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A remark on isolated singularities of surfaces with bounded mean curvature: the non-minimizing and non-perpendicular case

By

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Abstract. The length of the free boundary of twodimensional weak surfaces with bounded mean curvature is studied in the case of non-perpendicular contact angles and for non-minimizing stationary surfaces. Isolated singularities are excluded if the contact angle is bounded away from zero and if the solution is assumed to lie on one side of the supporting surface.

Given a vector field $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, $|\mathrm{div} Q| < H_0 < \infty$, we consider the functional

$$\mathscr{F}[Y] = \tfrac{1}{2} \int\int_{B_1(0)} |\nabla Y|^2 \, du\, dv + \int\int_{B_1(0)} Q(Y) \cdot (Y_u \wedge Y_v)\, du\, dv =: D[Y] + V^Q[Y]$$

and study regularity at the free boundary of twodimensional stationary points X in a suitable subclass $\mathscr{C}$ of the Sobolev space $H^{1,2}(B_1(0), \mathbb{R}^3)$. Here $\mathscr{C}$ defines partially free boundary values on a supporting surface S. A smooth solution X is a surface of mean curvature $H = \mathrm{div} Q/2$ satisfying a free boundary condition which is due to capillary forces:

$$|Q \cdot N| = \cos\alpha.$$

Here α denotes the angle in which X meets the supporting surface S at the free boundary and N is the outward normal unit vector of S (see, for instance, [1], [8]). Considering the perpendicular case, that is $Q \cdot N_{|S} \equiv 0$, Grüter, Hildebrandt and Nitsche ([8]) proved regularity of stationary points up to the free boundary by extending the interior results of Grüter ([6]) – see also [7] and [4] for the minimal surface case. In a recent paper ([2]), Hölder continuity of minima up to the free boundary was proved in the non-perpendicular case $|Q \cdot N_{|S}| < q < 1$ for a constant q. The last condition seems to be sharp since otherwise we have to expect unbounded solutions of bounded mean curvature and of bounded area.

Here we exclude the existence of isolated singularities of stationary points in the non-perpendicular case, where we assume that the solution does not penetrate the supporting surface (compare [3]/I, Chapter 6.4, pp. 396). Of course we also have to assume the contact angle to be bounded away from zero. The argumentation is based on length estimates of the free trace which are obtained in terms of the contact angle.

To fix notation, we always consider a rectifiable arc Γ with end points $P_1 \neq P_2$ on a supporting surface S. Then the class $\mathscr{C}(\Gamma, S)$ of admissible surfaces is given by

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Definition 1. For $B = B_1(0) \subset \mathbb{R}^2$ the class $\mathscr{C}(\Gamma, S)$ is the set of all functions $Y \in H^{1,2}(B, \mathbb{R}^3)$ with the following properties: there is an arc $C = \{e^{i\theta} : 0 \leqq \theta_1 \leqq \theta \leqq \theta_2 < 2\pi\}$ such that the well defined L^2-traces of Y satisfy:

(i.) Free boundary values: $Y(w) \in S$ for $\mathscr{H}^1$-almost all $w = (u, v) \in \partial B \sim C$;

(ii.) Plateau boundary values: $Y_{|C} : C \to \Gamma$ is a continuous, weakly monotonic mapping onto Γ with $Y(e^{i\theta_1}) = P_{i_1}$ and $Y(e^{i\theta_2}) = P_{i_2}$ for $\{i_1, i_2\} = \{1, 2\}$.

