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Local regularity of solutions of variational problems for the equilibrium configuration of an incompressible, multiphase elastic body

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Abstract. We consider a multiphase, incompressible, elastic body with $\mathbf{k}$ preferred states whose equilibrium configuration is described in terms of a nonconvex variational problem. We pass to a suitable relaxed variational integral whose solution has the meaning of the strain tensor and also study the associated dual problem for the stresses. At first we show that the strain tensor is smooth near any point of strict $\mathbf{J}^1_m$-quasiconvexity of the relaxed integrand. Then we use this result to get regularity of the stress tensor on the union of pure phases at least in the two-dimensional case.

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1 Introduction

Consider a multiphase elastic body with k preferred states which is in equilibrium under a given volume load f. Assume further that the temperature is fixed. Then the equilibrium configuration is described by the variational problem ($\mathcal{P}$): find a displacement u: $\Omega \to \mathbb{R}^d$ such that

$$\begin{aligned} I(u) &= \inf_{\mathcal{C}} I(v), \\ I(u) &= \int_\Omega (g(\varepsilon(u)) - f \cdot u)\, dx, \quad \varepsilon(u) \;=\; \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}) \ \text{(strain tensor)}, \\ g &= \min\{g_1, \dots, g_k\}, \end{aligned}$$

where g_i is the elastic potential of the ith phase, $i = 1, 2, \dots, k$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and $\mathcal{C} \subset u_0 + \overset{\circ}{W}{}^1_m(\Omega, \mathbb{R}^d)$ for a suitable $m > 1$. In addition, from now on the incompressible case is considered, i.e. $\operatorname{div} u = 0$ and therefore

$$\mathcal{C} \;=\; \{u \in u_0 + \overset{\circ}{W}{}^1_m(\Omega, \mathbb{R}^d) : \operatorname{div} u \equiv 0\}.$$

Problem ($\mathcal{P}$) may fail to have solutions, even without the incompressibility condition, and therefore one passes to a relaxed problem which means that a suitable quasiconvex envelope Qg is introduced taking care of the constraint $\operatorname{div} u = 0$. The relaxed problem then reads

$$(\mathcal{QP}) \quad \begin{cases} \text{to find a displacement field } u \in \mathcal{C} \text{ such that} \\ QI(u) \;=\; \inf_{\mathcal{C}} QI(v) \\ QI(u) \;:=\; \int_\Omega \{Qg(\varepsilon(u)) - f \cdot u\}\, dx \end{cases}$$

If u is a solution of ($\mathcal{QP}$), then one is interested in the regularity properties of u which is a quite delicate question since as a matter of fact one expects degeneracy of Qg, and since the representation formula obtained in [SE3] is not local, it is hard to decide where degeneracy occurs. So, much attention has been paid to get explicit formulas for Qg (compare [KO], [SE3]) and regularity with the help of some explicit formulas was proved in [SE3] and [FS1]. But due to the complex nature of the problem, success has been obtained only in very special cases.

In place of this we investigate the smoothness of solutions to ($\mathcal{QP}$) via local arguments in the spirit of [AG] and [AF]. To this purpose we first prove Theorem 2.1, assuming for simplicity that the volume load vanishes. Roughly speaking, Theorem 2.1 states that if g has m-growth ($m \geq 2$), g is of class C^2 in some neighbourhood of $\varepsilon_0 = \varepsilon(u)(x_0)$ and J^1_m-strictly quasiconvex at ε_0, then u is smooth in some neighbourhood of x_0 provided $\varepsilon(u)$ is close in measure to ε_0 on balls centered at x_0. Concerning the notion of J^1_m-quasiconvexity we refer the reader to [SE3]. The result is also true for sufficiently regular f.

Of course, Theorem 2.1 has a counterpart in the spirit of [AG] for globally convex integrands g formulated in Theorem 6.1. It is interesting that even in the case of global convexity of g Theorem 2.1 gives sometimes better results than Theorem 6.1 as it is shown by an example (see Proposition 6.5 I and Remark 6.6).

Let us come back to $(\mathcal{QP})$ and assume that $g_1, \dots, g_k$ are just quadratic potentials (a general version is given in Section 7). Suppose further that

$$Qg \;=\; g^{**}, \tag{1.1}$$

where g^{**} is the second Young transform of g defined on the space of all $d\times d$ matrices which are symmetric with zero trace. Condition (1.1) can be verified in the two-dimensional case (see [SE3], Theorem 2.3) but in general does not hold for $d = 3$ even if we just consider two wells, we refer to [SE3] for a counterexample. Under additional assumptions explicit formulas for Qg (implying (1.1)) were given in [FS1] and also in [SE3]. Here, we prove our main result (see Theorems 7.2 and 7.3) without using any explicit representation of Qg. In particular, in Theorem 7.3 it is stated that the stress tensor is regular on the union of pure phases which is an open set. Let us remark again that in the incompressible two-dimensional case (1.1) holds for any number k of quadratic or m-growth potentials. Thus we have a generalisation of Theorem 2.2 in [FS1]. Nevertheless Theorem 2.2 of [FS1] is slightly stronger in the sense that in this special setting x_0 is only required to be a Lebesgue point of the stress tensor σ which due to the weak differentiability of σ (compare Theorem 2.1 in [FS1]) holds up to a set of Hausdorff-dimension zero. For completeness we would like to mention that in the case of two wells a more refined analysis of the smoothness of the stress tensor σ is possible. According to [SE3], Theorem 2.7, we can define a quadratic function of σ which controls the distribution of phases and which is everywhere continuous on Ω.

2 Local regularity of the elastic displacement in points of strict quasiconvexity

As usual $\mathbb{M}^d$ denotes the space of all real $d \times d$ matrices, $\mathring{\mathbb{M}}^d$ the subspace of matrices with vanishing trace, $\mathbb{S}^d$ the subspace consisting of symmetric matrices, $\mathring{\mathbb{S}}^d$ the subspace of symmetric matrices with vanishing trace. We set for $u = (u_i)$, $v = (v_i) \in \mathbb{R}^d$, for $\varkappa = (\kappa_{ij})$, $\kappa = (\kappa_{ij}) \in \mathbb{M}^d$ and for $\varkappa^{\mathrm{T}} := (\varkappa_{ji}) \in \mathbb{M}^d$

$$\begin{aligned} u\cdot v \;&:=\; u_i v_i, \quad |u| \;:=\; \sqrt{u\cdot u}, \quad u\otimes v \;:=\; (u_i v_j) \in \mathbb{M}^d,\\ \varkappa : \kappa \;&:=\; \operatorname{tr}(\varkappa^{\mathrm{T}}\kappa) \;=\; \varkappa_{ij}\kappa_{ij}, \quad |\varkappa| \;:=\; \sqrt{\varkappa:\varkappa}, \quad \varkappa u \;:=\; (\varkappa_{ij}u_j) \in \mathbb{R}^d, \end{aligned}$$

where we always take the sum over repeated Latin indices from 1 to d. For balls in $\mathbb{R}^d$ the symbol $B(\cdot,\cdot)$ is used, balls in $\mathring{\mathbb{S}}^d$ are denoted by $\mathcal{B}(\cdot,\cdot)$. In the following $\Omega \subset \mathbb{R}^d$ is assumed to be a bounded Lipschitz domain and we consider

$$I(u,\Omega) \;=\; \int_\Omega g(\varepsilon(u))\,dx, \quad u \in J^1_m(\Omega,\mathbb{R}^d),$$

where $\varepsilon(u)$ is the symmetric part of the gradient of the vector-field u, $\varepsilon(u(x)) := \frac{1}{2}(\nabla u(x) + (\nabla u(x))^{\mathrm{T}})$, and the space $J^1_m(\Omega)$ is defined below. As a general hypothesis the integrand g, $g : \overset{\circ}{\mathbb{S}}{}^d \to \mathbb{R}$, is a locally Lipschitz function satisfying

$$\left|\frac{\partial g}{\partial \kappa}(\kappa)\right| \;\le\; c_1(1 + |\kappa|^{m-1}) \tag{2.1}$$

for some $m \ge 2$ and for a.e. $\kappa \in \overset{\circ}{\mathbb{S}}{}^d$. This immediately gives

$$|g(\kappa)| \;\le\; c_2(1 + |\kappa|^m) \quad \text{for all } \; \kappa \in \overset{\circ}{\mathbb{S}}{}^d. \tag{2.2}$$

The following spaces are used throughout this paper:

$$\begin{aligned}
\overset{\bullet}{\mathrm{C}}{}^\infty(\Omega, \mathbb{R}^d) \;&:=\; \{v \in C^\infty_0(\Omega, \mathbb{R}^d) : \;\; \operatorname{div} v = 0 \;\text{ in }\; \Omega\},\\
J^1_m(\Omega, \mathbb{R}^d) \;&:=\; \{v \in W^1_m(\Omega, \mathbb{R}^d) : \;\; \operatorname{div} v = 0 \;\text{ in }\; \Omega\},\\
\overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d) \;&:=\; \text{closure of } \; \overset{\bullet}{\mathrm{C}}{}^\infty(\Omega, \mathbb{R}^d) \;\text{ in }\; W^1_m(\Omega, \mathbb{R}^d).
\end{aligned}$$

Now the appropriate version of Theorem 2.1 in [AF] reads as follows:

Theorem 2.1 *Let $u \in J^1_m(\Omega, \mathbb{R}^d)$ be a minimizer of $I(\cdot, \Omega)$, that is*

$$I(u, \Omega) \;\le\; I(u + v, \Omega) \quad \textit{for all } \; v \in \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d).$$

Suppose that for $x_0 \in \Omega$ and for $\varkappa_0 \in \overset{\circ}{\mathbb{S}}{}^d$

$$\lim_{R \searrow 0} \;\; -\!\!\!\!\!\!\int_{B(x_0,R)} |\varepsilon(u) - \varkappa_0|^m \, dx = 0, \tag{2.3}$$

$$g \in C^2(\mathcal{B}(\varkappa_0, \rho_1)) \textit{ for some } \; \rho_1 > 0. \tag{2.4}$$

Assume further that g is $J^1_m(\Omega, \mathbb{R}^d)$-strictly quasiconvex at $\varkappa_0$, i.e. for any $v \in \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d)$ and for some constant $\nu > 0$ we have the inequality

$$\int_\Omega \{g(\varkappa_0 + \varepsilon(v)) - g(\varkappa_0)\} \, dx \;\ge\; 2\,\nu \int_\Omega \{|\varepsilon(v)|^2 + |\varepsilon(v)|^m\} \, dx. \tag{2.5}$$

Then the function ∇u is Hölder continuous in $B(x_0, R)$ for some $R > 0$.

Clearly, the same result is true if we drop the condition $\operatorname{div} u = 0$.

Remark 2.2 The notion of J^1_m-quasiconvexity was introduced in [SE3]. It is a natural modification of quasiconvexity introduced by Morrey [MO1] and W^1_p-quasiconvexity in the sense of Ball and Murat [BM] if solenoidal vector fields are considered.

