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Partial regularity for variational integrals with (s, μ, q)–Growth

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Abstract. We introduce integrands $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ of (s, μ, q)–type, which are, roughly speaking, of lower (upper) growth rate $s \geq 1$ ($q > 1$) satisfying in addition $D^2 f(Z) \geq \lambda\Big(1+|Z|^2\Big)^{-\mu/2}$ for some $\mu \in \mathbb{R}$. Then, if $q < 2-\mu+s\frac{2}{n}$, we prove partial C^1–regularity of local minimizers $u \in W^1_{1,loc}(\Omega, \mathbb{R}^N)$ by the way including integrands f being controlled by some N–function and also integrands of anisotropic power growth. Moreover, we extend the known results up to a certain limit and present examples which are not covered by the standard theory.

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1 Introduction

In this paper we study the problem of partial C^1–regularity for local minimizers $u \in W^1_{1,loc}(\Omega, \mathbb{R}^N)$ of strictly convex variational integrals

$$J(u) = \int_\Omega f(Du)\, dx \tag{1.1}$$

under rather general and also non–standard growth conditions. Here Ω is some domain in Euclidean space $\mathbb{R}^n$, $n \geq 2$, and we assume that the integrand f: $\mathbb{R}^{nN} \to [0, \infty)$ is a function of class C^2 whose second derivative $D^2 f(Z)$ has to satisfy certain coercivity conditions to be specified below. Thus, we do not touch the quasiconvex case (compare e. g. [EV], [FH], [EG1], [AF1], [AF2], [CFM]) and before presenting our results, we briefly summarize the conditions under which partial regularity is available in the framework of strong convexity. Roughly speaking, we can consider three different cases:

A. Power growth

For some number $m > 1$ and with constants λ, $\Lambda > 0$ the integrand f satisfies

$$\lambda\big(|Z|^m - 1\big) \leq f(Z) \leq \Lambda\big(|Z|^m + 1\big) \quad \text{for all } Z \in \mathbb{R}^{nN}, \tag{1.2}$$

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in particular, f has the same growth rate from above and from below. Then, if also $D^2 f(Z) > 0$ holds for any matrix Z, Anzellotti and Giaquinta proved in [AG] that for any local minimizer $u \in W^1_{1,loc}(\Omega, \mathbb{R}^N)$ of (1.1) there is an open set Ω_0 such that $|\Omega \sim \Omega_0| = 0$, i. e. the singular set has measure zero, and $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$. We emphasize that the paper [AG] also includes the case of linear growth ($m = 1$) with corresponding local minimizers in the space $BV_{loc}(\Omega, \mathbb{R}^N)$. Moreover, the reader will find there further comments on earlier results obtained under condition (1.2).

B. Growth conditions involving N–functions

The model $f(Z) = |Z| \ln(1 + |Z|)$ serves as a typical example for integrands f not satisfying (1.2) for any power $m \geq 1$. Generally speaking, the quantity $|Z|^m$ occuring in (1.2) is now replaced by $A(|Z|)$ for some arbitrary N–function A: $[0, \infty) \to [0, \infty)$ satisfying a Δ_2–condition. If we add an appropriate ellipticity and growth condition on $D^2 f(Z)$, then in [FO] partial regularity was shown to hold up to a certain dimension n. The particular class of integrands f with logarithmic structure (i. e. f is C^2–close to $|Z| \ln(1 + |Z|)$) was studied first in [FS] with the result that minimizers are partially C^1 provided that $n \leq 4$. Later on Esposito and Mingione [EM2] removed the restriction on n, moreover, Mingione and Siepe [MS] proved for $f(Z) = |Z| \ln(1 + |Z|)$ in fact that the singular set is empty which of course can not be expected in the general case. We would like to remark that some extensions of the results obtained in [MS] can be found in [FM].

C. Anisotropic power growth

was introduced by Marcellini [M1]–[M4] as a natural extension of (1.2) where now f is allowed to have different growth rates from above and from below, precisely: with numbers $1 < p < q$ we have

$$\lambda \big(|Z|^p - 1\big) \leq f(Z) \leq \Lambda \big(|Z|^q + 1\big) \quad \text{for all } Z \in \mathbb{R}^{nN} \tag{1.3}$$

(plus corresponding conditions involving $D^2 f(Z)$, for example $D^2 f(Z) \geq \lambda(1 + |Z|^2)^{(p-2)/2}$). Condition (1.3) is motivated by the integral ($n = 2$, $p \geq 2$)

$$J(u) = \int_\Omega \Big\{ \big(1 + |\partial_1 u|^2\big)^{\frac{p}{2}} + \big(1 + |\partial_2 u|^2\big)^{\frac{q}{2}} \Big\}\, dx$$

where the derivatives occur with different powers. It should be noted that B. is not a subcase of C. For formal reasons this should be obvious by considering energies of logarithmic type. On the other hand, partial regularity in the anisotropic case has been studied by Acerbi and Fusco [AF4] and later by Passarelli Di Napoli and Siepe [PS] under quite restrictive assumptions: in [PS] they impose the condition

$$2 \leq p < q < \min\Big\{p + 1, \frac{pn}{n-1}\Big\} \tag{1.4}$$

thus excluding any subquadratic growth.