Notice that stationarity with respect to inner variations forces X to be parametrized conformally (see [3]/I, pp. 242, [1], [2]), that is

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v \quad \text{almost everywhere in } B.$$

Having established this, we now (see once more [1], [2]) may use the standard notation

$$B := \{w \in \mathbb{R}^2 : |w| < 1, v > 0\}, \;\; C := \{w \in \mathbb{R}^2 : |w| = 1, v \geqq 0\}, \;\; I := \partial B \sim C,$$

as well as for a given $w_0 = (u_0, 0) \in I$ and for $0 < r < |1 - w_0|$

$$S_r(w_0) := \{w \in \mathbb{R}^2 : |w - w_0| < r, v > 0\},$$
$$C_r(w_0) := \{w \in \mathbb{R}^2 : |w - w_0| = r, v \geqq 0\}$$

and $I_r(w_0) := \partial S_r(w_0) \sim C_r(w_0)$. Finally $X \in \mathscr{C}(\Gamma, S)$ is a stationary point of $\mathscr{F}$ in this class (with respect to outer variations) if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{\mathscr{F}[X_\varepsilon] - \mathscr{F}[X]\} = 0$$

for any family of surfaces $X_\varepsilon \in \mathscr{C}(\Gamma, S)$, $|\varepsilon| < \varepsilon_0$ for some number $\varepsilon_0 > 0$, such that $X_\varepsilon(w) = X(w) + \varepsilon \Psi(w, \varepsilon)$, where the Dirichlet integrals $D[\Psi(\cdot, \varepsilon)]$ are uniformly bounded and where we have $\Phi \in H^{1,2} \cap L^\infty(B, \mathbb{R}^3)$ with $\Psi(w, \varepsilon) \to \Phi(w)$ for almost all $w \in B$ as $\varepsilon \to 0$.

The supporting surface S is assumed to be a regular surface of class C^2 which admits a normal vectorfield $N = (n^1, n^2, n^3)$ of class C^1. If threedimensional balls are denoted by $\mathscr{B}_\rho(y)$ then we have:

Proposition 2. *Given $x_0 \in S$, there is a neighbourhood $U \subset \mathbb{R}^3$ of x_0, a real number $\rho > 0$ and a C^2-diffeomorphism $x = h(y) = h_{x_0}(y)$, $\mathscr{B}_\rho(0) \to U$, satisfying:*

(i.) *$h^{-1}(x_0) = 0$ and $h^{-1}(S \cap U) = B_\rho(0) = \{y \in \mathbb{R}^3 : |y| < \rho,\ y^3 = 0\}$.*

(ii.) *$h_{x_0}(y^1, y^2, y^3) = R_{x_0}^{-1}\big(y^1, y^2, (1 + y^3)(1 + f_{x_0}(y^1, y^2))\big) + a_{x_0}$.*

Here $f = f_{x_0} \in C^2(B_\rho(0), \mathbb{R})$ satisfies $f(0,0) = 0$ and $Df(0,0) = (0,0)$. The rotation $R_{x_0} \in SO_3(\mathbb{R}^3, \mathbb{R}^3)$ is given by

$$R_{x_0} = \begin{pmatrix} \dfrac{(n^2)^2}{1+n^3} + n^3 & -\dfrac{n^1 n^2}{1+n^3} & -n^1 \\ -\dfrac{n^1 n^2}{1+n^3} & \dfrac{(n^1)^2}{1+n^3} + n^3 & -n^2 \\ n^1 & n^2 & n^3 \end{pmatrix} (x_0),$$

i.e. $R_{x_0} N(x_0) = (0,0,1) = e_3$. Finally $a_{x_0} \in \mathbb{R}^3$ is a translation.

Proof. Note that $|N| \equiv 1$ implies R_{x_0} to be well defined. Now, fix $x_0 \in S$ and choose R_{x_0} as above. With the obvious meaning of notation $\tilde{S} := R_{x_0}(S - x_0) + (0,0,1)$ is a regular surface satisfying $\tilde{x} := (0,0,1) \in S$ and $\tilde{N}(\tilde{x}) = e_3$. Since $\tilde{S}$ is locally a graph over its tangent plane at $\tilde{x}$ we find a parametrization $\tilde{h}(y^1, y^2, 0)$ of $\tilde{S}$ with the desired properties of f. Here $\tilde{h}$ is given by $\tilde{h}(y^1, y^2, y^3) = \big(y^1, y^2, (1+y^3)(1+f(y^1,y^2))\big)$. On account of $D\tilde{h}(0) = Id$ we may assume that $\tilde{h}$ is a C^2-diffeomorphism and turning back to the original surface S the proposition is proved. $\square$

To proceed further we need the following general hypotheses.