3 Two auxiliary lemmata

Let us place two auxiliary lemmata in front of the proof of Theorem 2.1. The first one follows the idea of [AF], Lemma 2.2. A proof can be also found in the preliminary version [BFS]. The second one is a simple but very useful observation.

Lemma 3.1 *Suppose that, besides the general hypotheses, g satisfies* (2.4) *and* (2.5)*. Then g is strictly J^1_m-quasiconvex in some neighbourhood of $\varkappa_0$, i.e.*

$$\int_\Omega \{g(\varkappa + \varepsilon(v)) - g(\varkappa)\}\, dx \geq \nu \int_\Omega \{|\varepsilon(v)|^2 + |\varepsilon(v)|^m\}\, dx \tag{3.1}$$

holds for any $v \in \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d)$, for any $\varkappa \in \mathcal{B}(\varkappa_0, \rho)$ and for some $\rho \in (0, \rho_1]$.

Lemma 3.2 *If* (2.4) *holds, then there is a constant A depending on $\varkappa_0$, ρ_1 and $\|\partial g/\partial\kappa\|_{C^1(\overline{\mathcal{B}}(\varkappa_0, 3\rho_1/4))}$ such that for a.e. $\tau \in \overset{\circ}{\mathbb{S}}{}^d$ and for all $\varkappa \in \overline{\mathcal{B}}(\varkappa_0, \frac{\rho_1}{2})$*

$$\left|\frac{\partial g}{\partial\kappa}(\tau) - \frac{\partial g}{\partial\kappa}(\varkappa)\right| \leq A(1 + |\tau|^{m-2})|\tau - \varkappa|. \tag{3.2}$$

Proof. Assume first that $\tau \in \overline{\mathcal{B}}(\varkappa, \frac{\rho_1}{4})$, i.e. $\tau \in \overline{\mathcal{B}}(\varkappa_0, \frac{3\rho_1}{4})$. Then we have

$$\left|\frac{\partial g}{\partial\kappa}(\tau) - \frac{\partial g}{\partial\kappa}(\varkappa)\right| = \left|\int_0^1 \frac{\partial^2 g}{\partial\kappa^2}(\varkappa + \theta(\tau - \varkappa))(\tau - \varkappa)\, d\theta\right| \leq \left\|\frac{\partial g}{\partial\kappa}\right\|_{C^1(\overline{\mathcal{B}}(\varkappa_0, 3\rho_1/4))} |\tau - \varkappa|.$$

Suppose now that $\tau \notin \overline{\mathcal{B}}(\varkappa, \frac{\rho_1}{4})$. We then introduce

$$\overline{\varkappa} = \varkappa + \frac{\rho_1}{8}\, \frac{\tau - \varkappa}{|\tau - \varkappa|},$$

where it is assumed w.l.o.g. that the following derivatives exist. So,

$$\begin{aligned}
\left|\frac{\partial g}{\partial\kappa}(\tau) - \frac{\partial g}{\partial\kappa}(\varkappa)\right| &\leq \left|\frac{\partial g}{\partial\kappa}(\tau)\right| + \left|\frac{\partial g}{\partial\kappa}(\overline{\varkappa})\right| + \left|\frac{\partial g}{\partial\kappa}(\overline{\varkappa}) - \frac{\partial g}{\partial\kappa}(\varkappa)\right| \\
&\leq c_5(2 + |\tau|^{m-1} + |\overline{\varkappa}|^{m-1}) + |\overline{\varkappa} - \varkappa| \left\|\frac{\partial g}{\partial\kappa}\right\|_{C^1(\overline{\mathcal{B}}(\varkappa_0, 3\rho_1/4))} \\
&\leq c_6(\varkappa_0, \rho_1)(1 + |\tau|^{m-1}) + |\overline{\varkappa} - \varkappa| \left\|\frac{\partial g}{\partial\kappa}\right\|_{C^1(\overline{\mathcal{B}}(\varkappa_0, 3\rho_1/4))}.
\end{aligned}$$

This together with

$$\frac{1}{|\tau - \varkappa|} \leq \frac{2}{1 + |\tau|} \cdot \begin{cases} 1 & \text{if } |\tau| > 2|\varkappa| + 1 \\ 4\rho_1^{-1}(1 + \rho_1 + |\varkappa_0|) & \text{if } |\tau| \leq 2|\varkappa| + 1 \end{cases}$$

proves (3.2) in the case $\tau \notin \overline{\mathcal{B}}(\varkappa, \frac{\rho_1}{4})$ as well and the lemma follows. □

4 A Caccioppoli-type inequality

In this section an inequality of Caccioppoli's type is proved which is the counterpart of [AF], Lemma 2.5. However, since Lemma 3.2 is used to prove this inequality, it is a slight improvement compared to the one of [AF]. Especially we do not have to impose new assumptions on the general situation.

Lemma 4.1 *Suppose that all the conditions of Theorem* 2.1 *hold, that* $B(x_0, R) \Subset \Omega$ *and that* $\pi \in \overset{\circ}{\mathbb{M}}{}^d$ *such that*

$$\varkappa := \frac{1}{2}(\pi + \pi^{\mathrm{T}}) \ \in \ \mathcal{B}\Big(\varkappa_0, \frac{\rho}{2}\Big)\,,$$

where ρ *is the number according to Lemma* 3.1. *Then for any* $a \in \mathbb{R}^d$ *we have*

$$\begin{aligned}\int_{B\left(x_0, \frac{R}{2}\right)} &\{|\nabla u - \pi|^2 + |\nabla u - \pi|^m\}\, dx \\ &\le \frac{c_7}{R^2}\int_{B(x_0,R)} |u - \pi(x - x_0) - a|^2\, dx \\ &\quad + \frac{c_7}{R^m}\int_{B(x_0,R)} |u - \pi(x - x_0) - a|^m\, dx,\end{aligned}$$

where the constant c_7 *does not depend on* x_0, R *and* π.

Proof. Consider $\varphi \in C_0^\infty(B(x_0, r))$ satifying $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B(x_0, r_1)$ and $|\nabla\varphi| \le \frac{c_8}{r - r_1}$, where $\frac{R}{2} \le r_1 < r \le R$ is assumed. We also let

$$\bar{u} \ = \ u - \pi(x - x_0) - a, \qquad \psi \ = \ 1 - \varphi.$$

According to [LS] there is a function $\hat{u} \in \overset{\circ}{W}{}^1_p(B(x_0, r))$ such that

$$\begin{aligned}\operatorname{div}\hat{u} \ &= \ \operatorname{div}(\varphi\bar{u}) \ = \ \bar{u}\cdot\nabla\varphi \quad \text{on } B(x_0, r),\\ \int_{B(x_0,r)} |\nabla\hat{u}|^2\, dx \ &\le \ c_9(m, d)\int_{B(x_0,r)} |\bar{u}\cdot\nabla\varphi|^2\, dx,\\ \int_{B(x_0,r)} |\nabla\hat{u}|^m\, dx \ &\le \ c_9(m, d)\int_{B(x_0,r)} |\bar{u}\cdot\nabla\varphi|^m\, dx.\end{aligned}$$

Now, by Korn's inequality, by strict quasiconvexity (see (3.1)) and by the relation $\varepsilon(\bar{u}) = \varepsilon(u) - \varkappa$ we obtain an upper bound for

$$A \ := \ c_{10}(m, d)\,\nu\int_{B(x_0,r)} \{|\nabla(\varphi\bar{u} - \hat{u})|^2 + |\nabla(\varphi\bar{u} - \hat{u})|^m\}\, dx,$$

$$
\begin{aligned}
A &\leq \nu \int_{B(x_0,r)} \{|\varepsilon(\varphi\bar{u}-\hat{u})|^2 + |\varepsilon(\varphi\bar{u}-\hat{u})|^m\}\, dx \\
&\leq \int_{B(x_0,r)} \{g(\varkappa + \varepsilon(\varphi\bar{u}-\hat{u})) - g(\varkappa)\}\, dx \\
&= \int_{B(x_0,r)} \{g(\varepsilon(u) - \varepsilon(\psi\bar{u}+\hat{u})) - g(\varepsilon(u))\}\, dx \\
&\quad + \int_{B(x_0,r)} \{g(\varepsilon(u)) - g(\varepsilon(u) - \varepsilon(\varphi\bar{u}-\hat{u}))\}\, dx \\
&\quad + \int_{B(x_0,r)} \{g(\varkappa + \varepsilon(\psi\bar{u}+\hat{u})) - g(\varkappa)\}\, dx \;=\; I + II + III.
\end{aligned}
$$

Observing $\operatorname{div}(\varphi\bar{u}-\hat{u}) = 0$ in $B(x_0,r)$, minimality of u gives a non-positive sign for the second integral on the right hand side, i.e.

$$
c_{10}(m,d)\,\nu \int_{B(x_0,r)} \{|\nabla(\varphi\bar{u}-\hat{u})|^2 + |\nabla(\varphi\bar{u}-\hat{u})|^m\}\, dx \leq I + III. \tag{4.1}
$$

We now define

$$
\begin{aligned}
\Phi(x_0,r) &:= \int_{B(x_0,r)} \{|\nabla\bar{u}|^2 + |\nabla\bar{u}|^m\}\, dx, \\
\Phi_1(x_0,r) &:= \int_{B(x_0,r)} \{|\nabla\varphi|^2|\bar{u}|^2 + |\nabla\varphi|^m|\bar{u}|^m\}\, dx,
\end{aligned}
$$

and claim that there is a constant $c_{11} > 0$ such that for every $\gamma > 0$ and for some other constant $c_{12} = c_{12}(\gamma)$

$$
I + III \leq c_{11}(\Phi(x_0,r) - \Phi(x_0,r_1)) + 2\,\gamma\,\Phi(x_0,r) + c_{12}(\gamma)\Phi_1(x_0,r). \tag{4.2}
$$

Let us assume for the moment that (4.2) holds. By the choice of $\hat{u}$

$$
\int_{B(x_0,r)} \{|\nabla\hat{u}|^2 + |\nabla\hat{u}|^m\}\, dx \leq c_9(m,d)\Phi_1(x_0,r) \tag{4.3}
$$

is seen to be true and this implies together with (4.1)

$$
\Phi(x_0,r_1) \leq c_{13}(\Phi(x_0,r) - \Phi(x_0,r_1)) + c_{14}\,\gamma\,\Phi(x_0,r) + c_{15}(\gamma)\Phi_1(x_0,r),
$$

respectively after “hole-filling”

$$
\Phi(x_0,r_1) \;\leq\; \frac{c_{13} + c_{14}\gamma}{c_{13}+1}\Phi(x_0,r) + \frac{c_{15}(\gamma)}{c_{13}+1}\Phi_1(x_0,r).
$$

