overline{A(D) _S}^{ au_{\ \cdot\ _{L^2(\mu)}}} \subset L^2$	$-2^{2}(\mu)$	
and			

$$H^{\infty}(\mu) = \overline{A(D)|_{S}}^{\tau_{w^*}} \subset L^{\infty}(\mu).$$

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The classical problem	Constraints and formulation of the problem 000●0 0000 0000	Main result 000000	Open questior
Toeplitz operators			
		_	
Let $\mu \in M^+$	(S) be a positive Borel measure o	n <i>S</i> .	

Definition We define $H^{2}(\mu) = \overline{A(D)|_{S}}^{\tau_{\|\cdot\|_{L^{2}(\mu)}}} \subset L^{2}(\mu)$ and $H^{\infty}(\mu) = \overline{A(D)|_{S}}^{\tau_{w^{*}}} \subset L^{\infty}(\mu).$ Furthermore, we denote by $I_{\mu} = \{f \in H^{\infty}(\mu) ; |f| = 1 \ \mu\text{-a.e.}\}$

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the set of inner functions with respect to μ .

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Toeplitz operators			

Let $f \in L^{\infty}(\mu)$. We call

$$T_f: H^2(\mu) \to H^2(\mu), \ g \mapsto P_{H^2(\mu)}(fg),$$

where $P_{H^2(\mu)}: L^2(\mu) \to H^2(\mu)$ is the orthogonal projection onto $H^2(\mu)$, the Toeplitz operator with symbol f.

Pagular triples	The classical problem	Constraints and formulation of the problem ○○○○○ ○○○○	Main result 000000	Open questions
incegural criptes	Regular triples			

Let $K \subset \mathbb{C}^n$ be a compact set, $A \subset C(K)$ be a closed subalgebra and $\nu \in M^+(K)$ a positive Borel measure.

The classical problem	Constraints and formulation of the problem • • • • • • • • • • • • • • • • • • •	Main result 000000	Open questions
Regular triples			

Let $K \subset \mathbb{C}^n$ be a compact set, $A \subset C(K)$ be a closed subalgebra and $\nu \in M^+(K)$ a positive Borel measure.

(i) The triple (A, K, ν) is called regular (in the sense of Aleksandrov) if for every φ ∈ C(K) with φ > 0, there exists a sequence of functions (φ_k)_{k∈N} in A with |φ_k| < φ on K for all k ∈ N and

$$\lim_{k\to\infty}|\varphi_k|=\varphi$$

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 ν -almost everywhere on K.

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Regular triples			

Let $K \subset \mathbb{C}^n$ be a compact set, $A \subset C(K)$ be a closed subalgebra and $\nu \in M^+(K)$ a positive Borel measure.

 (i) The triple (A, K, ν) is called *regular (in the sense of* Aleksandrov) if for every φ ∈ C(K) with φ > 0, there exists a sequence of functions (φ_k)_{k∈ℕ} in A with |φ_k| < φ on K for all k ∈ ℕ and

$$\lim_{k\to\infty}|\varphi_k|=\varphi$$

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 ν -almost everywhere on K.

(ii) The measure ν is called continuous if every one-point set has $\nu\text{-measure zero}$

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Regular triples			

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Examples

(i)
$$A = A(\mathbb{B}_n)|_{\partial \mathbb{B}_n}, K = \partial \mathbb{B}_n, \nu = \sigma.$$

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Regular triples			

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Examples

(i)
$$A = A(\mathbb{B}_n)|_{\partial \mathbb{B}_n}, K = \partial \mathbb{B}_n, \nu = \sigma$$
.
(ii) $A = A(\mathbb{D}^n)|_{\mathbb{T}^n}, K = \mathbb{T}^n, \nu = \otimes_n m$.

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Regular triples			

(i)
$$A = A(\mathbb{B}_n)|_{\partial \mathbb{B}_n}, K = \partial \mathbb{B}_n, \nu = \sigma.$$

(ii) $A = A(\mathbb{D}^n)|_{\mathbb{T}^n}, K = \mathbb{T}^n, \nu = \otimes_n m.$

Theorem (Aleksandrov, 1984)

Let (A, K, ν) be a regular triple with a continuous measure ν in $M^+(K)$. Then the weak* sequential closure of the set I_{ν} contains all $L^{\infty}(\nu)$ -equivalence classes of functions $f \in A$ with $|f| \leq 1$ on K.