The purpose of our paper is twofold: first, we would like to give a unified approach including all the different cases. Secondly, we present certain improvements by extending for example the results of [FO] (see Remark 5.) below) and by constructing integrands to which the results mentioned in A. – C. do not apply but which can be handled with the help of our techniques. We consider integrands f of (s, μ, q)–growth which are defined as follows: let F: $[0, \infty) \to [0, \infty)$ denote a continuous function such that for some $s \geq 1$ we have

$$\lim_{t\to\infty} \frac{F(t)}{t} = \infty\,, \qquad F(t) \geq c_0\, t^s \ \text{ for large values of } t\,. \tag{1.5}$$

The integrand f is a non–negative C^2–function such that for all $Z, Y \in \mathbb{R}^{nN}$

$$c_1\, F(|Z|) \leq f(Z)\,; \tag{1.6}$$

$$\lambda \big(1 + |Z|^2\big)^{-\frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y, Y) \leq \Lambda \big(1 + |Z|^2\big)^{\frac{q-2}{2}} |Y|^2\,. \tag{1.7}$$

Here $\mu \in \mathbb{R}$, $q > 1$ and c_0, c_1, λ, Λ denote positive constants. In addition, we require the (s, μ, q)–condition, i. e.

$$q < 2 - \mu + s\,\frac{2}{n}\,. \tag{1.8}$$

Note that on account of $q > 1$ (1.8) gives the upper bound

$$\mu < 1 + \frac{2}{n}\,. \tag{1.9}$$

In the case that f is C^2 close to $|Z| \ln(1+|Z|)$ we can take $s = 1$, $\mu = 1$, $q = 1+\varepsilon$ (for any $\varepsilon > 0$) and $F(t) = t \ln(1 + t)$, hence (1.8) holds. Now our main result reads as follows:

Theorem 1.1. *Let conditions (1.5)–(1.8) hold and let $u \in W^1_{1,loc}\big(\Omega, \mathbb{R}^N\big)$ denote a local minimizer of (1.1), i. e. $f(Du) \in L^1_{loc}(\Omega)$ and*

$$\int_{spt(u-v)} f(Du)\, dx \leq \int_{spt(u-v)} f(Dv)\, dx$$

for any $v \in W^1_{1,loc}\big(\Omega, \mathbb{R}^N\big)$ such that $spt(u - v) \Subset \Omega$. Then there is an open subset Ω_0 of Ω of full measure, i. e. $|\Omega \sim \Omega_0| = 0$, such that $u \in C^{1,\alpha}\big(\Omega_0, \mathbb{R}^N\big)$ for any $0 < \alpha < 1$.

Let us briefly comment on our conditions:

1.) The (s, μ, q)–condition was introduced in [BFM] where full regularity (i. e. $\Omega_0 = \Omega$) was established for the scalar case under exactly the same assumptions as stated here. The key ingredient in [BFM] is a local gradient bound in L^∞ which follows via Moser iteration technique or from DeGiorgi type arguments. In the vectorial setting $N > 1$ such a bound can not be expected to hold true, thus we could not benefit too much from the arguments in [BFM]. However, as it is shown in [BFM] for the scalar case, it is easy to check that the result of Theorem 1.1

continues to hold if we replace (1.8) by the weaker condition (note that $s \leq q$ on account of 2.))

$$(1.8^*) \qquad q \,<\, (2-\mu)\,\frac{n}{n-2}$$

provided we add the balancing condition (introduced in [FO])

$$\text{(B)} \qquad |D^2 f(Z)|\,|Z|^2 \,\leq\, const\big(f(Z)+1\big)\,.$$

Note that (1.8*) makes sense only in the case that $n \geq 3$ and then (1.8*) clearly implies (1.9). For the twodimensional case we have to replace (1.8*) by the requirement that $\mu < 2$. We leave the details of the proof of this variant of Theorem 1.1 to the reader.

2.) (1.5) together with the second inequality in (1.7) implies (see [AF3], Lemma 2.1, if $q < 2$) the bound $s \leq q$.