Since c_{13} and c_{14} are independent of γ, we can arrange

$$0 \;<\; \theta \;:=\; \frac{c_{13}+c_{14}\gamma}{c_{13}+1} \;<\; 1.$$

Finally, for $\frac{R}{2} \le r_1 < r \le R$ the upper bound

$$\theta\,\Phi(x_0,r) + c_{17}\left[\frac{1}{(r-r_1)^2}\int_{B(x_0,R)} |\bar u|^2\,dx + \frac{1}{(r-r_1)^m}\int_{B(x_0,R)} |\bar u|^m\,dx\right] \quad (4.4)$$

for $\Phi(x_0,r_1)$ is derived from the obvious inequality

$$\Phi_1(x_0,r) \;\le\; c_{16}\left[\frac{1}{(r-r_1)^2}\int_{B(x_0,R)} |\bar u|^2\,dx + \frac{1}{(r-r_1)^m}\int_{B(x_0,R)} |\bar u|^m\,dx\right].$$

Following [Gi], p. 161, or [AF] (see Lemma 2.4), Lemma 4.1 is proved by (4.4). So it remains to show (4.2):

$$\begin{aligned}
I \;&=\; -\int_{B(x_0,r)}\left\{\int_0^1 \frac{\partial g}{\partial\kappa}(\varepsilon(u)-\theta\varepsilon(\psi\bar u+\hat u)) : \varepsilon(\psi\bar u+\hat u)\,d\theta\right\} dx\\
&=\; -\int_{B(x_0,r)}\left\{\int_0^1 \left[\frac{\partial g}{\partial\kappa}(\varkappa+\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)) - \frac{\partial g}{\partial\kappa}(\varkappa)\right] : \varepsilon(\psi\bar u+\hat u)\,d\theta\right\} dx\\
&\quad -\int_{B(x_0,r)}\left\{\int_0^1 \frac{\partial g}{\partial\kappa}(\varkappa) : \varepsilon(\psi\bar u+\hat u)\,d\theta\right\} dx \;=:\; I_1 + I_2.
\end{aligned}$$

To estimate I_1, Lemma 3.2 is used:

$$\begin{aligned}
I_1 \;&\le\; c_{18}\int_{B(x_0,r)}\int_0^1 [1+|\varkappa+\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)|^{m-2}]|\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)|\\
&\quad \cdot|\varepsilon(\psi\bar u+\hat u)|\,d\theta\,dx\\
&\le\; c_{19}(\varkappa_0,\rho)\int_{B(x_0,r)}\int_0^1 (1+|\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)|^{m-2})|\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)|\\
&\quad \cdot|\varepsilon(\psi\bar u+\hat u)|\,d\theta\,dx\\
&=\; c_{20}\left\{\int_{B(x_0,r)}\int_0^1 |\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)|\,|\varepsilon(\psi\bar u+\hat u)|\,d\theta\,dx\right.\\
&\quad \left. + \int_{B(x_0,r)}\int_0^1 |\varepsilon(\bar u)-\theta\varepsilon(\psi\bar u+\hat u)|^{m-1}\,|\varepsilon(\psi\bar u+\hat u)|\,d\theta\,dx\right\}
\end{aligned}$$

$$\leq \; c_{21} \int_{B(x_0,r)} (|\varepsilon(\bar{u})| + |\varepsilon(\bar{u})|^{m-1}) |\varepsilon(\psi\bar{u} + \hat{u})| \, dx$$
$$+ c_{22} \int_{B(x_0,r)} (|\varepsilon(\psi\bar{u} + \hat{u})|^2 + |\varepsilon(\psi\bar{u} + \hat{u})|^m) \, dx.$$

Since $\psi \equiv 0$ in $B(x_0, r_1)$, we obtain

$$\begin{aligned} I_1 \;\leq\; & c_{23} \int_{B(x_0,r)\setminus B(x_0,r_1)} (|\nabla\bar{u}|^2 + |\nabla\bar{u}|^m) \, dx \\ & + c_{24} \int_{B(x_0,r)} (|\nabla\bar{u}| + |\nabla\bar{u}|^{m-1})(|\bar{u}| \, |\nabla\varphi| + |\nabla\hat{u}|) \, dx \\ & + c_{25} \int_{B(x_0,r)} \{(|\nabla\varphi|^2 |\bar{u}|^2 + |\nabla\varphi|^m |\bar{u}|^m) + |\nabla\hat{u}|^2 + |\nabla\hat{u}|^m\} \, dx \end{aligned}$$

and finally using Hölder's inequality

$$\begin{aligned} I_1 \;\leq\; & c_{23} \int_{B(x_0,r)\setminus B(x_0,r_1)} (|\nabla\bar{u}|^2 + |\nabla\bar{u}|^m) \, dx \\ & + c_{26} \left(\int_{B(x_0,r)} |\nabla\bar{u}|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B(x_0,r)} (|\bar{u}|^2 \, |\nabla\varphi|^2 + |\nabla\hat{u}|^2) \, dx \right)^{\frac{1}{2}} \\ & + c_{26} \left(\int_{B(x_0,r)} |\nabla\bar{u}|^m \, dx \right)^{\frac{m-1}{m}} \left(\int_{B(x_0,r)} (|\bar{u}|^m \, |\nabla\varphi|^m + |\nabla\hat{u}|^m) \, dx \right)^{\frac{1}{m}} \\ & + c_{25} \int_{B(x_0,r)} \{(|\nabla\varphi|^2 |\bar{u}|^2 + |\nabla\varphi|^m |\bar{u}|^m) + |\nabla\hat{u}|^2 + |\nabla\hat{u}|^m\} \, dx. \end{aligned}$$

Recalling (4.3) and the definitions of Φ and Φ_1, the following inequality is proved:

$$\begin{aligned} I_1 \;\leq\; & c_{23}(\Phi(x_0, r) - \Phi(x_0, r_1)) \\ & + c_{26}(\Phi^{\frac{1}{2}}(x_0, r)\Phi_1^{\frac{1}{2}}(x_0, r) + \Phi^{\frac{m-1}{m}}(x_0, r)\Phi_1^{\frac{1}{m}}(x_0, r)) \\ & + c_{27}\Phi_1(x_0, r). \end{aligned}$$

If $\gamma > 0$ is fixed, then Young's inequality gives

$$I_1 \;\leq\; c_{23}(\Phi(x_0, r) - \Phi(x_0, r_1)) + \gamma\Phi(x_0, r) + c_{28}(\gamma)\Phi_1(x_0, r).$$

Now, observe that I_2 has a negative counterpart arising from III:

$$\begin{aligned} III &= \int_{B(x_0,r)} \left\{ \int_0^1 \frac{\partial g}{\partial \kappa}(\varkappa + \theta\varepsilon(\psi\bar{u}+\hat{u})) : \varepsilon(\psi\bar{u}+\hat{u})\, d\theta \right\} dx \\ &= \int_{B(x_0,r)} \left\{ \int_0^1 \left[\frac{\partial g}{\partial \kappa}(\varkappa + \theta\varepsilon(\psi\bar{u}+\hat{u})) - \frac{\partial g}{\partial \kappa}(\varkappa) \right] : \varepsilon(\psi\bar{u}+\hat{u})\, d\theta \right\} dx \\ &\quad + \int_{B(x_0,r)} \left\{ \int_0^1 \frac{\partial g}{\partial \kappa}(\varkappa) : \varepsilon(\psi\bar{u}+\hat{u})\, d\theta \right\} dx \;:=\; III_1 + III_2. \end{aligned}$$

Thus $I_2 = -III_2$ and it only remains to estimate III_1 which can be done in the same manner as above and the whole Lemma is proved. □

5 Proof of Theorem 2.1

Theorem 2.1 will be a consequence of the following lemma.

Lemma 5.1 *Again suppose that the general hypotheses are satisfied for the integrand g and that u is a minimizer of $I(\cdot,\Omega)$ as described above. Suppose further that the conditions* (2.4) *and* (2.5) *hold for some $\varkappa_0 \in \mathring{\mathbb{S}}^d$ and let for $x_0 \in \Omega$*

$$\Psi(x_0,R) :\, = \left(\fint_{B(x_0,R)} |\nabla u - (\nabla u)_{x_0,R}|^2\, dx \right)^{\frac{1}{2}} + \left(\fint_{B(x_0,R)} |\nabla u - (\nabla u)_{x_0,R}|^m\, dx \right)^{\frac{1}{2}},$$

where $(\varphi)_{x,r}$ always denotes the mean value of φ on $B(x,r)$. Finally $\rho > 0$ is fixed according to Lemma 3.1*. Then, for any $t \in (0,1/8]$ there are numbers $\gamma_0 > 0$ and $R_0 > 0$ such that: if for $x_0 \in \Omega$ and for $0 < R < R_0$ the conditions $B(x_0,R) \Subset \Omega$,*

$$(\varepsilon(u))_{x_0,tR} \in \overline{\mathcal{B}}\left(\varkappa_0, \frac{\rho}{4}\right), \quad (\varepsilon(u))_{x_0,R} \in \overline{\mathcal{B}}\left(\varkappa_0, \frac{\rho}{4}\right), \quad \Psi(x_0,R) < \gamma_0$$

are satisfied, then the conclusion is

$$\Psi(x_0,tR) \;\le\; c_\oplus\, t\, \Psi(x_0,R)$$

where the constant $c_\oplus$ does not depend on x_0, R and t.

Proof. The lemma is proved by contradiction, so assume that there is a number $t \in (0,1/8]$ and that there are sequences $\{x^h\}$, $\{R_h\}$ and $\{\gamma_h\}$ such that

$B(x^h, R_h) \Subset \Omega$ and $R_h \to 0$, $\gamma_h = \Psi(x^h, R_h) \to 0$ as $h \to 0$, $\Psi(x^h, tR_h) \geq c_\oplus \, t \, \gamma_h$,

$$\varkappa_t^h \; = \; (\varepsilon(u))_{x^h, tR_h} \; \in \; \overline{\mathcal{B}}(\varkappa_0, \tfrac{\varrho}{4}), \quad \varkappa^h \; = \; (\varepsilon(u))_{x^h, R_h} \; \in \; \overline{\mathcal{B}}(\varkappa_0, \tfrac{\varrho}{4}),$$

where $c_\oplus$ is chosen below in an appropriate way to obtain the contradiction. We now consider the scaling $x = x^h + yR_h$,

$$v^h(y) \;=\; \frac{u(x^h + yR_h) - (\nabla u)_{x^h, R_h}(x - x^h) - (u)_{x^h, R_h}}{\gamma_h R_h}$$

and get after changing the variables

$$\nabla_x u = (\nabla_x u)_{x^h, R_h} + \gamma_h \nabla_y v^h(y), (\nabla_x u)_{x^h, tR_h} = (\nabla_x u)_{x^h, R_h} + \gamma_h (\nabla_y v^h)_{0,t},$$
$$(v^h)_{0,1} = 0, (\nabla_y v^h)_{0,1} \;=\; 0, \Psi(x^h, tR_h) = \gamma_h \Phi_h(t),$$

where we have abbreviated

$$\Phi_h(t) = \left(\fint_{B(0,t)} |\nabla v^h - (\nabla v^h)_{0,t}|^2 \, dy \right)^{\frac{1}{2}} + \gamma_h^{\frac{m}{2} - 1} \left(\fint_{B(0,t)} |\nabla v^h - (\nabla v^h)_{0,t}|^m \, dy \right)^{\frac{1}{2}}$$