Assumption 3. *There is a real number K and a function $\tilde{N}(x) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $\tilde{N}(x) = N(x)$ for all $x \in S$ and $|D\tilde{N}(x)| \leqq K$ for all $x \in \mathbb{R}^3$.*

Example 4. If S is an oriented strict C^2-surface in the sense of [8], Assumption (V), p. 122, and if the Gauss curvature of S is bounded, then S satisfies Assumption 3. The first observation to prove this is: there is a real number $\delta_1 > 0$ such that each ball $\mathscr{B}_{\delta_1}(x)$, $x \in S$, is contained in the image of one diffeomorphism h_x as defined in [8], Assumption (V). What is more, the second derivatives of all h_x are uniformly bounded since in addition to Assumption (V) the Gauss curvature is assumed to be bounded. The next step is to prove the existence of a real number $\delta_2 > 0$ such that: for each $x \in \mathbb{R}^3$ with $\operatorname{dist}(x, S) < \delta_2$ there is a unique decomposition $x = f(x) + tN(f(x))$, where $f(x) \in S$ satisfies $|x - f(x)| = \operatorname{dist}(x, S)$. With this decomposition the above assumption is immediately verified.

Now our main result reads as follows:

Theorem 5. *Consider a vectorfield $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, $|\operatorname{div} Q(z)| < H_0 < \infty$ for some constant $H_0 > 0$ and for all $z \in \mathbb{R}^3$, and a boundary configuration $\langle \Gamma, S \rangle$ with a supporting surface S as given above. Suppose that there is a real number $0 < q < 1$ with*

$$|Q(z) \cdot N(z)| < q \quad \text{for all } z \in S.$$

If X is a stationary point of $\mathscr{F}$ in the class $\mathscr{C}(\Gamma, S)$ satisfying

(i.) *$X \in C^2(\bar{B} \sim \{w_0\}, \mathbb{R}^3)$ for some $w_0 \in I$,*
(ii.) *$N\big(X(w)\big) \cdot X_v(w) \geqq 0$ for almost all $w \in I \sim \{w_0\}$,*

then we have

$$\lim_{\varepsilon \to 0} \int_{I \sim I_\varepsilon(w_0)} |X_v(u,0)|\, du < \infty .$$

Theorem 5 immediately implies continuity up to the free boundary:

Corollary 6. *With the above assumptions $X \in C^0(\bar{B}, \mathbb{R}^3)$ holds true.*

Remark 7. If we consider for example [8], Propositon 3′, pp. 136, then it is on account of the boundary integral in equation (4) not evident if the above result can be improved to Hölder continuity.

Proof of Corollary 6. Fix $\varepsilon > 0$. By assumption, $|X_v|$ is a smooth function on $I \sim \{w_0\}$ and Theorem 5 implies $|X_v| \in L^1(I)$. (For a rigorous proof apply for example Fatou's Lemma to the sequence $f_k := \min\{|X_v|_{|I}, k\}$.) By conformality, the same is true for $|X_u|$, that is there is a real number $\delta_1 > 0$ such that

$$\int\limits_{I_{3\delta_1}(w_0)} |X_u|\,du < \frac{\varepsilon}{9}.$$

Thus, w_0 splits $I_{3\delta_1}(w_0)$ into two parts $I^{\pm}_{3\delta_1}(w_0)$ with $\operatorname{osc}_{I^{\pm}_{3\delta_1}(w_0)} X < \frac{\varepsilon}{9}$. By the Courant-Lebesgue Lemma (see [7], Lemma 2, p. 393, for the situation considered here) we have proved

$$\text{(1)} \qquad \operatorname{osc}_{I_{3\delta_1}(w_0)\sim\{w_0\}} X < \frac{\varepsilon}{3}.$$

Again by the Courant-Lebesgue Lemma, there is a real number $\delta_2 > 0$ satisfying: for each $w = (u,v) \in S_{\delta_2}(w_0)$ there exits some $\tilde{v} \in (v, 3v/2)$ with

$$\text{(2)} \qquad \operatorname{osc}_{C_{\tilde{v}}((u,0))} X < \frac{\varepsilon}{3}\,.$$