3.) In the case $\mu \geq 1$ we have $2-\mu \leq s$. If $\mu \leq 0$ we clearly may assume that $2-\mu \leq s$ since $2-\mu$ is a lower bound for the growth of f, hence we can replace s by $\max\{s, 2-\mu\}$. For $0 < \mu < 1$ this inequality is also reasonable: from [AF3], Lemma 2.1, and the first inequality in (1.7) we get again that $2-\mu$ is a lower growth rate for f. Comparing this to (1.5) we may directly assume that $2-\mu \leq s$. In particular we have by 2.) that $2-\mu \leq q$.

4.) Suppose we are given numbers $1 < p < q$ such that for all $Z \in \mathbb{R}^{nN}$

$$\begin{aligned} a\big(|Z|^p - 1\big) \quad &\leq \quad f(Z) \quad \leq \quad b\big(|Z|^q + 1\big)\,, \\ \lambda\big(1+|Z|^2\big)^{\frac{p-2}{2}} \,&\leq D^2 f(Z) \leq \varLambda\big(1+|Z|^2\big)^{\frac{q-2}{2}}\,. \end{aligned}$$

Then we may let $\mu = 2-p$, $s = p$, and we deduce partial regularity if

$$q \,<\, p\,\frac{n+2}{n}$$

which is much weaker than (1.4). (Note: [PS] do not need an upper bound for $D^2 f(Z)$.)

5.) In [FO] partial regularity was established under the assumptions (1.5)–(1.7), $q \leq 2$, $\mu < 4/n$ together with condition (B) (see Remark 1.)). Clearly $q \leq 2$ and $\mu < 4/n$ imply (1.8*) so that we have included the result of [FO] on account of the first remark. But, what is even more important, Theorem 1.1 does not need any balancing condition of the form (B), the regularity of local minimizers follows from $\mu < 2 - q + s2/n$ which for q close to 1 and large values of n is a much weaker hypothesis than $\mu < 4/n$.

6.) Let us now sketch an example of an integrand of (s, μ, q)–growth which is not of type A., B. or C. For simplicity we assume $s \geq 2$, for a corresponding subquadratic example we refer to [BFM].

As shown in [BFM], Section 3, there exists for each $k \in \mathbb{N}$ and $t \geq 2$ a function $\Phi_t^k \colon \mathbb{R}^k \to [0, \infty)$ such that

$$\Phi_t^k(\eta) \,\geq\, a\,|\eta|^t \quad \text{for large values of } \eta \in \mathbb{R}^k\,, \tag{1.10}$$

$$\text{and} \quad 0 \,\leq\, D^2\Phi_t^k(\eta)(\tau,\tau) \,\leq\, b\big(1+|\eta|^2\big)^{\frac{t-2}{2}}\,, \quad \eta\,,\,\tau \in \mathbb{R}^k\,, \tag{1.11}$$

hold with positive constants a, b. Roughly speaking, the function Φ_t^k is constructed by first considering $(1+|Z|^2)^{t/2}$, then redefining equidistant parts to be linear and finally smoothing the result of the first two steps. By definition of Φ_t^k, the exponents in (1.10) and (1.11) can not be improved, moreover, due to the degeneracy of $D^2\Phi_t^k$, the lower bound of (1.11) is the best possible. Next consider numbers s, μ, q such that $2 \leq s < q$ and $2 - \mu < s$. Again, according to [BFM], Section 3, we can construct a function Φ: $\mathbb{R}^{nN} \to [0, \infty)$ satisfying

$$D^2\Phi(Z)(\tau,\tau) \,\geq\, c\big(1+|Z|^2\big)^{-\frac{\mu}{2}}\,|\tau|^2\,.$$

For instance, we may choose $\Phi(Z) = \varphi(|Z|)$ were φ is defined via

$$\varphi(r) \,=\, \int_0^r\int_0^s \big(1+|t|^2\big)^{-\frac{\mu}{2}}\,dt\,ds\,, \qquad r \in \mathbb{R}_0^+\,.$$

In the case $\mu \geq 1$ Φ is of lower growth than any power $|Z|^{1+\vartheta}$, $\vartheta > 0$, for $\mu < 1$ we get $\Phi(Z) \leq d\,(1+|Z|^2)^{(2-\mu)/2}$, and it is not possible to improve the exponents. We then define $\big(z = (z_1^i, \dots z_n^i)_{1\leq i\leq N} \in \mathbb{R}^{nN}\big)$

$$f(Z) \,=\, \Phi(Z) + \Phi_s^N(z_1) + \Phi_q^{(n-1)N}(z_2, \dots, z_n)\,.$$

Then (1.5)–(1.7) hold and if we also impose (1.8) then regularity of local minimizers follows which can not be deduced from the results stated in A. – C.