From our assumptions we get

$$\Phi_h(1) \;=\; \left(\fint_{B(0,1)} |\nabla v^h|^2 \, dy \right)^{\frac{1}{2}} + \gamma_h^{\frac{m}{2} - 1} \left(\fint_{B(0,1)} |\nabla v^h|^m \, dy \right)^{\frac{1}{2}} \;=\; 1,$$

i.e. $\Phi_h(t) \geq c_\oplus t$. Thus, after passing to subsequences (still denoted by the same symbols) without loss of generality it may be assumed that:

$$\begin{array}{rcll} v^h & \to & v & \text{in} \;\; L^2(B(0,1), \mathbb{R}^d), \\ \nabla v^h & \rightharpoonup & \nabla v & \text{in} \;\; L^2(B(0,1), \mathring{\mathbb{M}}^d), \\ \gamma_h^{1 - \frac{2}{m}} v^h & \to & 0 & \text{in} \;\; L^m(B(0,1), \mathbb{R}^d) \;\; \text{if} \;\; m > 2, \\ \gamma_h^{1 - \frac{2}{m}} \nabla v^h & \rightharpoonup & 0 & \text{in} \;\; L^m(B(0,1), \mathring{\mathbb{M}}^d) \;\; \text{if} \;\; m > 2 \end{array}$$

and $\varkappa^h \to \varkappa_*$ in $\mathring{\mathbb{S}}^d$ as $h \to 0$. Now, using the minimality of u, we will prove that

$$\int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*) \varepsilon(v) : \varepsilon(w) \, dy \;=\; 0 \quad \text{ for all } \; w \in \overset{\bullet}{\mathrm{C}}{}^\infty(B(0,1), \mathbb{R}^d). \tag{5.1}$$

To prove the claim (5.1), choose $w \in \overset{\bullet}{\mathrm{C}}{}^\infty(B(0,1), \mathbb{R}^d)$ and define

$$w^h \;=\; \gamma_h R_h w \left(\frac{x - x^h}{R_h} \right) \;\in\; \overset{\bullet}{\mathrm{C}}{}^\infty(B(x^h, R_h), \mathbb{R}^d).$$

As mentioned above, we use the minimality of u, i.e.

$$I(u, B(x^h, R_h)) \;\le\; I(u + w^h, B(x^h, R_h)).$$

This yields

$$\int_{B(x^h,R_h)} \left\{ \int_0^1 \frac{\partial g}{\partial \kappa}(\varepsilon(u) + \theta\varepsilon(w^h)) : \varepsilon(w^h)\, d\theta \right\} dx \;\ge\; 0, \;\text{ i.e.}$$

$$\int_{B(0,1)} \left\{ \int_0^1 \frac{\partial g}{\partial \kappa}(\varkappa^h + \gamma_h\varepsilon(v^h) + \theta\gamma_h\varepsilon(w)) : \varepsilon(w)\, d\theta \right\} dy \;\ge\; 0. \tag{5.2}$$

In order to pass to the limit in (5.2) we first observe that by condition (2.4) for every $\gamma > 0$ there is a real number $\delta(\gamma) > 0$ with the property

$$\left| \frac{\partial^2 g}{\partial \kappa^2}(\tau) - \frac{\partial^2 g}{\partial \kappa^2}(\tau') \right| \;<\; \gamma,$$

whenever $\tau,\ \tau' \in \overline{\mathcal{B}}(\varkappa_0, \frac{\varrho}{2})$ and $|\tau - \tau'| < \delta(\gamma)$. Now two sets are introduced setting $\hat\gamma = \min\{\frac{\varrho}{4}, \frac{\delta(\gamma)}{2}\}$: $B_h^1 = \{y \in B(0,1) : \gamma_h(|\varepsilon(v^h)(y)| + |\varepsilon(w)(y)|) \ge \hat\gamma\}$, $B_h^2 = B(0,1)\backslash B_h^1$. Then, by definition

$$\hat\gamma^2 |B_h^1| \;\le\; \gamma_h^2 \int_{B_h^1} (|\varepsilon(v^h)| + |\varepsilon(w)|)^2\, dy \;\le\; c_{29}\gamma_h^2 \left(1 + \int_{B(0,1)} |\varepsilon(w)|^2\, dy \right),$$

$$|B_h^1| \;\le\; \frac{c_{29}\gamma_h^2}{\hat\gamma^2} \left(1 + \int_{B(0,1)} |\varepsilon(w)|^2\, dy \right). \tag{5.3}$$

Going back to (5.2) we see

$$\begin{aligned} A \;:=&\; \int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(v) : \varepsilon(w)\, dy \\ \ge&\; \int_{B(0,1)} \left\{ \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(v) \right. \\ &\left. - \frac{1}{\gamma_h} \int_0^1 \left(\frac{\partial g}{\partial \kappa}(\varkappa^h + \gamma_h\varepsilon(v^h) + \theta\gamma_h\varepsilon(w)) - \frac{\partial g}{\partial \kappa}(\varkappa^h) \right) d\theta \right\} : \varepsilon(w)\, dy, \end{aligned}$$

i.e. $A = A_1 + A_2 + A_3 + A_4$ where the A_i are defined via

$$A_1 \;=\; -\frac{1}{\gamma_h} \int_{B_h^1} \int_0^1 \left(\frac{\partial g}{\partial \kappa}(\varkappa^h + \gamma_h\varepsilon(v^h) + \theta\gamma_h\varepsilon(w)) - \frac{\partial g}{\partial \kappa}(\varkappa^h) \right) d\theta : \varepsilon(w)\, dy,$$

$$A_2 \;=\; -\frac{1}{\gamma_h} \int_{B_h^2} \int_0^1 \left(\frac{\partial g}{\partial \kappa}(\varkappa^h + \gamma_h\varepsilon(v^h) + \theta\gamma_h\varepsilon(w)) - \frac{\partial g}{\partial \kappa}(\varkappa^h) \right) d\theta : \varepsilon(w)\, dy$$

$$
\begin{aligned}
& + \int_{B_h^2} \int_0^1 \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h)(\varepsilon(v^h) + \theta\varepsilon(w))\, d\theta : \varepsilon(w)\, dy, \\
A_3 &= \int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(v) : \varepsilon(w)\, dy - \int_{B_h^2} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h)\varepsilon(v^h) : \varepsilon(w)\, dy, \\
A_4 &= -\int_{B_h^2} \int_0^1 \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h)\theta\varepsilon(w)\, d\theta : \varepsilon(w)\, dy = -\frac{1}{2} \int_{B_h^2} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h)\varepsilon(w) : \varepsilon(w)\, dy.
\end{aligned}
$$

We have assumed that $\varkappa^h \in \overline{\mathcal{B}}(\varkappa_0, \frac{\rho}{4})$ and that $\frac{\partial^2 g}{\partial \kappa^2}$ is continuous in $\mathcal{B}(\varkappa_0, \rho)$. Thus

$$
A_4 \to -\frac{1}{2} \int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(w) : \varepsilon(w)\, dy \quad \text{as } h \to 0.
$$

is seen by (5.3). Next, we observe that

$$
\begin{aligned}
|A_3| \le & \left| \int_{B_h^2} \left(\frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*) - \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h) \right) \varepsilon(v) : \varepsilon(w)\, dy \right| \\
& + \left| \int_{B_h^2} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h)\varepsilon(v - v^h) : \varepsilon(w)\, dy + \int_{B_h^1} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(v) : \varepsilon(w)\, dy \right|,
\end{aligned}
$$

hence $|A_3| \to 0$ as $h \to 0$. By construction, $\varkappa^h + \gamma_h\varepsilon(v^h) + \theta\gamma_h\varepsilon(w) \in \overline{\mathcal{B}}(\varkappa_0, \frac{\rho}{2})$ for $y \in B_h^2$ and A_2 may be written in the following way

$$
\begin{aligned}
A_2 = & \int_{B_h^2} \int_0^1 \left[\frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h)(\varepsilon(v^h) + \theta\varepsilon(w)) : \varepsilon(w) \right. \\
& \left. - \int_0^1 \frac{\partial^2 g}{\partial \kappa^2}(\varkappa^h + \theta_1\gamma_h(\varepsilon(v^h) + \theta\varepsilon(w)))(\varepsilon(v^h) + \theta\varepsilon(w)) : \varepsilon(w)\, d\theta_1 \right] d\theta\, dy.
\end{aligned}
$$

Since we have on the other hand $\varkappa^h + \theta_1\gamma_h(\varepsilon(v^h) + \theta\varepsilon(w)) \in \overline{\mathcal{B}}(\varkappa_0, \frac{\rho}{2})$ and

$$
|\theta_1\gamma_h(\varepsilon(v^h) + \theta\varepsilon(w))| \le \hat{\gamma} \le \frac{\delta(\gamma)}{2} < \delta(\gamma)
$$

for $y \in B_h^2$, we obtain

$$
|A_2| \le \gamma \int_{B(0,1)} (|\varepsilon(v^h)| + |\varepsilon(w)|)|\varepsilon(w)|\, dy \le c_{30}\gamma \left(1 + \int_{B(0,1)} |\varepsilon(w)|^2\, dy \right).
$$

Finally, from Lemma 3.2 we get an upper bound for $|A_1|$:

$$\begin{aligned}
c_{31} \int_{B_h^1} & (1 + (|\varkappa^h| + \gamma_h|\varepsilon(v^h)| + \gamma_h|\varepsilon(w)|)^{m-2})(|\varepsilon(v^h)| + |\varepsilon(w)|)|\varepsilon(w)|\, dy \\
\leq\ & c_{32}\|\nabla w\|_{L^\infty(B(0,1))} \int_{B_h^1} (|\varepsilon(v^h)| + |\varepsilon(w)| + \gamma_h^{m-2}(|\varepsilon(v^h)| + |\varepsilon(w)|)^{m-1})\, dy \\
\leq\ & c_{33}(\nabla w) \Bigg\{ |B_h^1|^{\frac{1}{2}} \left(\int_{B(0,1)} (|\varepsilon(v^h)|^2 + |\varepsilon(w)|^2)\, dy \right)^{\frac{1}{2}} \\
& + \gamma_h^{m-2}|B_h^1|^{\frac{1}{m}} \left(\int_{B(0,1)} (|\varepsilon(v^h)|^m + |\varepsilon(w)|^m)\, dy \right)^{\frac{m-1}{m}} \Bigg\}.
\end{aligned}$$

Again (5.3) proves $A_1 \to 0$ as $h \to 0$. Summarizing these estimates we have proved

$$A \ \geq\ -c_{30}\gamma \left(1 + \int_{B(0,1)} |\varepsilon(w)|^2\, dy \right) - \frac{1}{2} \int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(w) : \varepsilon(w)\, dy,$$

or, since γ was an arbitrary positive number

$$\int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(v) : \varepsilon(w)\, dy + \frac{1}{2} \int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(w) : \varepsilon(w)\, dy \ \geq\ 0.$$

