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A UNIQUENESS THEOREM FOR THE DUAL PROBLEM ASSOCIATED TO A VARIATIONAL PROBLEM WITH LINEAR GROWTH

ABSTRACT. Uniqueness is proved for solutions of the dual problem which is associated to the minimum problem $\int_\Omega f(\nabla u)dx \to \min$ among mappings $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ with prescribed Dirichlet boundary data and for smooth strictly convex integrands f of linear growth. No further assumptions on f or its conjugate function f^* are imposed, in particular f^* is not assumed to be strictly convex. One special solution of the dual problem is seen to be a mapping into the image of ∇f which immediately implies uniqueness.

For strictly convex smooth integrands $f : \mathbb{R}^{nN} \to \mathbb{R}$ of linear growth the minimization problem

$$J(u) := \int_\Omega f(\nabla u)dx \to \min \text{ in } u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N) \tag{1}$$

in general fails to have a solution in the Sobolev class W^1_1. By way of contrast the associated dual problem is solvable (see [4]) and admits clear physical or geometrical interpretations, for example the solution σ corresponds to the normal vector field in the case of minimal surfaces or represents the stress tensor in the case of plastic materials (see [4, 5, 7, 10, 11]). Here we concentrate on the uniqueness of the dual solution which usually is ensured by assuming the conjugate function f^* to be strictly convex (see [4], (3.34), p. 146). Since this condition is formulated in terms of the conjugate function it might be difficult to be verified for a given class of integrands f. Thus, the condition was replaced in [3] by an assumption on f, namely that the closure of the image $\mathrm{Im}(\nabla f)$ should be a strictly convex set.

Here we show by means of more or less elementary arguments that strict convexity, smoothness and linear growth of f in the sense of Assumption 1 already imply uniqueness of the dual solution without additional restrictions. To prove this, we first recall the general setting including the notion of the δ-approximation. Passing to the limit $\delta \to 0$

we then show that the $W^1_{2,\mathrm{loc}}$ regularity result of [2] (see also [8] and [9]) yields one special maximizer σ of the dual problem which is almost everywhere seen to be a mapping into the open set $\mathrm{Im}(\nabla f)$ where f^* is known to be strictly convex by strict convexity of f. Thus, there is one solution σ for which we do not have to care about the fact that f^* on the closure of $\mathrm{Im}(\nabla f)$ might not be strictly convex and this is enough to prove uniqueness.

Throughout this paper it is assumed that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that the integrand satisfies the following general hypothesis:

Assumption 1. *The function f is smooth, strictly convex and of linear growth in the following sense:*

(i) $f \in C^2(\mathbb{R}^{nN}, \mathbb{R})$.

(ii) $f((1-\lambda)X + \lambda Y) < (1-\lambda)f(X) + \lambda(Y)$ *for all* $X \neq Y \in \mathbb{R}^{nN}$ *and for all* $0 < \lambda < 1$.

Suppose further that there is a real number M such that for all, $X, Y \in \mathbb{R}^{nN}$

$$0 \leqslant D^2 f(X)(Y,Y) \leqslant M \frac{1}{\sqrt{1+|X|^2}} |Y|^2 .$$

(iii) *There is a real number A such that $|\nabla f(X)| \leqslant A$ for all $X \in \mathbb{R}^{nN}$.*

(iv) *For numbers $a > 0$ and $b \in \mathbb{R}$ we have $f(X) \geqslant a|X| + b$ for all $X \in \mathbb{R}^{nN}$.*

Remark 2. According to the approximation arguments of [2], Remark (6.3), the boundary values u_0 in problem (1) are always supposed to be of class $W^1_2(\Omega, \mathbb{R}^N)$.

Now let us shortly introduce the dual problem – an intensive discussion of the general setting can be found, for instance, in [4, 6] or [13].

Definition 3. *Consider a Banach space V, its dual V^* and a function $G : V \to \bar{R}$. Then the polar or conjugate function of G is defined for all $v^* \in V^*$ by*

$$G^*(v^*) := \sup_{v \in V} \{\langle v, v^* \rangle - G(v)\}.$$

The following representation formula can be derived by elementary arguments:

$$J(u) = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^{nN})} \left\{ \int_\Omega \tau : \nabla u dx - \int_\Omega f^*(\tau) dx \right\}, \tag{2}$$

where the symbol $X : Y$ denotes the standard scalar product in $\mathbb{R}^{nN}$.

