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INTERIOR REGULARITY FOR FREE AND CONSTRAINED LOCAL MINIMIZERS OF VARIATIONAL INTEGRALS UNDER GENERAL GROWTH AND ELLIPTICITY CONDITIONS

ABSTRACT. We consider strictly convex energy densities $f\colon \mathbb{R}^n \to \mathbb{R}$ under nonstandard growth conditions. More precisely, we assume that for some constants λ, Λ and for all $Z, Y \in \mathbb{R}^n$ the inequality

$$\lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2$$

holds with exponents $\mu \in \mathbb{R}$ and $q > 1$. If u denotes a bounded local minimizer of the energy $\int f(\nabla w)\mathrm{d}x$ subject to a constraint of the form $w \ge \psi$ a.e. with a given obstacle $\psi \in C^{1,\alpha}(\Omega)$, then we prove local $C^{1,\alpha}$-regularity of u provided that $q < 4-\mu$. This result substantially improves what is known up to now (see, for instance, [8, 7, 13]), even for the case of unconstrained local minimizers.

**Dedicated to O.A. Ladyzhenskaya
on the occasion of her 80th birthday**

1. Introduction

Given a smooth, strictly convex integrand $f\colon \mathbb{R}^n \to \mathbb{R}$ and an open set $\Omega \subset \mathbb{R}^n$ we are interested in the smoothness properties of (local) minimizers of the energy

$$J[w] := \int_\Omega f(\nabla w)\mathrm{d}x \tag{1}$$

in a suitable energy class, where we also like to include obstacle problems into our considerations. Here and in the following we concentrate on scalar problems, the unconstrained vector-valued setting is discussed in [6] (compare [1] for more information on the subject).

Given a power growth integrand f, it is well known how to obtain local $C^{1,\alpha}$-regularity in the unconstrained case (we just mention the names of

De Giorgi, Moser, Nash, Ladyzhenskaya, and Ural'tseva), a discussion of (maybe degenerate) obstacle problems in this setting is found, for instance, in [25, 9, 20, 26, 11], and [12] – the classical quadratic case is extensively treated in the monographs [19, 15].

In recent years variational problems with nonstandard growth conditions became more and more popular. For example, we may assume that f is bounded from above and below by different growth rates $q > p > 1$ (together with the corresponding estimates for the second derivatives), typical examples are given by anisotropic integrands as considered in Example 1 below. Studies in this direction were forced in particular by Marcellini – starting with [22–24] – with the main result that an appropriate upper bound for the quantity q/p is sufficient for interior regularity. In fact, the existence of irregular solutions was already observed by Giaquinta ([17]) if p and q differ too much.

As a model case for a second class of nonstandard integrands one may think of $f(Z) = |Z| \ln(1 + |Z|)$, a function of nearly linear growth which is studied in the theory of Prandtl–Eyring fluids and of plastic materials with logarithmic hardening (an exhaustive overview is found in [16]). Motivated by this logarithmic example, the analysis of variational problems in general Orlicz-Sobolev energy classes starts with the papers [14] (partial regularity even for vector-valued local minimizers $u\colon \Omega \to \mathbb{R}^N$) and [13]. Note that [13] also covers the case of obstacle problems.

Finally, a unified and extended approach to "anisotropic energy densities defined on Orlicz–Sobolev classes and satisfying a quite bad ellipticity condition" is given in [7] (compare [3] for the vector-valued setting) by introducing the notion of (s, μ, q)-growth: the variational integrand f is bounded from below by some N-function F (which in turn has at least the growth rate $s \geq 1$) and the second derivatives are supposed to satisfy

$$\lambda(1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y, Y) \leq \Lambda(1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2 \tag{2}$$

for all Z, $Y \in \mathbb{R}^n$, for some positive constants λ, Λ, for $\mu \in \mathbb{R}$ and with the choice $q > 1$. Hence, anisotropic power growth is covered by letting $2 - \mu = p = s > 1$, the logarithmic integrand from above satisfies (2) by choosing $s = 1$, $\mu = 1$ and $q = 1 + \varepsilon$. For integrands of (s, μ, q)-growth, smoothness of local minimizers was proved under the so-called (s, μ, q)-condition relating the parameters s, μ and q in such a way that a variety of known results is included and extended (see also [1] for a detailed discussion).

From the technical point of view, a (first) restriction on the parameters enters through an application of Sobolev's inequality, which gives uniform local higher integrability of the gradients of some regularization.

Studying a linear growth situation it turned out in [2] that much better results can be obtained if the solution is supposed to be bounded (this assumption is reasonable, e.g., for Dirichlet problems with L^∞-boundary data). In this case, Sobolev's inequality may be replaced by an additional application of the (nondifferentiated) Euler equation. This method enabled us to reach (up to a certain extend) the limit case $\mu = 3$ and $q = 1$ in (2) by the way covering μ-elliptic integrands of linear growth with corresponding generalized minimizers. Moreover, as outlined in [1, 3], we do not expect regular solutions if (in the linear case) the left-hand side of (2) holds for some $\mu > 3$.

Let us focus again on variational problems with nonstandard but superlinear growth. If we restrict ourselves to the study of bounded solutions, then, as a formal correspondence to the results given in [2], the relation $1 < q < 4 - \mu$ (for anisotropic power growth integrands this condition reads as $1 < q < 2 + p$) is expected to be the best possible one inducing regular solutions. Note that the condition $q < 2 + p$ first appeared in [10], where higher integrability (up to a certain extend) in the anisotropic, superquadratic, vector-valued (p, q)-case was proved under some extra boundedness condition.