The same is true for any scaling of w and we arrive at (5.1). Concerning the linear system (5.1) with constant coefficients we first claim that strict J_m^1-quasiconvexity (3.1) implies for all $w \in \overset{\circ}{J}{}_2^1 (B(0,1), \mathbb{R}^d)$

$$\int_{B(0,1)} \frac{\partial^2 g}{\partial \kappa^2}(\varkappa_*)\varepsilon(w) : \varepsilon(w)\, dy \ \geq\ c_{34}(\nu) \int_{B(0,1)} |\nabla w|^2\, dy. \tag{5.4}$$

So by (5.1) and (5.4) the standard linear theory can be applied (compare for example [FS2], Lemma 3.0.5, p. 138, and notice that condition (5.4) is sufficient). Thus, setting

$$\Phi(s) \ =\ \left(\fint_{B(0,s)} |\nabla v - (\nabla v)_{0,s}|^2\, dy \right)^{\frac{1}{2}}$$

it is proved for all $s \in (0,1)$ that

$$\Phi(s) \ \leq\ c_{35} \left(\nu, \left\| \frac{\partial^2 g}{\partial \kappa^2} \right\|_{L^\infty(\mathcal{B}(\varkappa_0,\rho/4))} \right) s\, \Phi(1).$$

The uniform boundedness of $\Phi_h(1)$ gives in addition

$$\Phi(s) \ \le\ c_{35}\, s \quad \text{for all } \ s \in (0,1). \tag{5.5}$$

Then the contradiction will follow from the above assumption

$$\liminf_{h\to 0} \Phi_h(t) \ \ge\ c_{\oplus} t. \tag{5.6}$$

In fact, since $\varkappa_t^h \in \mathcal{B}(\varkappa_0, \frac{\varrho}{2})$, we can apply Lemma 4.1 replacing x_0 and R by x^h and tR_h with the result that $\Psi(x^h, tR_h)$ is bounded by

$$c_{36}\left\{\frac{1}{2tR_h}\left(\ \fint_{B(x^h,2tR_h)} \left|u-(\nabla u)_{x^h,tR_h}(x-x^h)-(u)_{x^h,2tR_h}\right|^2 dx\right)^{\frac12}\right.$$
$$\left.+\frac{1}{(2tR_h)^{\frac m2}}\left(\ \fint_{B(x^h,2tR_h)} \left|u-(\nabla u)_{x^h,tR_h}(x-x^h)-(u)_{x^h,2tR_h}\right|^m dx\right)^{\frac12}\right\}$$
$$\le\ c_{37}\left\{\frac{1}{2tR_h}\left(\ \fint_{B(x^h,2tR_h)} \left|u-(\nabla u)_{x^h,2tR_h}(x-x^h)-(u)_{x^h,2tR_h}\right|^2 dx\right)^{\frac12}\right.$$
$$\left.+\frac{1}{(2tR_h)^{\frac m2}}\left(\ \fint_{B(x^h,2tR_h)} \left|u-(\nabla u)_{x^h,2tR_h}(x-x^h)-(u)_{x^h,2tR_h}\right|^m dx\right)^{\frac12}\right\}$$
$$+c_{38}\{|(\nabla u)_{x^h,2tR_h}-(\nabla u)_{x^h,tR_h}|+|(\nabla u)_{x^h,2tR_h}-(\nabla u)_{x^h,tR_h}|^{\frac m2}\}.$$

By transformation we get

$$\Phi_h(t)\ \le\ c_{39}\left\{\frac{1}{2t}\left(\ \fint_{B(0,2t)} |v^h-(\nabla v^h)_{0,2t}y-(v^h)_{0,2t}|^2\,dy\right)^{\frac12}\right.$$
$$\left.+\frac{1}{(2t)^{\frac m2}}\left(\gamma_h^{\frac{m-2}{m}} \fint_{B(0,2t)} |v^h-(\nabla v^h)_{0,2t}y-(v^h)_{0,2t}|^m\,dy\right)^{\frac12}\right\}$$
$$+c_{40}\{|(\nabla v^h)_{0,2t}-(\nabla v^h)_{0,t}|+\gamma_h^{\frac{m-2}{m}}|(\nabla v^h)_{0,2t}-(\nabla v^h)_{0,t}|^{\frac m2}\},$$

$$\limsup_{h\to\infty}\Phi_h(t)\ \le\ \frac{c_{39}}{2t}\left(\ \fint_{B(0,2t)} |v-(\nabla v)_{0,2t}y-(v)_{0,2t}|^2\,dy\right)^{\frac12}$$
$$+c_{40}\{|(\nabla v)_{0,2t}-(\nabla v)_{0,t}|\}.$$

Finally, we notice that

$$\left| \,\fint_{B(0,t)} \{\nabla v - (\nabla v)_{0,2t}\}\, dy \right| \;\le\; c_{41} \left(\,\fint_{B(0,2t)} |\nabla v - (\nabla v)_{0,2t}|^2\, dy \right)^{\frac{1}{2}}$$

and by Poincaré's inequality (5.5) proves

$$\limsup_{h\to\infty} \Phi_h(t) \le c_{42}\Phi(2t) \;\le\; c_{42}\, c_{35}\, 2\, t.$$

So, the contradiction to (5.6) follows if $c_\oplus = 4\, c_{42}\, c_{35}$ was chosen at the beginning of the proof, i.e. Lemma 5.1 is proved. □

Now, we proceed as usual (see, for example [AF] or [BFS]) by iterating Lemma 5.1 and obtain:

Lemma 5.2 *With the assumptions of Lemma* 5.1 *suppose that the numbers* $\alpha \in (0,1)$ *and* $t \in (0, 1/8)$ *satisfy the condition*

$$c_\oplus\, t^{1-\alpha} \;\le\; 1.$$

If we assume for $x_0 \in \Omega$ *and for* $0 < R < R_0$ *that* $B(x_0, R) \Subset \Omega$,

$$\begin{aligned} (\varepsilon(u))_{x_0,tR} \;&\in\; \mathcal{B}\Big(\varkappa_0, \frac{\rho}{10}\Big), \qquad (\varepsilon(u))_{x_0,R} \;\in\; \mathcal{B}\Big(\varkappa_0, \frac{\rho}{10}\Big),\\ \Psi(x_0,R) \;&<\; \gamma_1 \;:=\; \min\Big\{\gamma_0, t^d(1-t^\alpha)\frac{3\rho}{20}\Big\}, \end{aligned}$$

where γ_0 *and* R_0 *are the numbers of Lemma* 5.1, *then for any* $k \in I\!N_0$, *we have*

$$(\varepsilon(u))_{x_0,t^{k+1}R} \;\in\; \mathcal{B}\Big(\varkappa_0, \frac{\rho}{4}\Big), \qquad \Psi(x_0, t^k R) \;\le\; t^{\alpha k}\Psi(x_0,R).$$

Now the proof of Theorem 2.1 follows in a standard way from Lemma 5.2.

6 The convex counterpart of Theorem 2.1 and its comparison with Theorem 2.1

As a corollary the main theorem will immediately imply a result in the spirit of [AG]. Of course $m \ge 2$ has to be assumed here.

Theorem 6.1 *Suppose that* $g : \mathring{\mathbb{S}}^d \to \mathbb{R}$ *satisfies our general hypotheses and that there is a convex function* $f : \mathring{\mathbb{S}}^d \to \mathbb{R}$ *and some* $\varkappa_0 \in \mathring{\mathbb{S}}^d$ *such that:*

(i) $f(\kappa) \le g(\kappa)$ *for all* $\kappa \in \mathring{\mathbb{S}}^d$.

(ii) $f(\kappa) \ge \tilde{c_1}|\kappa|^m - \tilde{c_2}$ *for all* $\kappa \in \mathring{\mathbb{S}}^d$ *and for some real numbers* $\tilde{c_1}$, $\tilde{c_2} > 0$.

(iii) $f \in C^2(\mathcal{B}(\varkappa_0, \rho_1))$ *for some* $\rho_1 > 0$ *and* $f(\kappa) = g(\kappa)$ *on* $\mathcal{B}(\varkappa_0, \rho_1)$.

(iv) $(\frac{\partial^2 f}{\partial \kappa^2}(\varkappa_0)\tau) : \tau \ge \lambda|\tau|^2$ *for all* $\tau \in \mathring{\mathbb{S}}^d$ *and for some real number* $\lambda > 0$.

Let $u \in J^1_m(\Omega, \mathbb{R}^d)$ *be a (local) minimizer of* $I(\cdot, \Omega)$ *and suppose further that* (2.3) *is true. Then the function* ∇u *is Hölder continuous in* $B(x_0, R)$ *for some* $R > 0$.

Proof. Notice that we may assume without loss of generality that

$$g(\varkappa_0) \;=\; 0 \quad \text{and} \quad \frac{\partial g}{\partial \kappa}(\varkappa_0) \;=\; 0. \tag{6.1}$$

In fact, if we consider $\tilde{g}(\kappa) := g(\kappa) - \frac{\partial g}{\partial \kappa}(\varkappa_0) : \kappa$ and $\tilde{\tilde{g}}(\kappa) := \tilde{g}(\kappa) - \tilde{g}(\varkappa_0)$, then $\tilde{\tilde{g}}$ satisfies the above assumptions and we have for all $\varphi \in \mathring{J}^1_m(\Omega, \mathbb{R}^d)$

$$I^*(u + \varphi, \tilde{\Omega}) := \int_{\tilde{\Omega}} \tilde{\tilde{g}}(\varepsilon(u + \varphi))\, dx \;=\; I(u + \varphi, \tilde{\Omega}) + \text{const.},$$

where the constant depends on the trace of u on the boundary of the domain under consideration. Observe that the conditions (i)–(iv) are also left unaltered.