Finally according to [6], Theorem 3.10, p. 8, and once again according to the Courant-Lebesgue Lemma there is a real number $\delta_3 > 0$ such that (compare also [6], Lemma 2.7, p. 6): for each $w = (u,v) \in S_{\delta_3}(w_0)$ there exits some $r \in (v/2, v)$ with

$$\text{(3)} \qquad \sup_{w^* \in \partial B_r((u,v))} |X(w^*) - X(w)| < \frac{\varepsilon}{3}\,.$$

So, if we choose $\delta < \min\{\delta_1, \delta_2, \delta_3\}$ and fix $\tilde{w} \in I_\delta(w_0)$, $\tilde{w} \neq w_0$, then (1)-(3) prove that $|X(w) - X(\tilde{w})| < \varepsilon$ for any $w \in S_\delta(w_0)$ and Corollary 6 follows. □

Proof of Theorem 5. For almost all (notice that S is not assumed to be complete) $\hat{w} \in I$, $\hat{w} \neq w_0$, we have $\hat{x} := X(\hat{w}) \in S$. Fix one of these points $\hat{w}$ and according to Proposition 2 fix $h = h_{\hat{x}}$, choose U small enough and set (repeated Latin indices are always to be summed from 1 to 3)

$$g_{ij}(y) := h^l_{y^i}(y) h^l_{y^j}(y) \ \text{ satisfying } \ \tilde{K}^{-1}|\xi|^2 \leqq g_{ij}(y)\xi^i\xi^j \leqq \tilde{K}|\xi|^2\,, \ \ i,j = 1\ldots 3,$$

for a real number $\tilde{K} > 1$, for all $\xi \in \mathbb{R}^3$ and for all $y \in \mathscr{B}_\rho(0)$. By the choice of $\hat{w}$ there is a real number $0 < \varepsilon_0 < |w_0 - \hat{w}|$ such that for all $\varepsilon \leqq \varepsilon_0$

$$X(S_\varepsilon(\hat{w})) \subset U \text{ and } X(w) \in S \cap U \ \text{ for a.a. } \ w \in I_\varepsilon(\hat{w}), \ \text{that is } \big(h^{-1}(X(w))\big)^3 = 0\,.$$

Now we can define for all $w \in S_{\varepsilon_0}(\hat{w})$:

$$Y(w) := h^{-1} \circ X(w) \quad \text{ and } \quad \|Y\|^2 := g_{ij}(Y) y^i y^j\,.$$

Conformality of X reads in terms of Y:

$$g_{ij}(Y) y^i_u y^j_u - g_{ij}(Y) y^i_v y^j_v = g_{ij}(Y) y^i_u y^j_v = 0.$$

Consider for fixed $\varepsilon < \varepsilon_0$ and $\delta \ll \varepsilon$ a smooth function $\lambda : B \to \mathbb{R}$,

$$\lambda(w) := \begin{cases} 1 & : \quad w \in S_{\varepsilon-\delta}(\hat{w}) \\ 0 & : \quad w \in B \sim S_\varepsilon(\hat{w}) \end{cases}$$

as well as a function $\tau \in C^1(\overline{S_\varepsilon(\hat{w})}, \mathbb{R}^3)$ satisfying $\tau^3(w) = 0$ for all $w \in I_\varepsilon(\hat{w})$. Then $\eta := \lambda\,\tau$ defines an admissible variation (see [8], p. 132) and in the same manner as outlined in [8], pp. 133–134, we see

$$\int\int_{S_\varepsilon(\hat w)} g_{ij}(Y)\nabla y^i \nabla\eta^j \, du\, dv$$

$$(4)\qquad = -\int\int_{S_\varepsilon(\hat w)} \left(\frac{1}{2}\frac{\partial g_{ik}(Y)}{\partial y^j}\nabla y^i\nabla y^k + 2H(h(Y))\sqrt{g(Y)}\,(Y_u\wedge Y_v)^j\right)\eta^j\, du\, dv$$