Our paper is organized as follows: in Section 2 we introduce a suitable regularization v_ε of our local minimizer u which converges weakly and in energy to u on compact subsets. Section 3 investigates higher weak differentiability of v_ε. As a consequence we obtain uniform local estimates in L^q for Dv_ε which allow us to give local apriori bounds for $\|Du\|_{L^q}$. Moreover, we prove certain Caccioppoli–type inequalities. Finally, Section 4 contains the proof of Theorem 1.1 via blow-up arguments by considering the cases $q \geq 2$ and $1 < q < 2$ more or less separately.

2 Approximation and some preliminary results

Let ε denote a sequence of positive real numbers converging to zero, where we do not care about relabelling if necessary. Then we define u_ε as the ε–mollification of u through φ_ε, where $\{\varphi_t\}_{t>0}$ is a family of smooth mollifiers. Moreover, let us fix $R > 0$ and $x_0 \in \Omega$. Letting $B_r := B_r(x_0)$ we assume $B_{2R} \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. For $\delta \in (0,1]$ we define

$$f_\delta(Z) := f(Z) + \delta\big(1 + |Z|^2\big)^{\frac{q}{2}}$$

and denote by $v_{\varepsilon,\delta}$ the unique solution of the variational problem

$$J_\delta(w) := \int_{B_{2R}} f_\delta(Dw)\,dx \rightsquigarrow \min \quad \text{in } u_\varepsilon + \overset{\circ}{W}{}^1_q\big(B_{2R}, \mathbb{R}^N\big)\,.$$

Lemma 2.1. *If ε and δ are connected via*

$$\delta = \delta(\varepsilon) := \frac{1}{1 + \varepsilon^{-1} + \|Du_\varepsilon\|^{2q}_{L^q(B_{2R})}}$$

and if $v_\varepsilon = v_{\varepsilon,\delta(\varepsilon)}$, $f_\varepsilon = f_{\delta(\varepsilon)}$, then we have as $\varepsilon \to 0$:

$$(i.) \qquad v_\varepsilon \rightharpoondown u \quad \textit{in } W^1_1\big(B_{2R}, \mathbb{R}^N\big)\,,$$

$$(ii.) \qquad \delta(\varepsilon)\int_{B_{2R}} \big(1 + |Dv_\varepsilon|^2\big)^{\frac{q}{2}}\,dx \to 0\,,$$

$$(iii.) \qquad \int_{B_{2R}} f(Dv_\varepsilon)\,dx \to \int_{B_{2R}} f(Du)\,dx\,,$$

$$(iv.) \qquad \int_{B_{2R}} f_\varepsilon(Dv_\varepsilon)\,dx \to \int_{B_{2R}} f(Du)\,dx\,.$$

Proof of Lemma 2.1. We argue as in [BFM], conclusion of Theorem 1.1, i. e. we use the minimality of v_ε as well as Jensen's inequality to get

$$\begin{aligned} \int_{B_{2R}} F(|Dv_\varepsilon|)\,dx \le \int_{B_{2R}} f(Dv_\varepsilon)\,dx &\le \int_{B_{2R}} f_\varepsilon(Du_\varepsilon)\,dx \\ &\le \int_{B_{2R}} f(Du)\,dx + o(\varepsilon)\,, \end{aligned} \tag{2.1}$$

i. e. we may suppose that

$$v_\varepsilon \rightharpoondown: v \quad \text{weakly in } W^1_1\big(B_{2R}, \mathbb{R}^N\big)\,.$$

Passing to the limit $\varepsilon \to 0$, lower semicontinuity implies

$$\int_{B_{2R}} f(Dv)\,dx \le \liminf_{\varepsilon\to 0} \int_{B_{2R}} f(Dv_\varepsilon)\,dx \le \int_{B_{2R}} f(Du)\,dx\,.$$

Finally, the minimality of u together with strict convexity of f (see (1.7)) ensure that $v = u$, thus, with (2.1) the lemma is proved. □

In the following δ is always assumed to be chosen according to Lemma 2.1. To finish this section, some well known properties of v_ε are summarized. Part a.) of the following lemma is proved in [AF3], Proposition 2.4 and Lemma 2.5, for the second part we refer the reader to [GM], especially formula (3.3), and to [CA] (compare Theorem 1.1).

