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Analytical Methods for PDEs (SoSe 2018)

Hometasks N 3 – 4

Ex. 9 Consider the Hilbert space $L^2[-1,1]$ and the system $(L_n)_{n=0}^{\infty}$ of the *Legendre polynomials*:

$$L_n(x) := \frac{1}{n!2^n} \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

(a) Show that $(L_n)_{n=0}^{\infty}$ is an orthogonal system in $L^2[-1,1]$.

(b) Show that L_n solves the following (Legendre's) differential equation:

$$(1-x^2)u''(x) - 2xu'(x) + n(n+1)u(x) = 0.$$

(c) Show that $\lambda_n := n(n+1)$, $n = 0, 1, \ldots$, are eigenvalues and L_n are corresponding eigenfunctions of the following singular Sturm–Liouville problem:

$$\begin{cases} -\frac{d}{dx}\left[(1-x^2)\frac{du(x)}{dx}\right] = \lambda u(x), & x \in (-1,1), \\ |u(\pm 1)| < \infty \end{cases}$$

Ex. 10 Solve the Sturm–Liouville problem

$$\begin{cases} -u''(x) = \lambda u(x), & x \in (0,l), \\ -\alpha_1 u'(0) + \beta_1 u(0) = 0 \\ \alpha_2 u'(l) + \beta_2 u(l) = 0 \end{cases}$$

in the following cases:

(a) $\alpha_1 = \alpha_2 = 0$;

(b) $\beta_1 = \beta_2 = 0$;

(c) $\alpha_1 = \beta_2 = 0$;

(d) $\alpha_2 = \beta_1 = 0$;

(e) $\alpha_1 = 0$, $\alpha_2 = -\beta_2$;

(f) $\beta_2 = 0$, $\alpha_1 = \beta_1$.

Ex. 11 Find the Fourier series of the function $\varphi(x)$, $x \in (0, l)$, in the system of eigenfunctions for each of the Sturm–Liouville problems (a) – (d) of Exercise 10, where

(a) $\varphi(x) = 1$;

(b) $\varphi(x) = x(l - x)$.

Ex. 12 Solve the following problems with the method of separation of variables (where $x \in (0, l)$, $t > 0$ and $a > 0$):

(a) $u'_t = a^2 u''_{xx}, \quad u'_x(t, 0) = u'_x(t, l) = 0, \quad u(0, x) = 1;$

(b) $u'_t = a^2 u''_{xx}, \quad u(t, 0) = u'_x(t, l) = 0, \quad u(0, x) = \sin \frac{7\pi x}{2l};$

(c) $u''_{tt} = 9u''_{xx}, \quad u(t, 0) = u'_x(t, l) = 0, \quad u(0, x) = x, \ u'_t(0, x) = 0;$

(d) $u''_{tt} = 9u''_{xx}, \quad u(t, 0) = u(t, l) = u(0, x) = 0, \ u'_t(0, x) = l^2 - x^2;$

(e) $u'_t = u''_{xx} - u, \quad u(t, 0) = u(t, l) = 0, \quad u(0, x) = 1;$

Ex. 13 Find harmonic functions in the rectangular $x \in (0, a)$, $y \in (0, b)$ by the method of separation of variables (for $u''_{xx} + u''_{yy} = 0$) if the following boundary conditions are given:

(a) $u(0, y) = A \sin \frac{\pi y}{b}, \quad u(a, y) = u(x, 0) = u(x, b) = 0;$

(b) $u(0, y) = A \sin \frac{\pi y}{b}, \quad u(a, y) = u(x, b) = 0, \quad u(x, 0) = B \sin \frac{\pi x}{a};$

Hint: For any A, B, z, $c \in \mathbb{R}$ there exit $\widetilde{A}$ and $\widetilde{B} \in \mathbb{R}$ such that

$$Ae^{cz} + Be^{-cz} = \widetilde{A} \cosh(cz) + \widetilde{B} \sinh(cz).$$