Now, since $f \in C^2(\mathcal{B}(\varkappa_0, \rho_1))$ and on account of (iv), there is a real number $\rho_2 \in (0, \rho_1]$ such that

$$\left(\frac{\partial^2 f}{\partial \kappa^2}(\varkappa)\tau\right) : \tau \ge \frac{1}{2}\lambda|\tau|^2 \quad \text{if } |\varkappa - \varkappa_0| \le \rho_2.$$

We may assume in addition $\rho_2 < 1$ and by Taylor's formula we therefore obtain real numbers δ_1, $\delta_2 > 0$ such that for all $\varkappa \in \mathcal{B}(\varkappa_0, \rho_2)$

$$f(\varkappa) \;\ge\; \delta_1|\varkappa - \varkappa_0|^2 \;\ge\; \delta_2(|\varkappa - \varkappa_0|^2 + |\varkappa - \varkappa_0|^m). \tag{6.2}$$

The growth condition (ii) also implies the existence of real numbers $0 < \delta_3$, δ_4 and $1 < \rho_3$ such that for all $|\varkappa - \varkappa_0| > \rho_3$

$$f(\varkappa) \;\ge\; \delta_3|\varkappa - \varkappa_0|^m \;\ge\; \delta_4(|\varkappa - \varkappa_0|^2 + |\varkappa - \varkappa_0|^m). \tag{6.3}$$

It remains to consider the case $\rho_2 \le |\varkappa - \varkappa_0| \le \rho_3$. To do this, fix $\kappa_1 \in \mathring{\mathbb{S}}^d$, $|\kappa_1| = 1$, and suppose $\varkappa = \varkappa_0 + (\frac{\rho_2}{2} + \alpha)\kappa_1$ for some real number $\frac{\rho_2}{2} \le \alpha \le \rho_3 - \frac{\rho_2}{2}$. Global convexity of f implies (again by Taylor's formula) for all $t \in \mathbb{R}$

$$f\left(\varkappa_0 + \frac{\rho_2}{2}\kappa_1 + t\kappa_1\right) \;\ge\; f\left(\varkappa_0 + \frac{\rho_2}{2}\kappa_1\right) + t\frac{\partial f}{\partial \kappa}\left(\varkappa_0 + \frac{\rho_2}{2}\kappa_1\right)\kappa_1.$$

Inserting $t = -\frac{\rho_2}{2}$ and recalling (6.1), (6.2) and assumption (iii) we see

$$\frac{\partial f}{\partial \kappa}\left(\varkappa_0 + \frac{\rho_2}{2}\kappa_1\right)\kappa_1 > 0,$$

i.e. there is a real number $\delta_5 > 0$ such that $f(\varkappa) > \delta_5$ for all $\varkappa$ as above. Since by the choice of $\varkappa$ the quantity $|\varkappa - \varkappa_0|$ is uniformly bounded, there is a real number $\delta_6 > 0$ satisfying for all $\varkappa$ with $\rho_2 \le |\varkappa - \varkappa_0| \le \rho_3$

$$f(\varkappa) \;\ge\; \delta_5 \;\ge\; \delta_6(|\varkappa - \varkappa_0|^2 + |\varkappa - \varkappa_0|^m). \tag{6.4}$$

Summarizing the results, (6.2)–(6.4) prove the existence of a real number $\delta_7 > 0$ satisfying

$$f(\varkappa_0 + \kappa) - f(\varkappa_0) \;\ge\; \delta_7(|\kappa|^2 + |\kappa|^m) \quad \text{for all } \kappa \in \overset{\circ}{\mathbb{S}}{}^d.$$

Thus, by assumption (i) and by $f(\varkappa_0) = g(\varkappa_0)$ the conclusion is

$$\begin{aligned}\int_\Omega \{g(\varkappa_0 + \varepsilon(v)) - g(\varkappa_0)\}\,dx \;&\ge\; \int_\Omega \{f(\varkappa_0 + \varepsilon(v)) - f(\varkappa_0)\}\,dx \\ &\ge\; \delta_7 \int_\Omega (|\varepsilon(v)|^2 + |\varepsilon(v)|^m)\,dx\end{aligned}$$

for any $v \in \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d)$ and Theorem 6.1 is proved. □

Remarks 6.2 (i) Of course, the above arguments neither depend on the incompressibility condition $\operatorname{div} u = 0$ nor they use the fact that only the symmetric part of ∇u is considered. Citing [AF], the corresponding results follow for functionals

$$I(u, \Omega) = \int_\Omega g(\nabla u)\,dx, \qquad u \in W^1_m(\Omega, \mathbb{R}^d). \tag{6.5}$$

(ii) The setting of [AG] requires $g = f$, that is only convex integrands are under consideration. The more general assumptions of Theorem 6.1 are adjusted to the quasiconvex case.

According to these remarks we finish this section with an example which compares typical regularity results of Anzellotti-Giaquinta's type to the corresponding ones of Acerbi-Fusco. For simplicity suppose $N = n = 2$ and consider the general situation (6.5). For a fixed $\bar{p} \in \mathbb{M}^2$ let

$$g_1(p) \;=\; \frac{1}{2}|p - \bar{p}|^2, \quad g_2(p) \;=\; \frac{1}{2}|p + \bar{p}|^2, \quad p \in \mathbb{M}^2,$$

$$\text{and then define} \quad g(p) \;=\; \min\{g_1(p), g_2(p)\}.$$

The above theorems cannot be applied to g directly, so consider the convex envelope g^{**} and the quasiconvex envelope Qg of g respectively. As outlined for example in [DA] we have the formulas

$$\begin{aligned} g^{**}(p) &= \sup\{p^* : p - g^*(p^*) : p^* \in \mathbb{M}^2\}, \\ g^*(p^*) &= \sup\{p^* : p - g(p) : p \in \mathbb{M}^2\}, \end{aligned}$$

and for the quasiconvex envelope we have

$$Qg(p) = \inf \left\{ \fint_\Omega g(p + \nabla v)\, dx : v \in C_0^\infty(\Omega, \mathbb{R}^2) \right\}.$$

In our particular case simple calculations prove

$$g^{**}(p) = \frac{1}{2} \begin{cases} |p + \bar{p}|^2 & \text{if } \ p : \bar{p} < -|\bar{p}|^2 \\ |p - \bar{p}|^2 & \text{if } \ p : \bar{p} > |\bar{p}|^2 \\ |p|^2 - \frac{(p:\bar{p})^2}{|\bar{p}|^2} & \text{if } \ |p : \bar{p}| \leq |\bar{p}|^2. \end{cases} \tag{6.6}$$

For an arbitrary tensor-valued parameter $\bar{p}$, an explicit formula for Qg was obtained by Kohn in [KO]. Here we are going to consider the two choices

$$\bar{p} = a \otimes a \quad \text{for some fixed } a \in \mathbb{R}^2, \tag{6.7}$$

$$\bar{p} = \mathrm{Id}, \tag{6.8}$$

where Id denotes the identity matrix in $\mathbb{M}^2$. Kohn's formula implies for all $p \in \mathbb{M}^2$

$$Qg(p) = g^{**}(p)$$

in the case (6.7), and in the case (6.8) we get

$$Qg(p) = \frac{1}{2} \left| p - \frac{1}{2} \mathrm{tr}\, p\, \mathrm{Id} \right|^2 + \frac{1}{4} \begin{cases} (\mathrm{tr}\, p + 2)^2 & \text{if } \ \mathrm{tr}\, p < -1 \\ (\mathrm{tr}\, p - 2)^2 & \text{if } \ \mathrm{tr}\, p > 1 \\ -(\mathrm{tr}\, p)^2 + 2 & \text{if } \ |\mathrm{tr}\, p| \leq 1 \end{cases} \tag{6.9}$$

$$g^{**}(p) = \frac{1}{2} \left| p - \frac{1}{2} \mathrm{tr}\, p\, \mathrm{Id} \right|^2 + \frac{1}{4} \begin{cases} (\mathrm{tr}\, p + 2)^2 & \text{if } \ \mathrm{tr}\, p < -2 \\ (\mathrm{tr}\, p - 2)^2 & \text{if } \ \mathrm{tr}\, p > 2 \\ 0 & \text{if } \ |\mathrm{tr}\, p| \leq 2 \end{cases} \tag{6.10}$$

Let us start considering the first case (6.7). Then we have the following

Proposition 6.3 *Suppose that $\bar{p} = a \otimes a$, $a \in \mathbb{R}^2$, and that $u \in W_2^1(\Omega, \mathbb{R}^2)$ is a local minimizer of $I(\cdot, \Omega)$, where*

$$I(v, \Omega) = \int_\Omega g^{**}(\nabla v)\, dx.$$

Then there exists an open set $\Omega_+ = \Omega_+(u) \subset \Omega$ *such that:*

(i) $\nabla u \in C^{0,\alpha}(\Omega_+, \mathbb{M}^2)$ *for all* $\alpha \in [0,1[$,

(ii) $|(\nabla u(x)a) \cdot a| > |a|^4$ *for all* $x \in \Omega_+$,

(iii) $|(\nabla u(x)a) \cdot a| \le |a|^4$ *for almost all* $x \in \Omega \backslash \Omega_+$.

Remarks 6.4 (i) We do not claim that the set Ω_+ is non-empty. Note that $\nabla u(\Omega_+)$ contains only points of strict quasiconvexity of g^{**}.

(ii) For the proof of this proposition, it will make no difference if Theorem 2.1 or Theorem 6.1 is applied.

Proof. The representations (6.6) and (6.7) of g^{**} respectively $\bar{p}$ immediately imply

$$\left(\frac{\partial^2 g^{**}}{\partial p^2}(p)\, q\right) : q \;=\; \begin{cases} |q|^2 & \text{if } |(pa)\cdot a| \;>\; |a|^4 \\ |q|^2 - \frac{((qa)\cdot a)^2}{|a|^4} & \text{if } |(pa)\cdot a| \;<\; |a|^4 \end{cases} . \tag{6.11}$$

Thus, the proof of Theorem 6.1 in case $m = 2$ and Lemma 3.1 show that g^{**} is strictly quasiconvex in some neighbourhood of any point p if $|(pa)\cdot a| > |a|^4$. On the other hand, strict quasiconvexity of g^{**} at some point p_0 gives (see [MO1], [MO2])

$$\left(\frac{\partial^2 g^{**}}{\partial p^2}(p_0)\, \tilde{a} \otimes \tilde{b}\right) : (\tilde{a} \otimes \tilde{b}) \;\ge\; \nu |\tilde{a}|^2 |\tilde{b}|^2 \tag{6.12}$$

for all $\tilde{a}$, $\tilde{b} \in \mathbb{R}^2$. Hence, by (6.11), g^{**} is not strictly quasiconvex at p if $|(pa)\cdot a| < |a|^4$, i.e. we have proved Proposition 6.3 and Remark 6.4, (ii). □

Now let us concentrate on the second case (6.8). Then the convex and quasiconvex envelopes do not coincide and we have

Proposition 6.5 *Suppose that* $\bar{p} = Id$. *Then we have the following*
I: *If* $u \in W_2^1(\Omega, \mathbb{R}^2)$ *is a local minimizer of* $I_c(\cdot, \Omega)$, *where*

$$I_c(v, \Omega) = \int_\Omega g^{**}(\nabla v)\, dx,$$

then there exists an open set $\Omega_+ = \Omega_+(u) \subset \Omega$ *such that:*

(i) $\nabla u \in C^{0,\alpha}(\Omega_+, \mathbb{M}^2)$ *for all* $\alpha \in [0,1[$,

(ii) $|\mathrm{div}\, u(x)| \ne 2$ *for any* $x \in \Omega_+$,

(iii) $|\mathrm{div}\, u(x)| = 2$ *almost everywhere on* $\Omega \backslash \Omega_+$.