Lemma 2.2.

a.) In the case $q < 2$ the approximative solution satisfies:
  (i.) $v_\varepsilon \in W^2_{q,loc}\big(B_{2R}, \mathbb{R}^N\big)$,
  (ii.) $Df_\varepsilon(Dv_\varepsilon) \in W^1_{2,loc}\big(B_{2R}, \mathbb{R}^{nN}\big)$,
  (iii.) $\big(1 + |Dv_\varepsilon|^2\big)^{\frac{q-2}{4}} Dv_\varepsilon \in W^1_{2,loc}\big(B_{2R}, \mathbb{R}^{nN}\big)$,
  (iv.) $\big|D^2 v_\varepsilon \mathbf{1}_{[|v_\varepsilon| \leq M]}\big| \in L^2_{loc}(B_{2R})$ *for all* $M > 0$.

b.) In the case $q \geq 2$ we have
  (i.) $v_\varepsilon \in W^2_{2,loc}\big(B_{2R}, \mathbb{R}^N\big)$,
  (ii.) $Df_\varepsilon(Dv_\varepsilon) \in W^1_{q/(q-1),loc}\big(B_{2R}, \mathbb{R}^{nN}\big)$,
  (iii.) $\big(1 + |Dv_\varepsilon|^2\big)^{\frac{q}{4}} \in W^1_{2,loc}(B_{2R})$,
  (iv.) $\big(1 + |Dv_\varepsilon|^2\big)^{\frac{q-2}{4}} Dv_\varepsilon \in W^1_{2,loc}\big(B_{2R}, \mathbb{R}^{nN}\big)$.

3 Apriori L^q–estimates and Caccioppoli–type inequalities

In this section we are going to prove the two main ingredients which will enable us to perform the blow–up procedure in Section 4. The starting point is the following Caccioppoli–type inequality for the approximative solutions.

Lemma 3.1. *There is a real number $c > 0$ such that for all $\eta \in C^1_0(B_{2R})$, $0 \leq \eta \leq 1$, and for all $Q \in \mathbb{R}^{nN}$*

$$\int_{B_{2R}} \eta^2 D^2 f_\varepsilon(Dv_\varepsilon)\big(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon\big)\, dx \\ \leq c\|D\eta\|^2_\infty \int_{B_{2R}\cap spt D\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big|\, |Dv_\varepsilon - Q|^2\, dx\,,$$

where summation with respect to $s = 1, \dots, n$ is always assumed in the following. In particular, for all $Q \in \mathbb{R}^{nN}$

$$\int_{B_{2R}} \eta^2 \big(1 + |Dv_\varepsilon|^2\big)^{-\frac{\mu}{2}} |D^2 v_\varepsilon|^2\, dx \\ \leq c\|D\eta\|^2_\infty \int_{B_{2R}\cap spt D\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big|\, |Dv_\varepsilon - Q|^2\, dx\,.$$

Proof of Lemma 3.1. First of all we recall that v_ε solves the regularized problem, i. e.

$$\int_{B_{2R}} Df_\varepsilon(Dv_\varepsilon) : D\varphi\, dx = 0 \quad \text{for all } \varphi \in \overset{\circ}{W}{}^1_q\big(B_{2R}, \mathbb{R}^N\big)\,. \tag{3.1}$$

Next, denote by $e_s \in \mathbb{R}^n$ the unit coordinate vector in x_s–direction and let for a function g on Ω

$$\Delta_h g(x) = \Delta^s_h g(x) = \frac{g(x + h e_s) - g(x)}{h}\,, \quad h \in \mathbb{R}\,,$$

denote the difference quotient of g at x in the direction e_s. Then, given $Q \in \mathbb{R}^{nN}$, $\varphi = \Delta_{-h}(\eta^2\, \Delta_h\, (v_\varepsilon - Qx))$, $\eta \in C_0^\infty(B_{2R})$, is admissible in (3.1) and by a "partial integration" we obtain

$$\begin{aligned}\int_{B_{2R}} &\eta^2\, \Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) : D\Delta_h v_\varepsilon\, dx \\ &= -2 \int_{B_{2R}} \eta\, \Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) : D\eta \otimes \Delta_h(v_\varepsilon - Qx)\, dx\,.\end{aligned} \tag{3.2}$$