Nevertheless, the full strength of the above stated correspondence could not be shown in the paper [4] on anisotropic variational integrals with convex hull property: instead of $1 < q < 2+p$ the exponents have to be related via $1 < q < p+2/3$ (note that [4] deals with the vector-valued situation, however, the restrictions on the exponents are the same in the scalar case). This is caused by an essential difference to the linear growth situation: in [2] we benefit from the growth rate $1 = q$ of the main quantity $\nabla f(Z) \cdot Z$ under consideration. Given an anisotropic power growth integrand, we just have the lower bound $p < q$ for this quantity.

As a consequence, the techniques again have to be changed in such a way that we do not have to rely on the growth of the quantity $\nabla f(Z) \cdot Z$. This leads to the study of Choe's article ([8]), where bounded solutions w.r.t. "anisotropic" integrands $f(Z) = g(|Z|^2)$ (the "anisotropy" being formulated in terms of the second derivatives) are handled under the restriction $1 < q < p+1$ (note that this special structure of the integrand is required for the scalar case as well as for vector-valued problems). As a third approach, his results depend on a partial integration combined with a Caccioppoli-type inequality (of course this type of inequality also enters the two other techniques mentioned above).

In [6] we gave a refinement of Choe's Ansatz in the vector-valued case, where the main assertion (local higher integrability of the gradient) in fact was shown for $q < 4-\mu$, thus the formal correspondence to the linear growth situation holds.

In our paper we are now interested in the question whether Choe's Ansatz can be improved in the case of scalar obstacle problems as well. Following the above listed references ([13, 7, 3–6, 1, 2]) we introduce a regularization satisfying a Caccioppoli-type inequality which is slightly different from the one given in [8]. This again enables us to refine Choe's reasoning with surprisingly strong results. Roughly speaking we otain

Main theorem. Consider a variational integrand f satisfying (2) with

$$q < 4 - \mu,$$

and a local minimizer u (with respect to the side-condition $u \geq \psi$) of the energy J given in (1). If u is of class $L^{\infty}_{loc}(\Omega)$, then u is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

Remark 1. It should be emphasized again that this result is new also in the unconstrained case.

Remark 2.

(i) Recall that anisotropic power growth energy densities (compare Example 1 below) satisfy our assumption whenever

$$q < 2 + p.$$

(ii) As mentioned above, the condition $q < 4 - \mu$ is in complete formal accordance with the requirement $\mu < 3$ in the case of linear growth problems (see [1, 2, 5]).

(iii) In contrast to [7], Theorem 1.1, the lower growth rate s of the variational integrand is not involved. We just make use of the bounds induced by (2).

(iv) In terms of anisotropic integrands with (p, q)-growth, the main assumption of Theorem 1.1, [7], reads as

$$q < p\frac{n+2}{n}, \tag{a}$$

and if (a) holds, then (according to [7]) there is no need to impose L^{∞}-bounds on the solution. Hence, at first sight one may wonder about the case

$$2 + p < p\frac{n+2}{n}, \tag{b}$$

since then the hypothesis $q < 2+p$ occurring in our Main Theorem implies (a), thus the conclusion of the Main Theorem holds without an additional boundedness condition. But (b) is equivalent to $p > n$, hence, by Sobolev's embedding theorem, boundedness becomes no restriction at all.

Example 1.

(i) Let us first have a look at the anisotropic energy density ($Z = (Z_1, Z_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \le k < n$)

$$f(Z) = (1 + |Z_1|^2)^{\frac{p}{2}} + (1 + |Z_2|^2)^{\frac{q}{2}},$$

with exponents $2 \le p < q$. Here, Theorem 1.1 of [7] yields regular solutions whenever

$$q < p\frac{n+2}{n}.$$

(ii) Let us now discuss the same example in the subquadratic case $1 < p < q < 2$. Then, by elementary calculations, the estimate

$$\lambda(1 + |Z|^2)^{\frac{p-2}{2}} |Y|^2 \le D^2 f(Z)(Y, Y) \le \Lambda |Y|^2$$

is seen to be the best possible one.

As a consequence, no regularity results have been available up to now if p is close to 1, even if $q - p$ becomes very small. Hence, on account of the trivial inequality $2 < p+2$, our theorem really covers a new class of variational integrals.

Our paper is organized as follows: after a precise formulation of the assumptions and main result, we introduce in Sec. 3 a regularization and prove two Caccioppoli-type inequalities. Uniform local higher integrability of the gradients is established in Sec. 4, and with the help of a modified De Giorgi-type technique we complete the proof of our main theorem in Sec. 5.

2. Assumptions and main results

In the following it is always supposed that the variational integrand under consideration satisfies

Assumption 1. *The energy density $f\colon \mathbb{R}^n \to [0, \infty)$ is a strictly convex function of class $C^2(\mathbb{R}^n)$ which satisfies $f(0) = 0$ and $\nabla f(0) = 0$. Its* *second derivative is estimated for all* $Y \in \mathbb{R}^n$ *and for all* $Z \in \mathbb{R}^n$, $|Z| > 1$, *by*

$$\lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \leq D^2 f(Z)(Y,Y) \leq \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2. \qquad (3)$$

Here λ *and* Λ *denote some positive constants and the exponents* $\mu \in \mathbb{R}$, $q > 1$ *are related by*

$$q < 4 - \mu. \qquad (4)$$

Let us finally assume that there is some continuous function $F\colon [0,\infty) \to [0,\infty)$ *of superlinear growth, i.e.,*

$$\lim_{t\to\infty} \frac{F(t)}{t} = \infty$$

such that for some real numbers $c_1 > 0$ *and* c_2

$$c_1 F(|Z|) - c_2 \leq f(Z) \quad \textit{for all} \quad Z \in \mathbb{R}^n. \qquad (5)$$

Remark 3.

(i) Clearly the setting is much more general in comparison to the one considered in [8]: we do not suppose $f(Z) = g(|Z|^2)$ and we merely assume (4).
As a formal difference, Choe studies energy densities admitting some kind of degeneracy as $|Z| \to 0$. This behaviour of the second derivative is covered by Assumption 1 since the validity of (3) is not supposed in the case $|Z| < 1$. We already like to remark that this causes no additional technical difficulties since in any way we make use of a cut-off function vanishing for small value of $|\nabla u|$ in order to study obstacle problems.