Remark 4. Formula (2) can also be derived from [4], Chapter III.4, pp. 58, if we set $J(u) = G(u, \Lambda u)$, $\Phi(u, p) = J(u, \Lambda u - p)$ and consider the case $\Lambda = \nabla$.

The next step is to define the Lagrangian $l(u, \tau)$ for all $(u, \tau) = (u_0 + \varphi, \tau)$ in the class $u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N) \times L^\infty(\Omega, \mathbb{R}^{nN})$:

$$l(u, \tau) := \int_\Omega \tau : \nabla u dx - \int_\Omega f^*(\tau) dx = l(u_0, \tau) + \int_\Omega \tau : \nabla \varphi dx.$$

Now, the dual function $R : L^\infty(\Omega, \mathbb{R}^{nN}) \to \bar{\mathbb{R}}$ reads as

$$R(\tau) := \inf_{u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)} l(u, \tau),$$

and we have (again see [4])

$$\inf_{u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)} J(u) = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^{nN})} R(\tau).$$

Finally, the dual problem is the maximization problem

$$R(\tau) \to \max \quad \text{among all functions} \quad \tau \in L^\infty(\Omega, \mathbb{R}^{nN}). \tag{3}$$

The existence of a maximizer is well known in duality theory (once more refer to [4]). To prove the uniqueness result one special maximizing sequence has to be introduced: consider for $\delta \in (0, 1)$

$$J_\delta(w) := \frac{\delta}{2} \int_\Omega |\nabla w|^2 dx + J(w), \quad w \in u_0 + \overset{\circ}{W}{}^1_2(\Omega, \mathbb{R}^N),$$

and denote by u_δ the unique minimizer of J_δ in the class $u_0 + \overset{\circ}{W}{}^1_2(\Omega, \mathbb{R}^N)$. Setting

$$\sigma_\delta := \delta \nabla u_\delta + \nabla f(\nabla u_\delta),$$

it is proved in [2], Section 3, (see also [9]) that (after passing to a subsequence)

$$\sigma_\delta \rightharpoonup: \sigma \quad \text{in} \quad L^2(\Omega, \mathbb{R}^{nN}) \quad \text{as} \quad \delta \to 0 \tag{4}$$

and that this weak limit σ is a solution of the dual Problem (3). The proof also gives

$$\delta \int_\Omega |\nabla u_\delta|^2 dx \to 0 \quad \text{as} \quad \delta \to 0. \tag{5}$$

Thus, the definition of σ_δ is in some sense an approximation of the classical duality relation $\sigma = \nabla f(\nabla u)$ (see [4], Chapter III, pp. 46).

The first regularity result of [2] was given in Section 4 where the sequence $\{\sigma_\delta\}$ was shown to be uniformly bounded in $W^1_{2,\mathrm{loc}}$ (see also [8, 10] and [11]). Note that the cited results do not depend on the stronger ellipticity condition which was assumed in [2].

Theorem 5 below is the main tool to prove the uniqueness of the dual solution. Here and in the following we let $U := \mathrm{Im}(\nabla f)$.

Theorem 5. *If σ denotes the limit of the δ-approximation (compare* (4)*), then*

$$|\{x \in \Omega : \sigma(x) \in \partial U\}| = 0.$$

For the proof we need an observation:

Lemma 6. *For all real numbers $K > 0$ there is an $\varepsilon > 0$ such that for all $X \in \mathbb{R}^{nN}$*

$$\mathrm{dist}(\nabla f(X), \partial U) < \varepsilon \Longrightarrow |X| > K.$$

Proof of Lemma 6. Notice that, by strict convexity, $X \neq Y \in \mathbb{R}^{nN}$ implies

$$[\nabla f(X) - \nabla f(Y)] : (X - Y) > 0,$$

i.e., ∇f is a one-to-one mapping. In fact, we have

$$[\nabla f(X) - \nabla f(Y)] : (X - Y) =$$

$$= \int_0^1 D^2 f(sX + (1-s)Y)((X-Y),(X-Y))ds \geqslant 0.$$

Setting $g(s) := f(sX + (1-s)Y)$, equality would give $g''(s) \equiv 0$ for all $s \in (0,1)$ which contradicts strict convexity. Now, fix a real number $K > 0$. Since ∇f is continuous and one-to-one we may apply the Theorem on Domain Invariance (compare [12], Corollary 3.22, p. 77) to see that U is an open set. Thus

$$\nabla f(\overline{B_K(0)}) \Subset U,$$

hence there is an $\varepsilon = \varepsilon(K)$ such that

$$\mathrm{dist}\,\big(\nabla f(\overline{B_K(0)}), \partial U\big) > \varepsilon,$$

and the lemma is proved. $\square$

Proof of Theorem 5. Suppose by contradiction that there is a real number $\gamma > 0$ and a set $\Sigma \subset \Omega$ satisfying

$$|\Sigma| > \gamma \quad \text{and} \quad \sigma(x) \in \partial U \quad \text{for all} \quad x \in \Sigma.$$

Here and in the following sets of measure zero are neglected. If necessary, we always choose suitable subsequences from our given sequence $\{\delta_m\}_{m\in N}$ – omitting the indices – such that all the limits below are well defined.