Consider now the case $q \geq 2$: by Lemma 2.2 and by (1.8) Dv_ε is known to be of class L^r_{loc} for some $r > q$ and if F_h denotes the integrand on the right–hand side of (3.2), then the existence of a real number $c(D\eta)$, independent of h, follows such that

$$|F_h| \,\leq\, c\Big\{\big|\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big)\big|^{l_1} + |\Delta_h v_\varepsilon|^{l_2}\Big\} \;\text{ for some }\; l_1 < \frac{q}{q-1}\,,\;\; q < l_2 < r\,,$$

thus, equiintegrability of F_h in the sense of Vitali's convergence theorem is ensured by Lemma 2.2, b.), (ii.), and passing to the limit $h \to 0$ the right–hand side of (3.2) tends to

$$-2 \int_{B_{2R}} \eta\, \partial_s\big(Df_\varepsilon(Dv_\varepsilon)\big) : D\eta \otimes (\partial_s v_\varepsilon - Q_s)\, dx \,\in\, (-\infty, +\infty)\,. \tag{3.3}$$

For the left–hand side of (3.2) we observe

$$\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) \,=\, \int_0^1 D^2 f_\varepsilon\big(Dv_\varepsilon + t\, h\, \Delta_h Dv_\varepsilon\big)\big(\Delta_h Dv_\varepsilon, \cdot\big)\, dt$$

and get using (3.3), Fatou's lemma and Young's inequality

$$\begin{aligned}\int_{B_{2R}} &\eta^2\, D^2 f_\varepsilon(Dv_\varepsilon)\big(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon\big)\, dx \\ &\leq \int_{B_{2R}} \eta^2\, \liminf_{h\to 0} \int_0^1 D^2 f_\varepsilon\big(Dv_\varepsilon + t\, h\, \Delta_h Dv_\varepsilon\big)\big(\Delta_h Dv_\varepsilon, \Delta_h Dv_\varepsilon\big)\, dt\, dx \\ &\leq \frac{1}{2} \int_{B_{2R}} \eta^2\, D^2 f_\varepsilon(Dv_\varepsilon)\big(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon\big)\, dx \\ &\qquad + c\, \|D\eta\|_\infty^2 \int_{B_{2R}\cap spt D\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big|\, |Dv_\varepsilon - Q|^2\, dx\,,\end{aligned}$$

i. e. the lemma is proved for $q \geq 2$. If $q < 2$ then we modify the truncation arguments given in [EM1]. To this purpose fix $M \gg 1$ and let for $t \geq 0$

$$\psi(t) \,:=\, \begin{cases} 0\,,\; t \geq M \\ 1\,,\; t \leq M/2 \end{cases}\,, \qquad |\psi'(t)| \leq 4/M\,.$$

Given η, Q as above, then, by Lemma 2.2, a.), (iv.), and by [EM1], Lemma 1, $\varphi = \Delta_{-h}\big(\eta^2\,\partial_s(v_\varepsilon - Qx)\,\psi(|Dv_\varepsilon|)\big)$ is seen to be admissible, hence

$$\begin{aligned} &\int_{B_{2R}} \eta^2\,\psi\,\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) : D\partial_s v_\varepsilon\,dx \\ &\quad = -2\int_{B_{2R}} \eta\,\psi\,\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) : D\eta\otimes\partial_s(v_\varepsilon - Qx)\,dx \\ &\qquad -2\int_{B_{2R}} \eta^2\,\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) : D\psi\otimes\partial_s(v_\varepsilon - Qx)\,dx\,. \end{aligned} \tag{3.4}$$

By the definition of ψ and again on account of Lemma 2.2, a.), (iv.), both integrals on the right–hand side of (3.4) can be written as

$$\begin{aligned} &\int_{spt\,\eta} \Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big) : \xi(x)\,dx \\ &\quad \le \int_{spt\,\eta} \big|\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big)\big|^2\,dx + \|\xi\|^2_{L^2(B_{2R},\mathbb{R}^{nN})}\,, \end{aligned} \tag{3.5}$$

for a suitable function ξ of class L^2. Since Lemma 2.2, a.), (ii.), shows $\partial_s(Df_\varepsilon(Dv_\varepsilon))$ to be of class L^2_{loc}, strong convergence of difference quotients (see [MO], Theorem 3.6.8 (b)) implies passing to the limit $h\to 0$

$$\begin{aligned} &\big|\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big)\big|^2 \to \big|\partial_s\big(Df_\varepsilon(Dv_\varepsilon)\big)\big|^2 \quad \text{almost everywhere}\,, \\ &\int_{spt\,\eta} \big|\Delta_h\big(Df_\varepsilon(Dv_\varepsilon)\big)\big|^2\,dx \to \int_{spt\,\eta} \big|\partial_s\big(Df_\varepsilon(Dv_\varepsilon)\big)\big|^2\,dx\,. \end{aligned} \tag{3.6}$$