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Regular triples			

Let $\mu \in M_1^+(S)$ and $(A(D)|_S, S, \mu)$ be a regular triple.

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Regular triples			

Let $\mu \in M^+_1(S)$ and $(A(D)|_S, S, \mu)$ be a regular triple.

Theorem

An operator $X \in B(H^2(\mu))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

for all $u \in I_{\mu}$.

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Henkin measures

Definition

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

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Henkin measures

Definition

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Proposition

The following statements hold:

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Henkin measures

Definition

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Proposition

The following statements hold:

(i) The space $H^{\infty}(D) \subset L^{\infty}(D) = (L^1(D))'$ is weak* closed.

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Henkin measures

Definition

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Proposition

The following statements hold:

- (i) The space $H^{\infty}(D) \subset L^{\infty}(D) = (L^1(D))'$ is weak* closed.
- (ii) The space $L^1(D)/{}^{\perp}H^{\infty}(D)$ is separable with

$$H^{\infty}(D)\cong \left(L^1(D)/^{\perp}H^{\infty}(D)\right)'.$$

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Henkin measures

Definition

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Proposition

The following statements hold:

- (i) The space $H^{\infty}(D) \subset L^{\infty}(D) = (L^1(D))'$ is weak* closed.
- (ii) The space $L^1(D)/{}^{\perp}H^{\infty}(D)$ is separable with

$$H^{\infty}(D) \cong \left(L^1(D) / {}^{\perp} H^{\infty}(D) \right)'.$$

(iii) The closed unit ball $\overline{B}_1^{H^{\infty}(D)}(0)$ equipped with the relative topology of the weak* topology of $H^{\infty}(D)$ is a compact metrizable space.

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Theorem

The map

$$r_m \colon H^{\infty}(\mathbb{D}) \to H^{\infty}(m), \ \theta \mapsto \tau_{w^*} - \lim_{r \to 1} [\theta(r \cdot)] =: \theta^*$$

is an isometric isomorphism and weak* homeomorphism with $r_m(\theta|_{\mathbb{D}}) = [\theta|_{\mathbb{T}}]$ for all $\theta \in A(\mathbb{D})$.

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Henkin measures			

We call μ a *(faithful) Henkin measure* if there is a contractive (isometric) weak* continuous algebra homomorphism

$$r_{\mu} \colon H^{\infty}(D) o L^{\infty}(\mu), \,\, heta \mapsto r_{\mu}(heta) =: heta^{*}$$

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with $r_{\mu}(\theta|_D) = [\theta|_S]$ for all $\theta \in A(D)$.

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Henkin measures			

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$$r_{\mu} \colon H^{\infty}(D) o L^{\infty}(\mu), \ heta \mapsto r_{\mu}(heta) =: heta^{*}$$

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with
$$r_{\mu}(\theta|_D) = [\theta|_S]$$
 for all $\theta \in A(D)$.

Let $\mu \in M_1^+(S)$ be a faithful Henkin measure.

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Henkin measures			

We call μ a *(faithful) Henkin measure* if there is a contractive (isometric) weak* continuous algebra homomorphism

$$\mathit{r}_{\mu}\colon \mathit{H}^{\infty}(\mathcal{D})
ightarrow \mathit{L}^{\infty}(\mu), \; heta\mapsto \mathit{r}_{\mu}(heta)=: heta^{*}$$

with
$$r_{\mu}(\theta|_D) = [\theta|_S]$$
 for all $\theta \in A(D)$.

Let $\mu \in M_1^+(S)$ be a faithful Henkin measure.

Remark

The map $r_{\mu} \colon H^{\infty}(D) \to \operatorname{Im}(r_{\mu})$ is an isometric isomorphism and weak* homeomorphism with weak* closed range.