(ii) To have the existence of minimizers of Dirichlet boundary value problems in Orlicz–Sobolev spaces one should assume in addition that F is a N-function having the Δ_2-property (see, for instance, [14] or [1]).

(iii) If $\mu < 1$, then ellipticity is good enough to improve 5 to the power growth estimate

$$c_1|Z|^{2-\mu} - c_2 \leq f(Z) \quad \text{for all} \quad Z \in \mathbb{R}^n$$

and with some constants $c_1 > 0$ and c_2. In fact, the convexity of f yields on account of $f(0) = 0$

$$f(Z) \geq f(Z/2) + \nabla f(Z/2) \cdot Z/2.$$

This, together with the inequality (recall $\nabla f(0) = 0$)

$$\nabla f(Z) \cdot Z = \int_0^1 D^2 f(\theta Z)(Z, Z) \mathrm{d}\theta$$

proves the claim.

(iv) Note that $\nabla f(0) = 0$ may be assumed w.l.o.g. since we may replace f by $\bar{f}(Z) := f(Z) - \nabla f(0) \cdot Z$. Of course, normalization also gives w.l.o.g. that $f(0) = 0$.

(v) Note that (3) implies the obvious inequality $2 - \mu \leq q$.

We now formulate

Theorem 1. *Let f be given as above and consider a function $\psi \in W^1_{\infty,loc}(\Omega)$. Moreover, denote by $u \in W^1_{1,\mathrm{loc}}(\Omega)$ a local minimizer of (1) subject to the constraint $u \geq \psi$ a.e., i.e.,*

$$\int_{\widehat{\Omega}} f(\nabla u) \mathrm{d}x < \infty \quad \textit{for any} \quad \widehat{\Omega} \Subset \Omega,$$

$$\int_{\mathrm{spt}\,(u-v)} f(\nabla u) \mathrm{d}x \leq \int_{\mathrm{spt}\,(u-v)} f(\nabla v) \mathrm{d}x$$

for any $v \in W^1_{1,\mathrm{loc}}(\Omega)$, $\mathrm{spt}\,(u - v) \Subset \Omega$, such that $v \geq \psi$, a.e. Suppose further that u is of class $L^\infty_{loc}(\Omega)$. Then we have:

(i) *u is of class $W^1_{\infty,\mathrm{loc}}(\Omega)$;*

(ii) *if in addition inequality (3) holds for any $Z \in \mathbb{R}^n$, then u is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$, if so is the obstacle.*

Remark 4.

(i) If we consider degenerate energy densities with anisotropic (p, q)-growth, then local Lipschitz continuity can be improved to $C^{1,\alpha}_{\mathrm{loc}}$-regularity following [7] (compare [26]). Here an additional hypothesis is needed to control the kind of degeneration of $D^2 f$.

(ii) Following Remark 2, iv), our results extend to the case

$$q < \max\Big\{4 - \mu, (2 - \mu)\frac{n + 2}{n}\Big\}.$$

Note that, if $4 - \mu < (2 - \mu)(n + 2)/n$, then $2 - \mu > n$, in particular $\mu < 1$ and f is at least of growth rate $s = 2 - \mu$. Hence, in this case we may apply Theorem 1.1 of [7] to obtain Theorem 1 (without imposing a local L^∞-bound for u).

(iii) Note that Assumption 1 is strong enough to imply boundedness of solutions of Dirichlet problems whenever the boundary data u_0 are of class L^∞. In fact, we take $w := \min\{u, \sup_\Omega u_0\}$ as a comparison function and recall that $f(0) = 0$ and $\nabla f(0) = 0$. Hence, the strictly convex integrand f attains its minimum in 0 and we get $u \le \sup_\Omega u_0$ by standard arguments.

3. Regularization and Caccioppoli-type inequalities

We denote by $(u)^\varepsilon$, $(\psi)^\varepsilon$ the ε-mollification through a family of smooth mollifiers of the local minimizer u under consideration and the obstacle ψ, respectively; we fix $B := B_R(x_0) \Subset \Omega$ and assume that $B \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ for any small $\varepsilon > 0$ as above. Next we let for any $\delta \in (0,1)$

$$f_\delta(Z) := f(Z) + \delta(1 + |Z|^2)^{\frac{q}{2}},$$

and define v_δ^ε as the unique solution of the Dirichlet problem

$$J_\delta[w, B] := \int_B f_\delta(\nabla w)\mathrm{d}x \to \min, \quad w \in (u)^\varepsilon_{|B} + \overset{\circ}{W}{}^1_q(B), \quad w \ge (\psi)^\varepsilon \quad a.e.$$

If $\delta = \delta(\varepsilon)$ is chosen sufficiently small (see, for instance, [4]), then we obtain writing $v_\varepsilon = v^\varepsilon_{\delta(\varepsilon)}$ and $f_\varepsilon = f_{\delta(\varepsilon)}$

Lemma 1. *With the above notation we have (for some constants c_1, c_2 not depending on ε)*

(i) $\displaystyle\int_B F(|\nabla v_\varepsilon|)\mathrm{d}x \le c_1 < \infty;$

(ii) $v_\varepsilon \rightharpoondown u$ *in* $W^1_1(B)$ *and a.e. as* $\varepsilon \to 0$;

(iii) $\displaystyle\sup_B |v_\varepsilon| \le \sup_{B_{R+\varepsilon}(x_0)} |u| < c_2 < \infty;$

(iv) $\displaystyle\delta(\varepsilon)\int_B (1 + |\nabla v_\varepsilon|^2)^{q/2}\mathrm{d}x \to 0$ *as* $\varepsilon \to 0$;

(v) $\displaystyle\int_B f(\nabla v_\varepsilon)\mathrm{d}x \to \int_B f(\nabla u)\mathrm{d}x$ *as* $\varepsilon \to 0$;

(vi) $\displaystyle\int_B f_\varepsilon(\nabla v_\varepsilon)\mathrm{d}x \to \int_B f(\nabla u)\mathrm{d}x$ *as* $\varepsilon \to 0$.