II: *If* $u \in W_2^1(\Omega, \mathbb{R}^2)$ *is a local minimizer of* $I_q(\cdot, \Omega)$*, where*

$$I_q(v, \Omega) = \int_\Omega Qg(\nabla v)\, dx,$$

then there exists an open set $\Omega_+ = \Omega_+(u) \subset \Omega$ *such that:*

(i) $\nabla u \in C^{0,\alpha}(\Omega_+, \mathbb{M}^2)$ *for all* $\alpha \in [0, 1[$,

(ii) $|\operatorname{div} u(x)| > 1$ *for any* $x \in \Omega_+$,

(iii) $|\operatorname{div} u(x)| \le 1$ *almost everywhere on* $\Omega \backslash \Omega_+$.

Remark 6.6 Although the first part of the proposition deals with a globally convex integrand, the proof will be an application of Theorem 2.1. This gives better results than Theorem 6.1. The reason is that $\frac{\partial^2 g^{**}}{\partial p^2}$ is degenerated if $|\operatorname{tr} p| < 2$, that is Theorem 6.1 cannot be applied in that case.

Proof. First of all notice that $\partial^2 g^{**}/\partial p^2$ is of class C^2 on the set $\{p \in \mathbb{M}^2 : |\operatorname{tr} p| \neq 2\}$. Next, we have by (6.10) the decomposition

$$g^{**}(p) = g_0(p) + g_+(p), \qquad g_0(p) \;=\; \frac{1}{2}\left|p - \frac{1}{2}\operatorname{tr} p\, \mathrm{Id}\right|^2,$$

where $g_+(p)$ is a convex function. Thus, for any $v \in \overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}^2)$ and for any $p \in \mathbb{M}^2$ convexity of g_+ implies with some elementary calculations

$$\begin{aligned} \int_\Omega \{g^{**}(p + \nabla v) - g^{**}(p)\}\, dx \;&\ge\; \frac{1}{2}\int_\Omega \{|\nabla v|^2 - \frac{1}{2}\operatorname{div}^2 v\}\, dx \qquad (6.13)\\ =\; \frac{1}{2}\int_\Omega &\left\{\frac{1}{2}|\nabla v|^2 - \det(\nabla v) + \frac{1}{2}(D_1 v^2 - D_2 v^1)^2\right\} dx \;\ge\; \frac{1}{4}\int_\Omega |\nabla v|^2\, dx. \end{aligned}$$

So, the first part of the proposition is proved by Theorem 2.1. Now consider the quasiconvex envelope Qg which is not globally convex. The representation formula (6.10) shows that $\partial^2 Qg/\partial p^2$ is of class C^2 on the set $\{p \in \mathbb{M}^2 : |\operatorname{tr} p| \neq 1\}$. First we observe that our result is optimal in the sense that Qg is not strictly quasiconvex at $p \in \mathbb{M}^2$ if $|\operatorname{tr} p| < 1$. This follows from (6.12) and from

$$\left(\frac{\partial^2 Qg}{\partial p^2}(p)(\tilde a \otimes \tilde b)\right) : (\tilde a \otimes \tilde b) \;=\; |\tilde a|^2 |\tilde b|^2 - (\tilde a \cdot \tilde b)^2$$

for any $\tilde a$, $\tilde b \in \mathbb{R}^2$ and for any $p \in \mathbb{M}^2$ with $|\operatorname{tr} p| < 1$. Now we want to prove quasiconvexity of Qg at any point $p_0 \in \mathbb{M}^2$ with $|\operatorname{tr} p_0| > 1$, more precisely we claim

$$\int_\Omega \{Qg(p_0 + \nabla v) - Qg(p_0)\}\, dx \;\ge\; \frac{1}{4}\min\left\{1, \frac{3|\operatorname{tr} p_0| - 2}{2|\operatorname{tr} p_0|}\right\}\int_\Omega |\nabla v|^2\, dx \qquad (6.14)$$

for any $v \in \overset{\circ}{W}{}^1_2(\Omega, \mathbb{R}^2)$ and for any $p_0 \in \mathbb{M}^2$ with $|\mathrm{tr}\, p_0| > 1$. To prove this claim,

$$\bar{g}_+(p) \;=\; Qg(p) - g_0(p) \quad \text{and} \quad g_0(p) \;=\; \frac{1}{2}\left|p - \frac{1}{2}\mathrm{tr}\, p\, \mathrm{Id}\right|^2$$

are introduced. Considering $\bar{g}_+(p)$, the idea of construction is to find a parabola which touches the parabolas $(t-2)^2$ and $(t+2)^2$, $t \in \mathbb{R}$, at the points $t_0 := |\mathrm{tr}\, p_0| > 1$ and $-t_0$ respectively. This leads to the definition

$$\hat{g}_+(p) \;=\; -\frac{2-t_0}{4t_0}(\mathrm{tr}\, p)^2 + 2(2-t_0).$$

Then, by construction $\bar{g}_+(p) \geq \hat{g}_+(p)$ for any $p \in \mathbb{M}^2$ and $\bar{g}_+(p_0) = \hat{g}_+(p_0)$. Recalling (6.13) and using Taylor's formula we obtain

$$\begin{aligned} J \;&:=\; \int_\Omega \{Qg(p_0 + \nabla v) - Qg(p_0)\}\, dx \\ &=\; \int_\Omega g_0(\nabla v)\, dx + \int_\Omega \{\bar{g}_+(p_0 + \nabla v) - \bar{g}_+(p_0)\}\, dx \\ &\geq\; \frac{1}{4}\int_\Omega |\nabla v|^2\, dx + \int_\Omega \{\hat{g}_+(p_0 + \nabla v) - \hat{g}_+(p_0)\}\, dx \\ &=\; \frac{1}{4}\int_\Omega |\nabla v|^2\, dx + \frac{1}{2}\int_\Omega \int_0^1 (1-\theta)\left(\frac{\partial^2 \hat{g}_+}{\partial p^2}(p_0 + \theta \nabla v)\nabla v\right) : \nabla v\, d\theta\, dx \\ &\geq\; \frac{1}{4}\int_\Omega |\nabla v|^2\, dx - \frac{2-t_0}{8t_0}\int_\Omega \mathrm{div}^{\,2} v\, dx. \end{aligned}$$

So, if $t_0 \geq 2$ we are done and in the case $1 < t_0 < 2$ we see

$$\begin{aligned} J \;&\geq\; \frac{1}{4}\int_\Omega \left\{\frac{3t_0 - 2}{2t_0}|\nabla v|^2 + \frac{2-t_0}{2t_0}(D_1 v^2 - D_2 v^1)^2 - \frac{2-t_0}{t_0}\det(\nabla v)\right\} dx \\ &\geq\; \frac{3t_0 - 2}{8t_0}\int_\Omega |\nabla v|^2\, dx. \end{aligned}$$

Thus (6.14) and by Theorem 2.1 the whole proposition is proved. □

7 Applications: local regularity of the stress tensor for the k-well problem

Consider the energy density of a k-phase body given by

$$g(\kappa) = \min_{i=1\dots k}\{g_i(\kappa)\}, \kappa \in \overset{\circ}{\mathbb{S}}{}^d.$$

We assume the densities to satisfy for all $i = 1 \dots k$ and for some $m \geq 2$:

(i) g_i is smooth and strictly convex,

(ii) $c_1|\kappa|^m - c_2 \leq g_i(\kappa) \leq c_3(1 + |\kappa|^m)$,

(iii) $\dfrac{\partial g_i^*}{\partial \tau}(\cdot)$ is an open mapping, (7.1)

(iv) $g(\kappa) = \min_{i=1\dots k}\{g_i(\kappa)\}$ satisfies the general hypotheses (see (2.1) and (2.2)) of our paper.

Here g_i^*, g^* (g_i^{**}, g^{**}) denote, as usual, the first (second) Young transforms of g_i and g respectively on $\overset{\circ}{\mathbb{S}}{}^d$, for example

$$\begin{aligned} g^*(\tau) &= \sup\{\kappa : \tau - g(\kappa) : \kappa \in \overset{\circ}{\mathbb{S}}{}^d\}, \quad \tau \in \overset{\circ}{\mathbb{S}}{}^d, \\ g^{**}(\kappa) &= \sup\{\kappa : \tau - g^*(\tau) : \tau \in \overset{\circ}{\mathbb{S}}{}^d\}, \quad \kappa \in \overset{\circ}{\mathbb{S}}{}^d. \end{aligned}$$

From the definition of g we immediately get $g^*(\tau) = \max_{i=1\dots k}\{g_i^*(\tau)\}$. Following [SE3] (see Theorem 2.4 and 2.5) we now pass to a suitable relaxed variational problem $\mathcal{QP}$: find a function $u \in u_0 + \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d)$ such that

$$QI(u) = \inf\{QI(v) : v \in u_0 + \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d)\}.$$

Here Qg denotes the J^1_m-quasiconvex envelope for g intoduced in [SE3] and the relaxed energy QI is given by

$$QI(v) = \int_\Omega Qg(\varepsilon(v))\, dx.$$

In the following it is assumed that $Qg = g^{**}$.

Example 7.1 (i) This hypothesis is fulfilled in the case $d = 2$ (compare [SE3], Theorem 2.3). Thus the situation of [FS1] is generalized by admitting k-wells of m-growth.

(ii) Since the arguments of this paper are not limited to the incompressible case, the (compressible) setting of [SE1] is also covered, where the compatible structure of two wells in three dimensions implies $Qg = g^{**}$. A discussion of the incompatible case can be found in [SE2].

Now let u be a solution of $\mathcal{QP}$ and denote by Ω_u the set of all $x_0 \in \Omega$ such that (2.3) holds for some $\varkappa_0 \in \overset{\circ}{\mathbb{S}}{}^d$. Then, as an immediate consequence of Theorem 6.1 one obtains

Theorem 7.2 *If $x_0 \in \Omega_u$ and if $g^{**}(x_0) = g(x_0) = g_i(x_0)$ for all x near x_0 and for only one $i = 1, \dots, k$, then ∇u is Hölder continuous in $B(x_0, R)$ for some $R > 0$.*

On the other hand, consider the dual variational problem $\mathcal{P}^*$: find a tensor $\sigma \in Q$ such that σ maximizes the dual functional R:

$$\begin{aligned} R(\sigma) &= \sup\{R(\tau) : \tau \in Q\}, \\ R(\tau) &= \int_\Omega (\varepsilon(u_0) : \tau - g^*(\tau))\, dx, \\ \tau \in Q &:= \left\{ \tau \in L^{\frac{m}{m-1}}(\Omega, \overset{\circ}{\mathbb{S}}{}^d) : \int_\Omega \tau : \varepsilon(v)\, dx = 0 \text{ for all } v \in \overset{\circ}{J}{}^1_m(\Omega, \mathbb{R}^d) \right\}. \end{aligned}$$

We recall that $\mathcal{P}^*$ has a unique solution σ. If u denotes a solution of $\mathcal{QP}$, and if ∂ denotes the subdifferential, then the duality relation (see [ET], Prop. 5.1, p. 115)

$$\sigma(x) \in \partial g^{**}(\varepsilon(u)(x)) \quad \text{for almost all } x \in \Omega \tag{7.2}$$

is known as well as the equation

$$QI(u) = R(\sigma). \tag{7.3}$$

Now introduce the set of (σ, u) Lebesgue points, i.e.

$$\Omega_{\sigma,u} = \left\{ x \in \Omega_u : \lim_{R \downarrow 0} (\sigma)_{x,R} \text{ exists and (7.2) holds} \right\}.$$

Furthermore, let $A = \{1 \dots k\}$ and

$$\begin{aligned} A(\tau) &= \{i \in A : g^*(\tau) = g_i^*(\tau)\}, \\ a_i &= \{\tau \in \overset{\circ}{\mathbb{S}}{}^d : g^*(\tau) = g_i^*(\tau) \text{ and } \operatorname{card} A(\tau) = 1\}, \\ a(\sigma) &= \{x \in \Omega_{\sigma,u} : \operatorname{card} A(\sigma(x)) = 1\}. \end{aligned}$$

The physical meaning of the set $a(\sigma)$ is that it can be seen as the union of single phases and that no microstructure occurs. Then our regularity result reads as:

Theorem 7.3 *The set $a(\sigma)$ is open and σ is Hölder continuous on $a(\sigma)$ for any exponent $0 < \alpha < 1$. Moreover,* $\operatorname{card} A(\sigma(x)) > 1$ *for almost all $x \in \Omega \sim a(\sigma)$.*

Remark 7.4 For the particular case studied in [FS1] we have a slightly stronger result, i.e. $a(\sigma)$ can be replaced by the set of all Lebesgue points x of σ for which $\mathrm{card}A(\sigma(x)) = 1$.