Since the approximating sequence σ_δ is uniformly bounded in $W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ we may assume

$$\sigma_\delta(x) \to \sigma(x) \quad \text{for all} \quad x \in \Sigma \quad \text{as} \quad \delta \to 0.$$

Because of (5), in addition

$$\delta \nabla u_\delta(x) \to 0 \quad \text{for all} \quad x \in \Sigma \quad \text{as} \quad \delta \to 0$$

can be assumed. Now Egoroff's Theorem gives a measurable set $E \subset \Omega$ such that

$$|\Omega \sim E| < \frac{\gamma}{2}$$

and such that

$$\sigma_\delta \overset{\text{uniformly}}{\longrightarrow} \sigma \quad \text{and} \quad \delta\nabla u_\delta \overset{\text{uniformly}}{\longrightarrow} 0 \quad \text{on} \quad E \quad \text{as} \quad \delta \to 0. \tag{6}$$

Since E is measurable,

$$|\Sigma| = |\Sigma \cap E| + |\Sigma \sim E|$$

immediately yields

$$|\Sigma \cap E| > \frac{\gamma}{2}. \tag{7}$$

By construction $\{u_\delta\}$ is a J-minimizing sequence (see [2]), in particular there is a real number $c_0 > 0$ such that for all δ sufficiently small

$$\int_\Omega |\nabla u_\delta| dx < c_0. \tag{8}$$

Given c_0 we choose $K > 2c_0/\gamma$ and $0 < \varepsilon = \varepsilon(K)$ according to Lemma 6. The uniform convergence (6) proves that for all $x \in E$ and for all δ sufficiently small

$$|\sigma_\delta(x) - \sigma(x)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\delta\nabla u_\delta(x)| < \frac{\varepsilon}{2},$$

hence the definition of σ_δ gives

$$|\nabla f(\nabla u_\delta(x)) - \sigma(x)| < \varepsilon \quad \text{for all} \quad x \in E.$$

Moreover, if $x \in \Sigma \cap E$ then $\sigma(x) \in \partial U$ implies

$$\operatorname{dist}(\nabla f(\nabla u_\delta(x)), \partial U) < \varepsilon.$$

Since $\varepsilon = \varepsilon(K)$ was chosen according to Lemma 6 we arrive at

$$|\nabla u_\delta(x)| > K \quad \text{for all} \quad x \in \Sigma \cap E$$

and finally get, using (7), a contradiction to (8):

$$\int\limits_\Omega |\nabla u_\delta(x)| dx \geqslant \int\limits_{\Sigma\cap E} |\nabla u_\delta(x)| dx > \int\limits_{\Sigma\cap E} K\, dx > \frac{\gamma}{2} K = c_0.$$

Thus the theorem is proved. □

Theorem 7. *With the above assumptions on f the dual problem* (3) *has a unique solution.*

Proof. The weak limit σ of the δ-approximation (see (4)) was proved in [2] (see Lemma 3.1) to be a solution of the dual problem. Suppose by contradiction that the dual problem admits a second maximizer $\tilde{\sigma} \neq \sigma$ and recall that by the above theorem $\sigma(x) \in U$ for almost every $x \in \Omega$. Now Ω is devided into four parts,

$$\Omega = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3,$$

where we have by definition

$$\begin{aligned}
&\sigma(x) = \tilde{\sigma}(x) && \text{for } x \in \Sigma_0,\\
&\sigma(x) \neq \tilde{\sigma}(x),\ \sigma(x) \in U,\ \tilde{\sigma}(x) \in U && \text{for } x \in \Sigma_1,\\
&\sigma(x) \neq \tilde{\sigma}(x),\ \sigma(x) \in U,\ \tilde{\sigma}(x) \in \partial U && \text{for } x \in \Sigma_2,\\
&|\Sigma_3| = 0.
\end{aligned}$$