Proof. As mentioned above, the proof is standard and outlined in detail, for instance, in [3] and [4] – in [7] the reader will find the modifications which are necessary to include obstacles: in the constrained case we first show weak W^1_1 and a.e. convergence of the sequence $\{v_\varepsilon\}$. As a result, the limit v is seen to respect the obstacle by a.e. convergence, hence lower semicontinuity and the uniqueness of solutions yield $v = u$. ■

Next we state a suitable Euler-Lagrange equation, a proof of the following lemma is given in [11].

Lemma 2. *Let Assumption* 1 *hold. Then the functions* v_ε *introduced above are of class* $W^2_{t,\mathrm{loc}}(B)$ *for any* $t < \infty$ *and*

$$\nabla f_\varepsilon(\nabla v_\varepsilon) \in W^1_{t,loc}(B).$$

Moreover, the equation

$$\int_B \nabla f_\varepsilon(\nabla v_\varepsilon)\cdot\nabla\varphi\mathrm{d}x = \int_B \varphi g \mathrm{d}x$$

is valid for any $\varphi \in C^1_0(B)$*, where*

$$g := \mathbf{1}_{\{x\in B:\, v_\varepsilon=(\psi)^\varepsilon\}}\Big(-\operatorname{div}\big[\nabla f_\varepsilon(\nabla(\psi)^\varepsilon)\big]\Big).$$

Remark 5. In the unconstrained case we refer to [21], Chapter 4, in order to obtain the appropriate starting integrability needed in the following.

Lemma 2 is the main tool for proving the following Caccioppoli-type inequalities in the presence of an obstacle.

Lemma 3. *Suppose that Assumption* 1 *holds and fix a number* $L > 1$ *such that for any* ε *as above*

$$L > 1 + \|\nabla(\psi)^\varepsilon\|^2_{L^\infty(B,\mathbb{R}^n)}.$$

(i) *Let* $B_\varkappa := \{x \in B :\, \Gamma_\varepsilon := 1 + |\nabla v_\varepsilon|^2 > \varkappa\}$, $\varkappa > 1$. *Then there is a constant* c*, independent of* ε*, such that for any* $\varkappa > L$ *and for any real number* $s \geq 0$*, and for any* $\eta \in C^\infty_0(B)$, $0 \leq \eta \leq 1$,

$$\begin{aligned}&\int_{B_{2\varkappa}} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon, \partial_\gamma\nabla v_\varepsilon)\Gamma^s_\varepsilon\eta^2\mathrm{d}x\\ &\leq c\int_{B_\varkappa} \big|D^2 f_\varepsilon(\nabla v_\varepsilon)\big|\Gamma^{1+s}_\varepsilon|\nabla\eta|^2\mathrm{d}x.\end{aligned}$$

Here and in the following we always take the sum w.r.t. repeated Greek indices $\gamma = 1, \dots, n$.

(ii) *Let*

$$A(k,r) = A_\varepsilon(k,r) = \big\{x \in B_r(x_0) \colon \Gamma_\varepsilon > k\big\}, \quad k > 1 + L,\ 0 < r < R.$$

Then there is a real number $c > 0$ *such that for any* $\eta \in C_0^\infty(B_r(x_0))$, $0 \le \eta \le 1$, *and for any* $\varepsilon > 0$

$$\int\limits_{A(k,r)} \Gamma_\varepsilon^{-\frac{\mu}{2}} |\nabla \Gamma_\varepsilon|^2 \eta^2 \,\mathrm{d}x \le c \int\limits_{A(k,r)} \big|D^2 f_\varepsilon(\nabla v_\varepsilon)\big| |\nabla \eta|^2 (\Gamma_\varepsilon - k)^2 \,\mathrm{d}x.$$

Proof. ad i). This time we shortly sketch the proof following the idea given in [7], Lemma 2.3: fix $\varkappa > L$ and let for all $t \in \mathbb{R}$

$$\tilde{h}(t) := \min\big\{\max[t-1, 0], 1\big\}, \qquad h(t) = h_\varkappa(t) = \tilde{h}(\varkappa^{-1} t), \tag{7}$$

i.e., $h(t) \equiv 0$ if $t < \varkappa$ and $h(t) \equiv 1$ if $t > 2\varkappa$. Now observe that integrability is good enough (see Lemma 2) to differentiate the Euler equation (6) with the result

$$\begin{aligned} &\int\limits_B D^2 f_\varepsilon(\nabla v_\varepsilon)\big(\partial_\gamma \nabla v_\varepsilon, \nabla(\eta^2 \partial_\gamma v_\varepsilon h(\Gamma_\varepsilon)\Gamma_\varepsilon^s)\big)\,\mathrm{d}x \\ &= -\int\limits_B g \partial_\gamma \big(\eta^2 \partial_\gamma v_\varepsilon h(\Gamma_\varepsilon)\Gamma_\varepsilon^s\big)\,\mathrm{d}x. \end{aligned}$$

On the set of coincidence we have a.e. $\nabla v_\varepsilon = \nabla(\psi)^\varepsilon$ (see [18], Lemma 7.7, p. 152), hence the auxiliary function $h(\Gamma_\varepsilon)$ vanishes on account of $\varkappa > L$. This, together with

$$\int\limits_B D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma v_\varepsilon \partial_\gamma \nabla v_\varepsilon, \nabla \Gamma_\varepsilon) h'(\Gamma_\varepsilon)\Gamma_\varepsilon^s \eta^2 \,\mathrm{d}x \ge 0,$$

$$s\int\limits_B D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma v_\varepsilon \partial_\gamma \nabla v_\varepsilon, \nabla \Gamma_\varepsilon)\Gamma_\varepsilon^{s-1} h(\Gamma_\varepsilon)\eta^2 \,\mathrm{d}x \ge 0$$