Proof. Fix i and $\tau_0 \in a_i$ such that

$$g^*(\tau_0) = g_i^*(\tau_0) \neq g_j^*(\tau_0)$$

for all $j \in \{1, \dots k\}$, $j \neq i$. Since $g^*(\tau) = \max_{j=1\dots k}\{g_j^*(\tau)\}$ and since each g_j^* is a smooth function, there exists a real number $0 < \rho_0$ and a ball $\mathcal{B}(\tau_0, \rho_0)$ such that

$$g^*(\tau) = g_i^*(\tau) \neq g_j^*(\tau) \quad \text{for all } \tau \in \mathcal{B}(\tau_0, \rho_0)$$

and again for all $j \in \{1, \dots k\}$, $j \neq i$. In particular, g^* is seen to be smooth on $\mathcal{B}(\tau_0, \rho_0)$ and we have

$$\frac{\partial g^*}{\partial \tau}(\tau) = \frac{\partial g_i^*}{\partial \tau}(\tau) \quad \text{for all } \tau \in \mathcal{B}(\tau_0, \rho_0). \tag{7.4}$$

In general, given a convex function F and its polar function F^*, it follows that $v^* \in \partial F(v)$ if and only if $F(v) + F^*(v^*) = \langle v, v^* \rangle$. Setting $\varkappa = \frac{\partial g^*}{\partial \tau}(\tau)$ on $\mathcal{B}(\tau_0, \rho_0)$ we have on this ball $g^*(\tau) + g^{**}(\varkappa) = \tau : \varkappa$ and the same relation holds for g_i^*, that is one obtains

$$\begin{aligned} g^*(\tau) + g^{**}\left(\frac{\partial g^*}{\partial \tau}(\tau)\right) &= \tau : \frac{\partial g^*}{\partial \tau}(\tau), \\ g_i^*(\tau) + g_i^{**}\left(\frac{\partial g_i^*}{\partial \tau}(\tau)\right) &= \tau : \frac{\partial g_i^*}{\partial \tau}(\tau). \end{aligned} \tag{7.5}$$

Notice that only the local smoothness of g^* and no further properties of g^{**} are used to prove (7.5). Now let

$$V = \frac{\partial g^*}{\partial \tau}(\mathcal{B}(\tau_0, \rho_0)).$$

By assumption, $\frac{\partial g_i^*}{\partial \tau}$ is an open mapping, hence V is known to be an open neighbourhood of $\varkappa_0 := \frac{\partial g^*}{\partial \tau}(\tau_0)$. By definition, for any $\varkappa \in V$ there exists $\tau = \tau(\varkappa) \in \mathcal{B}(\tau_0, \rho_0)$ such that

$$\varkappa = \frac{\partial g^*}{\partial \tau}(\tau) = \frac{\partial g_i^*}{\partial \tau}(\tau),$$

i.e. for any $\varkappa \in V$ we have by (7.5)

$$g^{**}(\varkappa) = \tau : \frac{\partial g^*}{\partial \tau}(\tau) - g^*(\tau) = \tau : \frac{\partial g_i^*}{\partial \tau}(\tau) - g_i^*(\tau) = g_i^{**}(\varkappa).$$

So far the existence of an open neighbourhood $V(\varkappa_0) = \frac{\partial g^*}{\partial \tau}(\mathcal{B}(\tau_0, \rho_0))$ such that

$$g^{**}(\varkappa) \ = \ g_i^{**}(\varkappa) \ = \ g_i(\varkappa) \quad \text{for all } \ \varkappa \in V \tag{7.6}$$

is proved. In particular, g^{**} is seen to be smooth and strictly convex on V. Thus, on $\mathcal{B}(\tau_0, \rho_0)$ it is allowed to take the derivatives of (7.5) and we get

$$\frac{\partial g^*}{\partial \tau}(\tau) + \frac{\partial g^{**}}{\partial \tau}\left(\frac{\partial g^*}{\partial \tau}(\tau)\right)\frac{\partial^2 g^*}{\partial \tau^2}(\tau) \ = \ \frac{\partial g^*}{\partial \tau}(\tau) + \tau\,\frac{\partial^2 g^*}{\partial \tau^2}(\tau).$$

Since g_i^* is strictly convex the second derivatives are not degenerated at least on a dense set and by smoothness we obtain on $\mathcal{B}(\tau_0, \rho_0)$

$$\tau \ = \ \frac{\partial g^{**}}{\partial \tau}\left(\frac{\partial g^*}{\partial \tau}(\tau)\right). \tag{7.7}$$

At this point, consider the dual solution and fix $x_0 \in a(\sigma)$. On one hand, the above reasoning can be applied to $\sigma(x_0)$ and (7.7) gives

$$\sigma(x_0) \ = \ \frac{\partial g^{**}}{\partial \tau}\left(\frac{\partial g^*}{\partial \tau}(\sigma(x_0))\right). \tag{7.8}$$

On the other hand, by (7.2) we have

$$\sigma(x_0) \ \in \ \partial g^{**}(\varepsilon(u)(x_0)). \tag{7.9}$$

Let $\kappa_1 := \frac{\partial g^*}{\partial \tau}(\sigma(x_0))$ and $\kappa_2 := \varepsilon(u)(x_0)$. We claim that $\kappa_1 = \kappa_2$. In fact, g^{**} is smooth in an open neighbourhood $V = V(\kappa_1)$ and we can choose $0 < \delta$ sufficiently small such that $\tilde{\kappa} := \kappa_1 + \delta(\kappa_2 - \kappa_1) \in V$. By construction, we have

$$\kappa_1 - \tilde{\kappa} \ = \ \delta(\kappa_1 - \kappa_2), \qquad \tilde{\kappa} - \kappa_2 \ = \ (1-\delta)(\kappa_1 - \kappa_2).$$

On account of (7.8) and (7.9) one gets

$$\begin{aligned} 0 &= \left(\frac{\partial g^{**}}{\partial \tau}(\kappa_1) - \sigma(x_0)\right) : (\kappa_1 - \kappa_2) \\ &= \left(\frac{\partial g^{**}}{\partial \tau}(\kappa_1) - \frac{\partial g^{**}}{\partial \tau}(\tilde{\kappa})\right) : \frac{\kappa_1 - \tilde{\kappa}}{\delta} + \left(\frac{\partial g^{**}}{\partial \tau}(\tilde{\kappa}) - \sigma(x_0)\right) : \frac{\tilde{\kappa} - \kappa_2}{1-\delta}. \end{aligned} \tag{7.10}$$

Since g^{**} is convex and smooth at $\tilde{\kappa}$, we obtain

$$\frac{\partial g^{**}}{\partial \tau}(\tilde{\kappa}) : (\kappa_2 - \tilde{\kappa}) + g^{**}(\tilde{\kappa}) \ \le \ g^{**}(\kappa_2).$$

Although g^{**} is not necessarily smooth at κ_2, $\sigma(x_0)$ at least is known to be a subgradient of g^{**} at κ_2, which means

$$\sigma(x_0) : (\tilde{\kappa} - \kappa_2) + g^{**}(\kappa_2) \ \leq \ g^{**}(\tilde{\kappa}).$$

Combining these relations we see

$$\left(\frac{\partial g^{**}}{\partial \tau}(\tilde{\kappa}) - \sigma(x_0) \right) : (\tilde{\kappa} - \kappa_2) \ \geq \ 0.$$

Thus together with (7.10) it is proved that

$$0 \ = \ \left(\frac{\partial g^{**}}{\partial \tau}(\kappa_1) - \frac{\partial g^{**}}{\partial \tau}(\tilde{\kappa}) \right) : (\kappa_1 - \tilde{\kappa}). \tag{7.11}$$

However, on the line $(\kappa_1, \tilde{\kappa})$ the function g^{**} is known to be smooth and strictly convex, that is we can write

$$\begin{aligned}
&\left(\frac{\partial g^{**}}{\partial \tau}(\kappa_1) - \frac{\partial g^{**}}{\partial \tau}(\tilde{\kappa}) \right) : (\kappa_1 - \tilde{\kappa}) \\
&= \int_0^1 \frac{d}{ds} \left\{ \frac{\partial g^{**}}{\partial \tau}(s\kappa_1 + (1-s)\tilde{\kappa}) : (\kappa_1 - \tilde{\kappa}) \right\} ds \\
&= \int_0^1 \frac{\partial^2 g^{**}}{\partial \tau^2}(s\kappa_1 + (1-s)\tilde{\kappa})((\kappa_1 - \tilde{\kappa}), (\kappa_1 - \tilde{\kappa}))\, ds \ > \ 0
\end{aligned}$$

by strict convexity if $\kappa_1 \neq \kappa_2$. In other words, we have proved that

$$\varepsilon(u)(x_0) \ = \ \frac{\partial g^{*}}{\partial \tau}(\sigma(x_0)).$$

Then, as above, there is a ball $\mathcal{B}(\sigma(x_0), \rho)$ such that for some $i \in \{1 \dots k\}$

$$g^{**}(\varkappa) \ = \ g_i^{**}(\varkappa) \ = \ g_i(\varkappa) \quad \text{for all } \ \varkappa \in V,$$

where V is some open neighbourhood of $\varepsilon(u)(x_0)$. Again, we can apply Theorem 6.1 and the theorem is proved since $\varepsilon(u)$ is smooth near x_0, since g^{**} is smooth near $\varepsilon(u)(x_0)$ and since we have (7.2). ☐

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