The last equality is justified by Theorem 5 and by the fact that $f^* \equiv \infty$ outside the set $\bar{U}$. By definition f^* is a lower semicontinuous convex function on $\bar{U}$ and by

$$f(u) + f^*(\nabla f(u)) = u : \nabla f(u)$$

(see [4], Proposition 5.1, p.21) it is also seen to be strictly convex on U. This implies

$$|\Sigma_1| = 0. \tag{9}$$

In fact, setting

$$\tau(x) := \frac{\sigma(x) + \tilde{\sigma}(x)}{2},$$

convexity gives for almost every $x \in \Omega$

$$f^*(\tau(x)) \leqslant \frac{1}{2} f^*(\sigma(x)) + \frac{1}{2} f^*(\tilde{\sigma}(x))$$

and the strict inequality holds for all $x \in \Sigma_1$. Now observe that by definition $\operatorname{div} \sigma_\delta = 0$ in the sense of distributions and the same holds for σ. So we obtain

$$\begin{aligned} R(\tau) &= l(u_0, \tau) \\ &= \int\limits_\Omega \left(\frac{\sigma + \tilde{\sigma}}{2}\right) : \nabla u_0 dx - \int\limits_{\Omega \sim \Sigma_1} f^*(\tau) dx - \int\limits_{\Sigma_1} f^*(\tau) dx \\ &\geqslant \int\limits_\Omega \left(\frac{\sigma + \tilde{\sigma}}{2}\right) : \nabla u_0 dx - \frac{1}{2} \int\limits_\Omega f^*(\sigma) dx - \frac{1}{2} \int\limits_\Omega f^*(\tilde{\sigma}) dx \\ &= \frac{1}{2} R(\sigma) + \frac{1}{2} R(\tilde{\sigma}) = \sup R. \end{aligned} \tag{10}$$

Since the strict inequality would hold for $|\Sigma_1| > 0$, assertion (9) is proved by contradiction. We next claim that

$$|\Sigma_2| = 0. \tag{11}$$

To prove (11), fix $x \in \Sigma_2$, that is

$$P := \sigma(x) \in U \quad \text{and} \quad Q := \tilde{\sigma}(x) \in \partial U.$$

Now U is an open set and Theorem 6.1, [6], p. 45, proves

$$[P, Q) := \{(1-t)P + tQ : 0 \leqslant t < 1\} \subset U. \tag{12}$$

If we set for $m \in \mathbb{N}$ sufficiently large

$$\begin{aligned} Q_m &:= P + \frac{m-1}{m}(Q - P) \quad \text{and} \\ R_m &:= P + \frac{1}{2}(Q_m - P) = P + \frac{m-1}{2m}(Q - P), \end{aligned}$$

and if we introduce for $t \in (0,1)$ the function

$$g(t) = f^*(P + t(Q - P)),$$

which is by (12) known to be strictly convex, then we see (compare, for instance, the proof of Theorem 4.4, [6], p. 26) that there is a real number $0 < \varepsilon = \varepsilon(P,Q)$ such that

$$f^*(R_m) - f^*(P) = \int\limits_0^{\frac{m-1}{2m}} g'(s)ds \leqslant g'\left(\frac{m-1}{2m}\right)\frac{m-1}{2m} - \varepsilon,$$

$$f^*(Q_m) - f^*(R_m) = \int\limits_{\frac{m-1}{2m}}^{\frac{m-1}{m}} g'(s)ds \geqslant g'\left(\frac{m-1}{2m}\right)\frac{m-1}{2m} + \varepsilon.$$

Combining these inequalities it is proved that

$$f^*(R_m) \leqslant \frac{1}{2}f^*(P) + \frac{1}{2}f^*(Q_m) - \varepsilon. \tag{13}$$

Since a lower semicontinuous convex function is continuous on lines up to the boundary and since ε is not depending on m we may pass in (13) to the limit $m \to \infty$ and obtain

$$f^*(\tau(x)) = f^*\left(\frac{\sigma(x) + \tilde{\sigma}(x)}{2}\right) < \frac{1}{2}f^*(\sigma(x)) + \frac{1}{2}f^*(\tilde{\sigma}(x))$$

$$\text{for all} \quad x \in \Sigma_2.$$

The same computation as in (10) gives (11), we have shown

$$\Omega = \Sigma_0 \cup (\Omega \sim \Sigma_0) \quad \text{with} \quad |\Omega \sim \Sigma_0| = 0$$

and the theorem is proved by the definition of Σ_0. □

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