(which follows from $h' \ge 0$, $s \ge 0$ and $2\partial_\gamma v_\varepsilon \partial_\gamma \nabla v_\varepsilon = \nabla \Gamma_\varepsilon$) yields

$$\int\limits_B D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon) h(\Gamma_\varepsilon)\Gamma_\varepsilon^s \eta^2 \,\mathrm{d}x$$

$$\leq -2 \int_B D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \nabla \eta)\eta \partial_\gamma v_\varepsilon h(\Gamma_\varepsilon)\Gamma_\varepsilon^s \mathrm{d}x.$$

Finally, Young's inequality proves the claim after absorbing terms.

ad ii). Following the reasoning of [2], Lemma 3.2, (ii), we now have to include the side condition. If we are given $k > 1 + L$, then we choose $\varphi = \eta^2 \partial_\gamma v_\varepsilon \max\left[\Gamma_\varepsilon - k, 0\right]$, η as above. Again Lemma 2 shows the validity of the Euler equation (6) and its differentiated version. As before the right-hand side vanishes since k is large enough, thus

$$\begin{aligned}
&\int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)(\Gamma_\varepsilon - k)\eta^2 \mathrm{d}x \\
&+ \int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \nabla \Gamma_\varepsilon)\partial_\gamma v_\varepsilon \eta^2 \mathrm{d}x \\
&= -2 \int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \nabla \eta)\eta \partial_\gamma v_\varepsilon(\Gamma_\varepsilon - k)\mathrm{d}x. \qquad (8)
\end{aligned}$$

Here the nonnegative first integral on the left-hand side is neglected, the second one satisfies

$$\begin{aligned}
&\int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \nabla \Gamma_\varepsilon)\partial_\gamma v_\varepsilon \eta^2 \mathrm{d}x \\
&= \frac{1}{2} \int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\nabla \Gamma_\varepsilon, \nabla \Gamma_\varepsilon)\eta^2 \mathrm{d}x.
\end{aligned} \qquad (9)$$

The right-hand side of (8) is estimated from above by

$$\begin{aligned}
&c\gamma \int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\nabla \Gamma_\varepsilon, \nabla \Gamma_\varepsilon)\eta^2 \mathrm{d}x \\
&+ c\gamma^{-1} \int_{A(k,r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\nabla \eta, \nabla \eta)(\Gamma_\varepsilon - k)^2 \mathrm{d}x,
\end{aligned} \qquad (10)$$

where we made use of Young's inequality for $\gamma > 0$ sufficiently small. Absorbing terms, the lemma is proved by (8)–(10) and the ellipticity condition (3), which can be applied on account of $k > 1 + L$. ∎

4. Uniform local higher integrability

In contrast to the discussion of the vector-valued case given in [4], we now can apply an iteration procedure to get the following theorem on uniform local higher integrability.

Theorem 2. *Assume that f satisfies Assumption 1 and consider the regularization $\{v_\varepsilon\}$ from above. Then, for any $1 < s < \infty$ and for any ball $B_r(x_0)$, $r < R$, there is a constant c, just depending on the data, $\sup_B |(u)^\varepsilon|$, r, and s such that*

$$\int_{B_r(x_0)} |\nabla v_\varepsilon|^s \,\mathrm{d}x \le c < \infty .$$

Proof. We fix some nonnegative number $\alpha \ge 0$ and let

$$\beta = 4 - \mu - q > 0 ,$$

where the positive sign follows from assumption (4). As a consequence, we may define

$$0 < \sigma := 2 + \frac{\alpha - \beta - \mu}{2} < 2 + \frac{\alpha - \mu}{2} =: \sigma' ,$$

and choose $k \in \mathbb{N}$ sufficiently large such that

$$2k \frac{\sigma}{\sigma'} < 2k - 2. \tag{11}$$

Next we recall the definition of the auxiliary function h given in (7), where we now define h w.r.t. $2\varkappa$, $\varkappa > L + 1$, L given in Lemma 3. Moreover, we choose $r < \rho < \rho' < R$, $\eta \in C_0^\infty(B_{\rho'}(x_0))$, $\eta \equiv 1$ on $B_\rho(x_0)$, $|\nabla \eta| \le c(\rho' - \rho)^{-1}$. With this notation, our starting inequality is derived by performing a partial integration, which is admissible on account of the regularity results of Lemma 2

$$
\begin{aligned}
&\int_B |\nabla v_\varepsilon|^2 \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} h(\Gamma_\varepsilon)\eta^{2k}\mathrm{d}x \\
&= -\int_B v_\varepsilon \partial_\gamma \left[\partial_\gamma v_\varepsilon \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} h(\Gamma_\varepsilon)\eta^{2k}\right]\mathrm{d}x \\
&\le c\int_B |\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} h(\Gamma_\varepsilon)\eta^{2k}\mathrm{d}x \\
&\quad + c\int_B \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} h(\Gamma_\varepsilon)\eta^{2k-1}|\nabla\eta|\mathrm{d}x \\
&\quad + c\int_B \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} h'(\Gamma_\varepsilon)|\nabla v_\varepsilon||\nabla^2 v_\varepsilon|\eta^{2k}\mathrm{d}x. \qquad (12)
\end{aligned}
$$

Note that at this point the uniform bound for v_ε (recall Lemma 1, (ii)) was used in an essential way. The left-hand side of inequality (12) is bounded from below by

$$
\begin{aligned}
\int_B |\nabla v_\varepsilon|^2 \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} h(\Gamma_\varepsilon)\eta^{2k}\mathrm{d}x &\ge c_1(\varkappa)\int_{B\cap[|\nabla v_\varepsilon|\ge 2\varkappa]} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x \\
&\ge c_1(\varkappa)\int_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x - c_2(\varkappa).
\end{aligned}
$$