$$+ \int_{I_\varepsilon(\hat w)} \tilde Q^3(Y)(Y_u\wedge\eta)^3\, du.$$

Here we have $g(y) := \det(g_{ij}(y))$ and $\tilde Q^3(Y) := Q(h(Y))\cdot\big(\partial h(Y)/\partial y^1 \wedge \partial h(Y)/\partial y^2\big)$ is not necessarily vanishing for $w\in I_\varepsilon$. On the other hand,

$$g_{ij}(Y)\Delta y^i + \frac{\partial g_{ij}(Y)}{\partial y^k}\nabla y^i\nabla y^k = \frac{1}{2}\frac{\partial g_{ik}(Y)}{\partial y^j}\nabla y^i\nabla y^k + 2H(h(Y))\sqrt{g(Y)}\,(Y_u\wedge Y_v)^j$$

is known (see [3]/II p. 64) since X is assumed to be of class C^2. A partial integration proves

$$\int\int_{S_\varepsilon(\hat w)} g_{ij}(Y)\nabla y^i\nabla\eta^j\, du\, dv$$

$$= -\int\int_{S_\varepsilon(\hat w)}\left(g_{ij}(Y)\Delta y^i + \frac{\partial g_{ij}(Y)}{\partial y^k}\nabla y^i\nabla y^k\right)\eta^j\, du\, dv - \int_{I_\varepsilon(\hat w)} g_{ij}(Y)y^i_v\eta^j\, du$$

$$(5)\qquad = -\int\int_{S_\varepsilon(\hat w)}\left(\frac{1}{2}\frac{\partial g_{ik}(Y)}{\partial y^j}\nabla y^i\nabla y^k + 2H(h(Y))\sqrt{g(Y)}\,(Y_u\wedge Y_v)^j\right)\eta^j\, du\, dv$$

$$- \int_{I_\varepsilon(\hat w)} g_{ij}(Y)y^i_v\eta^j\, du\,.$$

Thus (4) and (5) imply

$$(6)\qquad - \int_{I_\varepsilon(\hat w)} g_{ij}(Y)y^i_v\eta^j\, du = \int_{I_\varepsilon(\hat w)} \tilde Q^3(Y)(Y_u\wedge\eta)^3\, du\,.$$

Because of $X\in C^2(\overline{S_\varepsilon(\hat w)},\mathbb{R}^3)$ we now may choose $\tau(w) := \big(-g_{2m}(Y)y^m_u, g_{1m}(Y)y^m_u, 0\big)$, giving $(Y_u\wedge\tau)^3 = g_{ij}(Y)y^i_u y^j_u = \|Y_u\|^2 = \|Y_v\|^2$ on $I_\varepsilon(\hat w)$. Since no derivatives of λ are involved in (6), it is allowed to pass to the limit $\delta\to 0$, that is λ can be replaced by the characteristic function of $S_\varepsilon(\hat w)$, then we may pass to the limit $\varepsilon\to 0$ and finally conclude

$$(7)\qquad - g_{ij}(Y)y^i_v\tau^j{}_{|\hat w} = \tilde Q^3(Y)\|Y_v\|^2{}_{|\hat w} \quad \text{for almost all } \hat w\in I,\ \ \hat w\neq w_0\,.$$

Since $\hat x$ was chosen as the reference point of the diffeomorphism $h = h_{\hat x}$ we have

$$g_{ij}(Y(\hat w)) = \delta_{ij}\,, \qquad \tau(\hat w) = (-y^2_u(\hat w), y^1_u(\hat w), 0)$$

$$\|Y_v(\hat w)\|^2 = |Y_v(\hat w)|^2 \quad\text{and}\quad \tilde Q^3(Y(\hat w)) = Q(\hat x)\cdot N(\hat x)\,.$$

Together with (7) this proves

$$(8)\qquad - Y_v(\hat w)\cdot\big(-y^2_u(\hat w), y^1_u(\hat w), 0\big) = Q(\hat x)\cdot N(\hat x)\,|Y_v(\hat w)|^2\,.$$