With (3.5) and (3.6) the variant of the dominated convergence theorem, given for example in [EG2], Theorem 4, p. 21, is applicable (note that almost everywhere convergence in (3.6) is needed for a proof of this variant). Thus, we may pass to the limit $h\to 0$ on the right–hand side of (3.4). The left–hand side is handled as in the case $q\ge 2$ and summarizing the results we arrive at (again after applying Young's inequality to the bilinear form $D^2 f_\varepsilon(Dv_\varepsilon)$)

$$\begin{aligned} &\int_{B_{2R}} \eta^2\,\psi\,D^2 f_\varepsilon(Dv_\varepsilon)\big(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon\big)\,dx \\ &\quad \le \frac{1}{2}\int_{B_{2R}} \eta^2\,\psi\,D^2 f_\varepsilon(Dv_\varepsilon)\big(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon\big)\,dx \\ &\qquad + c\,\|D\eta\|^2_\infty \int_{sptD\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big|\,|Dv_\varepsilon - Q|^2\,dx \\ &\qquad + c\int_{sptD\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big|\,|D^2 v_\varepsilon|^2\,\mathbf{1}_{[M/2\le|Dv_\varepsilon|\le M]}\,dx\,. \end{aligned}$$

Here we use the fact that $|Dv_\varepsilon - Q| < 2\,M$ on $[M/2 \le |Dv_\varepsilon| \le M]$ for M sufficiently large and that $D\big(\psi(|Dv_\varepsilon|)\big) \le c\,|D^2 v_\varepsilon|/M$. Before passing to the limit $M \to \infty$ we use Proposition 2.4 of [AF3] once again, i. e. we observe the estimate

$$\int_{B_t} \big(1+|Dv_\varepsilon|^2\big)^{\frac{q-2}{2}} |D^2 v_\varepsilon|^2\, dx \,\le\, c(t,t') \int_{B_{t'}} \big(1+|Dv_\varepsilon|^2\big)^{\frac{q}{2}}\, dx$$

being valid for all $0 < t < t' < 2R$. Recalling the growth of $|D^2 f(Dv_\varepsilon)|$ we immediately get that

$$\int_{spt D\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big|\, |D^2 v_\varepsilon|^2\, \mathbf{1}_{[M/2\le |Dv_\varepsilon|\le M]}\, dx \;\overset{M\to\infty}{\longrightarrow}\; 0$$

on account of $\mathbf{1}_{spt\, D\psi} \to 0$ as $M \to \infty$ and the claim of the lemma follows. □

Besides Lemma 3.1 the following technical proposition is needed to prove uniform L^q–estimates for Dv_ε. So let us introduce $\Theta(t) := (1+t^2)^{(2-\mu)/4}$, $t \ge 0$, and let $h_\varepsilon := \Theta(|Dv_\varepsilon|)$.

Proposition 3.2. *With this notation* $h_\varepsilon \in W^1_{2,loc}(B_{2R})$ *and*

$$Dh_\varepsilon \,=\, \Theta'\big(|Dv_\varepsilon|\big)\, D\big|Dv_\varepsilon\big|\, .$$

Remark 3.3. If we consider for instance the case $q \ge 2$, then the fact that h_ε is of class W^1_2 follows from Lemma 2.2, b.), (iii.). However, in Lemma 3.4 we need an explicit formula for the derivative.

Proof of Proposition 3.2. In order to reduce the problem to an application of the usual chain rule for Lipschitz functions, let $L \gg 1$ be some real number and let

$$\Theta_L(t) \,=\, \begin{cases} (1+t^2)^{\frac{2-\mu}{4}}\,, & 0 \le t \le L \\ (1+L^2)^{\frac{2-\mu}{4}}\,, & t \ge L \end{cases} \quad , \qquad h^L_\varepsilon \,:=\, \Theta_L(|v_\varepsilon|)\, .$$

As a consequence of Lemma 2.2, h^L_ε is immediately seen to be of class W^1_1 satisfying

$$Dh^L_\varepsilon \,=\, \Theta'_L\big(|Dv_\varepsilon|\big)\, D\big|Dv_\varepsilon\big|\, . \tag{3.7}$$

In addition, for $0 < r < 2R$ we have the estimate

$$\begin{aligned} \int_{B_r} \big|Dh^L_\varepsilon\big|^2\, dx &\le \int_{B_r\cap[|Dv_\varepsilon|\le L]} |\Theta'|^2\, \big|D^2 v_\varepsilon\big|^2\, dx \\ &\le c \int_{B_r} \big(1+|Dv_\varepsilon|^2\big)^{-\frac{\mu}{2}} \big|D^2 v_\varepsilon\big|^2\, dx\, , \end{aligned}$$

hence, by Lemma 3.1, $\|Dh^L_\varepsilon\|_{L^2(B_r,\mathbb{R}^n)}$ is uniformly bounded with respect to L and we may assume

$$Dh^L_\varepsilon \,\rightharpoondown\!:\, W_\varepsilon \quad \text{in } L^2(B_r,\mathbb{R}^n) \text{ as } L\to\infty\, .$$