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Henkin measures			
Proposition			

We have

 $H^{\infty}(\mu) \subset \operatorname{Im}(r_{\mu}).$

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Henk	in measures			
	Proposition			
	We have			
		$H^\infty(\mu)\subset {\sf Im}(r_\mu).$		
	Examples			
	(i) $D = \mathbb{B}_n$,	$\mu = \sigma$		

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Henkin measures			
Proposition			

We have

$$H^{\infty}(\mu) \subset \operatorname{Im}(r_{\mu}).$$

Examples

(i)
$$D = \mathbb{B}_n, \mu = \sigma$$

(ii) $D = \mathbb{D}^n, \mu = \otimes_n m$

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Main result

Open questions

Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\mu \in M_1^+(S)$ be a continuous faithful Henkin probability measure such that $(A(D)|_S, S, \mu)$ is a regular triple in the sense of Aleksandrov.

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Open questions

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Theorem

Let $p \in [1, \infty)$. An operator $X \in B(H^2(\mu))$ is a $S_p(H^2(\mu))$ Toeplitz perturbation (i.e. there exists a $f \in L^{\infty}(\mu)$ and $S \in S_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

$$T_u^*XT_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_{\mu}$.

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Proof

Proposition

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $H^{\infty}(\mu)$ with

$$\tau_{w^*} - \lim_{k \to \infty} \alpha_k = \alpha \in \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$$

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and $X \in B(H^2(\mu))$ an operator such that

$$Y = au_{\mathsf{WOT}} - \lim_{k o \infty} T^*_{lpha_k} X T_{lpha_k} \in B(H^2(\mu))$$

exists.

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exists. If $T_u^*XT_u - X \in S_{\infty}(H^2(\mu))$ for all $u \in I_{\mu}$, then there exists a function $f \in L^{\infty}(\mu)$ such that

$$X = T_f + \frac{1}{1 - |\alpha|^2} (X - Y).$$

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Proof

Proposition (Hiai, 1997)

The map

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$$\|\cdot\|_{p}: (B(H^{2}(\mu)), \tau_{\mathsf{WOT}}) \to [0, \infty], \ S \mapsto \|S\|_{p}$$

is lower semi-continuous.

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Proof

We denote by

$$I_D = \{ \theta \in H^\infty(D) ; \ \theta^* \in I_\mu \}$$

the set of inner functions with respect to D.

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Proposition

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$$T_u^*XT_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_{\mu}$. Then, for all sequences $(\theta_k)_{k \in \mathbb{N}}$ in I_D with

$$au_{w^*}$$
- $\lim_{k o \infty} heta_k^* = 1,$

we have

$$\tau_{\|\cdot\|_{\rho}} - \lim_{k \to \infty} T^*_{\theta^*_k} X T_{\theta^*_k} - X = 0.$$

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Proof			

Proposition

Let $(\theta_k)_{k \in \mathbb{N}}$ be a sequence in I_D . Then the following statements are equivalent:

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(i)
$$\tau_{w^*}$$
- $\lim_{k\to\infty} \theta_k^* = 1$,

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Proof

Proposition

Let $(\theta_k)_{k \in \mathbb{N}}$ be a sequence in I_D . Then the following statements are equivalent:

(i)
$$\tau_{w^*} - \lim_{k \to \infty} \theta_k^* = 1$$
,
(ii) $\lim_{k \to \infty} \int_{\mathcal{S}} \theta_k^* d\mu = 1$,

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Proof

Proposition

Let $(\theta_k)_{k \in \mathbb{N}}$ be a sequence in I_D . Then the following statements are equivalent:

(i) $\tau_{w^*} - \lim_{k \to \infty} \theta_k^* = 1$, (ii) $\lim_{k \to \infty} \int_S \theta_k^* d\mu = 1$, (iii) There exists $w \in D$ such that $\lim_{k \to \infty} \theta_k(w) = 1$.

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Let $p \in [1,\infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

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$$T_u^*XT_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_{\mu}$. Then, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\|T_{\theta^*}^* X T_{\theta^*} - X\|_p \le \varepsilon$$

for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$.

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Questions

(i) For which regular triple (A, K, μ) does the result holds?

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Questions

- (i) For which regular triple (A, K, μ) does the result holds?
- (ii) Are there other ideals for which the result holds?

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Questions

- (i) For which regular triple (A, K, μ) does the result holds?
- (ii) Are there other ideals for which the result holds?
- (iii) What about $p = 0, \infty$?

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