On the right-hand side of (12) we observe that $h'(\Gamma_\varepsilon)$ identically vanishes outside the set $[2\varkappa \le \Gamma_\varepsilon \le 4\varkappa]$. From this we obtain as an immediate consequence

$$
\int_B \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} h'(\Gamma_\varepsilon)|\nabla v_\varepsilon||\nabla^2 v_\varepsilon|\eta^{2k}\mathrm{d}x \le c(\varkappa)\int_{B_{2\varkappa}} |\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x,
$$

where the definition of $B_{2\varkappa}$ is the same as introduced in Lemma 3. Since it is also obvious that

$$
\int_B |\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} h(\Gamma_\varepsilon)\eta^{2k}\mathrm{d}x \le \int_{B_{2\varkappa}} |\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x,
$$

and since an analogous estimate holds for the remaining integral, we arrive at

$$\begin{aligned}
&\int\limits_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x \\
&\leq c\Big\{1+\int\limits_{B_{2\varkappa}}|\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x+\int\limits_{B_{2\varkappa}}\Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}}\eta^{2k-1}|\nabla\eta|\mathrm{d}x\Big\} \qquad (13)\\
&=: c\big\{1+I+II\big\}.
\end{aligned}$$

Now, for $\gamma > 0$ sufficiently small, Young's inequality yields a bound for II

$$\begin{aligned}
II &\leq \gamma\int\limits_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x+\gamma^{-1}\int\limits_B \Gamma_\varepsilon^{-2-\frac{\alpha-\mu}{2}}\Gamma_\varepsilon^{3+\alpha-\mu}\eta^{2k-2}|\nabla\eta|^2\mathrm{d}x\\
&\leq \gamma\int\limits_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x+\frac{c\gamma^{-1}}{(\rho'-\rho)^2}\int\limits_B \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}}\eta^{2k-2}\mathrm{d}x,
\end{aligned}$$

where the first integral on the right-hand side of (14) may be absorbed on the left-hand side of (13). The discussion of I starts by observing that we have for $\gamma > 0$

$$\begin{aligned}
I &\leq \gamma\int\limits_{B_{2\varkappa}} \Gamma_\varepsilon^{\frac{\alpha+\beta}{2}}\Gamma_\varepsilon^{-\frac{\mu}{2}}|\nabla^2 v_\varepsilon|^2\eta^{2k+2}\mathrm{d}x\\
&\quad+\gamma^{-1}\int\limits_{B_{2\varkappa}} \Gamma_\varepsilon^{2+\frac{\alpha-\beta-\mu}{2}}\eta^{2k-2}\mathrm{d}x =: \gamma I_1+\gamma^{-1}I_2.
\end{aligned}$$

At this point we have to check that I_1 can be handled via Lemma 3, (i): by definition it is clear that $\alpha+\beta \geq 0$. Moreover, the choice of $\varkappa$ implies the validity of assumption (3) on the set $B_\varkappa$ (recall that (3) is only supposed to be true whenever $|Z| > 1$), hence one gets

$$\begin{aligned}
I_1 &\leq c\int\limits_{B_{2\varkappa}} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma\nabla v_\varepsilon)\Gamma_\varepsilon^{\frac{\alpha+\beta}{2}}\left(\eta^{k+1}\right)^2\mathrm{d}x\\
&\leq c\int\limits_{B_\varkappa} \left|D^2 f_\varepsilon(\nabla v_\varepsilon)\right|\Gamma_\varepsilon^{1+\frac{\alpha+\beta}{2}}\eta^{2k}|\nabla\eta|^2\mathrm{d}x \leq \frac{c}{(\rho'-\rho)^2}\int\limits_B \Gamma_\varepsilon^{\frac{\alpha+\beta}{2}}\Gamma_\varepsilon^{\frac{q}{2}}\eta^{2k}\mathrm{d}x.
\end{aligned}$$

Finally, the choice of β implies

$$I \leq \frac{c\gamma}{(\rho'-\rho)^2}\int\limits_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\mathrm{d}x+\gamma^{-1}\int\limits_B \Gamma_\varepsilon^{2+\frac{\alpha-\beta-\mu}{2}}\eta^{2k-2}\mathrm{d}x. \qquad (15)$$

If we let $\gamma = \hat{\gamma}(\rho' - \rho)^2$ and if $\hat{\gamma} > 0$ is sufficiently small, then the first integral on the right-hand side of (15) is absorbed on the left-hand side of (13), and it remains to find a uniform bound for the second one. Here the choice of k, i.e., (11), comes into play: for $1 \gg \tilde{\gamma} > 0$ we get with a final application of Young's inequality

$$\begin{aligned} &\hat{\gamma}^{-1}(\rho' - \rho)^{-2} \int_B \Gamma_\varepsilon^{2+\frac{\alpha-\beta-\mu}{2}} \eta^{2k-2} \mathrm{d}x \\ &\leq c\hat{\gamma}^{-1}(\rho' - \rho)^{-2} \left\{ \tilde{\gamma} \int_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k} \mathrm{d}x + \tilde{\gamma}^{-\frac{\sigma}{\sigma'-\sigma}} |B| \right\}. \end{aligned} \tag{16}$$

Following (13)–(16), letting $\tilde{\gamma} = \gamma'\hat{\gamma}(\rho' - \rho)^2$, $1 \gg \gamma' > 0$ and absorbing terms for a last time we have found a real number $c = c(\varkappa, \alpha, \rho' - \rho, \sup_B |(u)^\varepsilon|)$, independent of ε such that

$$\int_B \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k} \mathrm{d}x \leq c \left\{ 1 + \int_B \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k-2} \mathrm{d}x \right\}. \tag{17}$$