Using (8) we now want to obtain an upper bound of $|X_v|$ in terms of q. To do this, we observe that $e_1 := Y_u(\hat w)/|Y_u(\hat w)|$, $e_2 := \tau(\hat w)/|\tau(\hat w)|$ and $e_3 := (0,0,1)$ define by construction an orthonormal base of $\mathbb{R}^3$. Conformality yields

$$Y_v^\top(\hat w) := (Y_v(\hat w)\cdot e_1)\,e_1 + (Y_v(\hat w)\cdot e_2)\,e_2 = (Y_v(\hat w)\cdot e_2)\,e_2\,.$$

So $|Q(\hat{x}) \cdot N(\hat{x})|^2 < q^2 < 1$ and $|Y_v^\top(\hat{w})|^2 = |Y_v(\hat{w})|^2 - |y_v^3(\hat{w})|^2$ prove

$$|Y_v(\hat{w})|^2 < \frac{1}{1-q^2}\, |y_v^3(\hat{w})|^2 .$$

To get an estimate for $X_v(\hat{w})$, we recall $f(0,0) = 0$ and $Df(0,0) = (0,0)$ and obtain

$$X_v(\hat{w}) = R_{\hat{x}}^{-1}\big(y_v^1(\hat{w}),\, y_v^2(\hat{w}),\, y_v^3(\hat{w})\big)\,,$$

that is $|Y_v(\hat{w})| = |X_v(\hat{w})|$. Furthermore, the setting $B(x) := R_{\hat{x}}(x - a_{\hat{x}})$ yields

$$Y = \left(B^1(X), B^2(X), \frac{B^3(X)}{1 + f(B^1(X), B^2(X))} - 1\right).$$

Using $\big(B^1(X(\hat{w})), B^2(X(\hat{w}))\big) = \big(y^1(\hat{w}), y^2(\hat{w})\big) = (0,0)$, the assumption on f also proves

$$y_v^3(\hat{w}) = N(\hat{x}) \cdot X_v(\hat{w})\,.$$

Summarizing the results we arrive at

$$\text{(9)} \qquad |X_v(\hat{w})| \leqq \left(\frac{1}{1-q^2}\right)^{\frac{1}{2}} |N(\hat{x}) \cdot X_v(\hat{w})|\,.$$

Notice that (9) holds for all $\hat{w} \in I$, $\hat{w} \neq w_0$, satisfying $\hat{x} \in S$, that is for almost all $\hat{w} \in I$. Now we substitute the fixed $N(\hat{x})$ by its extension $\tilde{N}(x)$ and recall the assumption

$$\tilde{N}\big(X(\hat{w})\big) \cdot X_v(\hat{w}) \geqq 0\,,$$

that is: for almost all $w \in I$

$$\text{(10)} \qquad |X_v(w)| \leqq \left(\frac{1}{1-q^2}\right)^{\frac{1}{2}} \tilde{N}(X(w)) \cdot X_v(w)$$

holds true. The fixed sign on the right hand side of (10) will give the result: on one hand for any real number $0 < \varepsilon < 1$

$$\text{(11)} \qquad \begin{aligned} &\left| \int\!\!\int_{B \sim S_\varepsilon(w_0)} \big\{(\tilde{N}(X) \cdot X_u)_u + (\tilde{N}(X) \cdot X_v)_v\big\}\, du\, dv \right| \\ &\leqq \int\!\!\int_{B \sim S_\varepsilon(w_0)} |D\tilde{N}(X)|\, |\nabla X|^2\, du\, dv + \int\!\!\int_{B \sim S_\varepsilon(w_0)} |\tilde{N}(X)|\, |\Delta X|\, du\, dv \\ &= \int\!\!\int_{B \sim S_\varepsilon(w_0)} |D\tilde{N}(X)|\, |\nabla X|^2\, du\, dv + \int\!\!\int_{B \sim S_\varepsilon(w_0)} |\tilde{N}(X)|\, |2H(X)(X_u \wedge X_v)|\, du\, dv \\ &\leqq c \int\!\!\int_{B \sim S_\varepsilon(w_0)} |\nabla X|^2\, du\, dv < c_1, \end{aligned}$$