To start an iteration of (17) let

$$\rho_k = r + (R - r)2^{-k}, \quad k = 0, 1, 2, \ldots,$$

as well as

$$\alpha_k = 2k, \text{ i.e., } \alpha_{k+1} = 2 + \alpha_k, \quad k = 0, 1, 2, \ldots,$$

where for any k as above α_k is nonnegative, hence admissible in the above calculations. Then we obtain (17) for any $k = 0, 1, 2, \ldots,$ with the choices $\rho = \rho_{k+1}$, $\rho' = \rho_k$, $\alpha = \alpha_k$, i.e.,

$$\int_{B_{\rho_{k+1}}(x_0)} \Gamma_\varepsilon^{1+\frac{\alpha_{k+1}-\mu}{2}} \mathrm{d}x \leq c \left\{ 1 + \int_{B_{\rho_k}(x_0)} \Gamma_\varepsilon^{1+\frac{\alpha_k-\mu}{2}} \mathrm{d}x \right\}.$$

Iteration completes the proof since the choice $\alpha_0 = 0$ gives a uniformly bounded right-hand side, which is immediate for $\mu \geq 1$, in case $\mu < 1$ we use Remark 3, (iii). ∎

5. Proof of Theorem 1

Once Theorem 1 is established, one may apply a Moser-type iteration (similar to [8]) to obtain uniform local a priori gradient bounds for the regularization $\{v_\varepsilon\}$. We prefer a De Giorgi-type technique (corresponding to [2]), which seems to be much more convenient in the case of "bad" ellipticity. Moreover, the side condition is easily eliminated.

Theorem 3. *Consider a ball $B_{R_0}(x_0) \Subset B$ and an energy density f as classified in Assumption 1. Then there is a positive local constant c such that for any ε as above the regularizations v_ε satisfy the estimate*

$$\|\nabla v_\varepsilon\|_{L^\infty(B_{R_0/2},\mathbb{R}^n)} \le c.$$

Before proving Theorem 3 we are going to establish an auxiliary lemma which was shown in [2], Lemma 6.2, in the case $q = 2$.

Lemma 4. *Consider radii $0 < r < \hat{r} < R_0$ such that $B_{R_0}(x_0) \Subset B$. Then there is a real number c, independent of r, $\hat{r}$, R_0, k and ε, satisfying for any $k > 1 + L$ (L as above)*

$$\int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}} \,\mathrm{d}x \le \frac{c}{(\hat{r}-r)^{\frac{n}{n-1}}} \Bigg[\int_{A(k,\hat{r})} \Gamma_\varepsilon^{\frac{q-2}{2}} (\Gamma_\varepsilon - k)^2 \,\mathrm{d}x\Bigg]^{\frac{1}{2}\frac{n}{n-1}} \Bigg[\int_{A(k,\hat{r})} \Gamma_\varepsilon^{\frac{\mu}{2}} \,\mathrm{d}x\Bigg]^{\frac{1}{2}\frac{n}{n-1}}, \tag{18}$$

where the sets $A(k,r) = \{x \in B_r(x_0) : \Gamma_\varepsilon > k\}$ are introduced in Lemma 3.

Proof of Lemma 4. With the notion $w^+ = \max[w,0]$, Sobolev's inequality yields for $\eta \in C_0^\infty(B_{\hat{r}}(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla \eta| \le c/(\hat{r}-r)$,

$$\begin{aligned}
\int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}} \,\mathrm{d}x &\le \int_{B_{\hat{r}}(x_0)} \big[\eta(\Gamma_\varepsilon - k)^+\big]^{\frac{n}{n-1}} \,\mathrm{d}x \\
&\le c\Bigg[\int_{B_{\hat{r}}(x_0)} \big|\nabla\big[\eta(\Gamma_\varepsilon - k)^+\big]\big| \,\mathrm{d}x\Bigg]^{\frac{n}{n-1}} \le c\Bigg[\int_{A(k,\hat{r})} \big|\nabla\big[\eta(\Gamma_\varepsilon - k)\big]\big| \,\mathrm{d}x\Bigg]^{\frac{n}{n-1}} \\
&\le c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big],
\end{aligned} \tag{19}$$

where we have (recall $2-\mu \leq q$)

$$\begin{aligned}
I_1^{\frac{n}{n-1}} &:= \Big[\int_{A(k,\hat r)} |\nabla\eta|(\Gamma_\varepsilon - k)\mathrm{d}x\Big]^{\frac{n}{n-1}}\\
&\leq \Big[\int_{A(k,\hat r)} |\nabla\eta|^2\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{2-q}{2}}\,dx\Big]^{\frac{1}{2}\frac{n}{n-1}}\\
&\leq \frac{c}{(\hat r - r)^{\frac{n}{n-1}}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{\mu}{2}}\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}},
\end{aligned}$$

thus $I_1^{\frac{n}{n-1}}$ is seen to be bounded from above by the right-hand side of (18). Estimating I_2 we recall the choice $k > 1+L$, hence it is possible to refer to Lemma 3, (ii), with the result

$$\begin{aligned}
I_2^{\frac{n}{n-1}} &:= \Big[\int_{A(k,\hat r)} \eta|\nabla\Gamma_\varepsilon|\mathrm{d}x\Big]^{\frac{n}{n-1}}\\
&\leq \Big[\int_{A(k,\hat r)} \eta^2|\nabla\Gamma_\varepsilon|^2\Gamma_\varepsilon^{-\frac{\mu}{2}}\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{\mu}{2}}\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}}\\
&\leq c\Big[\int_{A(k,\hat r)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\nabla\eta,\nabla\eta)(\Gamma_\varepsilon-k)^2\Big]^{\frac{1}{2}\frac{n}{n-1}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{\mu}{2}}\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}}\\
&\leq \frac{c}{(\hat r - r)^{\frac{n}{n-1}}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}}\Big[\int_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{\mu}{2}}\mathrm{d}x\Big]^{\frac{1}{2}\frac{n}{n-1}},
\end{aligned}$$