where the constant c_1 does not depend on ε. On the other hand, since $X \in C^2(\bar{B} \sim \{w_0\}, \mathbb{R}^3)$ and since the boundary of $B \sim S_\varepsilon(w_0)$ is piecewise smooth, a partial integration can be applied to obtain

$$\text{(12)} \qquad \begin{aligned} \int\!\!\int_{B \sim S_\varepsilon(w_0)} (\tilde{N}(X) \cdot X_u)_u\, du\, dv &= \int_{\partial(B \sim S_\varepsilon(w_0))} \tilde{N}(X) \cdot X_u\, \nu^1\, d\mathscr{H}^1 \\ &= \int_C \tilde{N}(X) \cdot X_u\, \nu^1\, d\mathscr{H}^1 + \int_{C_\varepsilon(w_0)} \tilde{N}(X) \cdot X_u\, \nu^1\, d\mathscr{H}^1\,, \end{aligned}$$

where ν denotes the outward unit normal of $\partial(B \sim S_\varepsilon(w_0))$. In addition we have

$$\int\limits_{B\sim S_\varepsilon(w_0)}\int (\tilde{N}(X)\cdot X_v)_v\,du\,dv = \int\limits_C \tilde{N}(X)\cdot X_v\,\nu^2\,d\mathscr{H}^1 + \int\limits_{C_\varepsilon(w_0)} \tilde{N}(X)\cdot X_v\,\nu^2\,d\mathscr{H}^1 - \int\limits_{I\sim I_\varepsilon(w_0)} \tilde{N}(X)\cdot X_v\,du\,. \tag{13}$$

Finally observe that by Hölder's inequality and by conformality (see (8) of [3]/II, p. 50)

$$\begin{aligned}\int\limits_{C_\varepsilon(w_0)} |\nabla X|\,d\mathscr{H}^1 &\leqq \varepsilon\sqrt{\pi}\left(\int\limits_0^\pi |\nabla X|^2(w_0+\varepsilon e^{i\theta})\,d\theta\right)^{\frac{1}{2}}\\ &\leqq \sqrt{2\pi}\left(\int\limits_0^\pi |X_\theta|^2\,d\theta\right)^{\frac{1}{2}},\end{aligned} \tag{14}$$

i.e. in the same manner as proving the Courant-Lebesgue Lemma (see [7], Lemma 2, p. 393) we find a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, such that

$$\int\limits_{C_{\varepsilon_n}(w_0)} |\nabla X|\,d\mathscr{H}^1 \to 0 \quad \text{as } n \to \infty\,.$$

Summarizing the results, (11) – (14) give

$$\left|\lim_{\varepsilon\to 0}\int\limits_{I\sim I_\varepsilon(w_0)} \tilde{N}(X)\cdot X_v\,du\right| = \lim_{\varepsilon\to 0}\int\limits_{I\sim I_\varepsilon(w_0)} \tilde{N}(X)\cdot X_v\,du < \infty$$

and (10) proves the theorem. □

Remark 8.

- The contact angle $\alpha \in [0, \pi/2]$ is proved in (8) to satisfy $\cos(\alpha) = |Q(\hat{x})\cdot N(\hat{x})|$.
- Due to Dziuk (see [5]) (compare also [3]/I, p. 411) length estimates in the (smooth) perpendicular case are possible without assumption (ii.) of Theorem 5. In this case for all $w \in I$ we have $|X_v(w)| = |D_\nu\rho(X(w))|$, where ρ denotes the orientated distance with respect to the supporting surface. Here, this relation is not true and the above assumption can only be dropped in the case of plane supporting surfaces, where $|x_v^3(w)| < \varphi_v(w)$ for almost all $w \in I$ can be proved by Dziuks arguments. To do this, we have to set $\varphi(w) := \left(\delta^2 v^2 + (x^3(w))^2\right)^{1/2}$ for a fixed real number $\delta > 0$.

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