and the lemma follows from (19). ■

Proof of Theorem 3. Again we follow the reasoning of [2]. Starting with the left-hand side of (18) we fix a real number $s > 1$ and observe that Hölder's inequality implies

$$\begin{aligned}
&\int_{A(k,r)}\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x = \int_{A(k,r)}(\Gamma_\varepsilon-k)^{\frac{n}{n-1}\frac{1}{s}}\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^{2-\frac{n}{n-1}\frac{1}{s}}\mathrm{d}x\\
&\leq \Big[\int_{A(k,r)}(\Gamma_\varepsilon-k)^{\frac{n}{n-1}}\mathrm{d}x\Big]^{\frac{1}{s}}\Big[\int_{A(k,r)}\Gamma_\varepsilon^{\frac{q-2}{2}\frac{s}{s-1}}(\Gamma_\varepsilon-k)^{\left(2-\frac{n}{n-1}\frac{1}{s}\right)\frac{s}{s-1}}\Big]^{\frac{s-1}{s}}.
\end{aligned}$$

Theorem 2 ensures the existence of a real number $c_1(s,n,B_{R_0}(x_0))$, independent of ε,

$$c_1(s,n,B_{R_0}(x_0)) := \sup_{\varepsilon>0}\Bigg[\int\limits_{B_{R_0}(x_0)} \Gamma_\varepsilon^{\frac{s}{s-1}\left(\frac{q-2}{2}+2-\frac{n}{n-1}\frac{1}{s}\right)}\mathrm{d}x\Bigg]^{\frac{s-1}{s}} < \infty$$

such that

$$\int\limits_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x \le c_1\Bigg[\int\limits_{A(k,r)}(\Gamma_\varepsilon-k)^{\frac{n}{n-1}}\mathrm{d}x\Bigg]^{\frac{1}{s}}. \tag{20}$$

In a similar way one obtains

$$\int\limits_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{\mu}{2}}\mathrm{d}x \le c_2(t,\mu,B_{R_0}(x_0))\Bigg[\int\limits_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{q-2}{2}}\mathrm{d}x\Bigg]^{\frac{1}{t}}, \tag{21}$$

where $t>1$ is a fixed second parameter. Combining (18), (20), and (21) it is proved that

$$\begin{aligned}&\int\limits_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x\\ &\le \frac{c}{(\hat r-r)^{\frac{n}{n-1}\frac{1}{s}}}\Bigg[\int\limits_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x\Bigg]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}}\Bigg[\int\limits_{A(k,\hat r)}\Gamma_\varepsilon^{\frac{q-2}{2}}\mathrm{d}x\Bigg]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}.\end{aligned} \tag{22}$$

For $k>1+L$ and $r<\hat r$ as above we now let

$$\tau(k,r) := \int\limits_{A(k,r)}\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\mathrm{d}x,\quad a(k,r) := \int\limits_{A(k,r)}\Gamma_\varepsilon^{\frac{q-2}{2}}\mathrm{d}x,$$

hence (22) can be rewritten as

$$\tau(k,r) \le \frac{c}{(\hat r-r)^{\frac{n}{n-1}\frac{1}{s}}}\big[\tau(k,\hat r)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}}\big[a(k,\hat r)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}. \tag{23}$$

Given two real numbers $h>k>1+L$ we have the obvious estimate

$$a(h,\hat r) \le \frac{1}{(h-k)^2}\tau(k,\hat r),$$

which together with (23) implies

$$\begin{aligned}\tau(h,r) &\le \frac{c}{(\hat{r}-r)^{\frac{n}{n-1}\frac{1}{s}}}\big[\tau(h,\hat{r})\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}}\frac{1}{(h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}}\big[\tau(k,\hat{r})\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}\\ &\le \frac{c}{(\hat{r}-r)^{\frac{n}{n-1}\frac{1}{s}}}\frac{1}{(h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}}\big[\tau(k,\hat{r})\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left(1+\frac{1}{t}\right)}.\end{aligned}$$

Finally s and t are chosen sufficiently close to 1 (depending on n) such that

$$\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right]=:\beta>1,$$

moreover we let

$$\alpha := \frac{n}{n-1}\frac{1}{s}\frac{1}{t}>0, \qquad \gamma := \frac{n}{n-1}\frac{1}{s}>0.$$

Then Theorem 3 is an immediate application of the following well known lemma (compare, for instance, [27], Lemma 5.1, p. 219) to the function $\tau(h,r)$, where we once more benefit from Theorem 2. ■

Lemma 5. *Assume that $\varphi(h,\rho)$ is a nonnegative real valued function defined for $h>k_0$ and $\rho<R_0$. Suppose further that for fixed ρ the function is nonincreasing in h and that it is nondecreasing in ρ if h is fixed. Then*

$$\varphi(h,\rho)\le \frac{C}{(h-k)^{\alpha}(R-\rho)^{\gamma}}\big[\varphi(k,R)\big]^{\beta},\quad h>k>k_0,\quad \rho<R<R_0,$$

with some positive constants C, α, $\beta>1$, and γ implies for all $0<\sigma<1$

$$\varphi(k_0+d,R_0-\sigma R_0)=0,$$

where the quantity d is given by

$$d^{\alpha}=\frac{2^{(\alpha+\beta)\beta/(\beta-1)}C\big[\varphi(k_0,R_0)\big]^{\beta-1}}{\sigma^{\gamma}R_0^{\gamma}}.$$

Since the data of the obstacle just enter through the choice of the constant L, the Proof of Theorem 1, (i), is an immediate consequence of Lemma 1. Now, having established (i), the second assertion follows from the arguments outlined in the well known paper [26] (compare also [13] for details). ■

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