On the other hand, the obvious convergence $h^L_\varepsilon \to h_\varepsilon$ in $L^2(B_r)$ as $L \to \infty$ implies $W_\varepsilon = Dh_\varepsilon$, thus $h_\varepsilon \in W^1_2(B_r)$. (3.7) also gives

$$Dh^L_\varepsilon \,\to\, \Theta'\big(|Dv_\varepsilon|\big)\, D\big|Dv_\varepsilon\big| \quad \text{almost everywhere}\, ,$$

hence we can identify the limit and the proposition is proved. □

As mentioned above, we now turn our attention to (uniform) higher integrability of Dv_ε. Let us remark, that with uniform growth estimates for f, but even without any control on the derivatives, integrability of the gradient can be slightly improved (compare, for instance, [CF]). In the situation at hand, Lemma 3.4 can be proved following the lines of [BFM], Lemma 2.4.

Lemma 3.4. *Assume again (1.5)–(1.8) and let* $\chi = \frac{n}{n-2}$ *if* $n > 2$, *in the case* $n = 2$ *let* $\chi > \frac{2s}{s+2-\mu-q}$. *Then there are real numbers* c, β, *independent of* ε *such that for all* $r < 2R$

$$\int_{B_r} \big(1 + |Dv_\varepsilon|^2\big)^{\frac{(2-\mu)\chi}{2}} \, dx \,\leq\, c(r,R) \left\{ \int_{B_{2R}} \big(1 + f_\varepsilon(Dv_\varepsilon)\big) \, dx \right\}^\beta .$$

In particular, by Lemma 2.1, $Dv_\varepsilon \in L^{(2-\mu)\chi}_{loc}(B_{2R}, \mathbb{R}^{nN}) \subset L^q_{loc}(B_{2R}, \mathbb{R}^{nN})$ *uniformly with respect to* ε, *i. e.*

$$Du \in L^{(2-\mu)\chi}_{loc}\big(B_{2R}, \mathbb{R}^{nN}\big) \subset L^q_{loc}\big(B_{2R}, \mathbb{R}^{nN}\big) .$$

Proof of Lemma 3.4. We consider the case $n \geq 3$, let $\alpha = \frac{(2-\mu)n}{2(n-2)}$ and assume without loss of generality $R < r < 3R/2$. Moreover, fix $0 < \rho < R/2$ and $\eta \in C^1_0(B_{r+\rho/2})$, $\eta \equiv 1$ on B_r, $D\eta \leq 4/\rho$. Since h_ε was proved in Proposition 3.2 to be of class W^1_2, we obtain using Sobolev's inequality

$$\begin{aligned}
\int_{B_r} \big(1 + |Dv_\varepsilon|^2\big)^\alpha \, dx &\leq \int_{B_{2R}} \Big(\eta \big[1 + |Dv_\varepsilon|^2\big]^{\alpha \frac{n-2}{2n}}\Big)^{\frac{2n}{n-2}} \, dx \\
&= \int_{B_{2R}} \big(\eta \, h_\varepsilon\big)^{\frac{2n}{n-2}} \, dx \\
&\leq c \left(\int_{B_{2R}} \big|D(\eta \, h_\varepsilon)\big|^2 \, dx \right)^{\frac{n}{n-2}} \,\leq\, c \big\{T_1 + T_2\big\}^{\frac{n}{n-2}} ,
\end{aligned}$$

where we have set

$$T_1 \,=\, \int_{B_{2R}} |D\eta|^2 \, h_\varepsilon^2 \, dx \, , \qquad T_2 \,=\, \int_{B_{2R}} \eta^2 \, |Dh_\varepsilon|^2 \, dx \, .$$

T_1 is directly seen to satisfy

$$T_1 \,\leq\, \frac{c}{\rho^2} \int_{B_{2R}} \big(1 + |Dv_\varepsilon|^2\big)^{\frac{2-\mu}{2}} \, dx \, ,$$

whereas T_2 has to be handled via the representation formula for the derivative of h_ε given in Proposition 3.2:

$$T_2 \,\leq\, c \int_{B_{r+\rho/2}} \big(1 + |Dv_\varepsilon|^2\big)^{-\frac{\mu}{2}} \, \big|D^2 v_\varepsilon\big|^2 \, dx \, .$$

With Lemma 3.1 (choosing $Q = 0$) and (1.7) we obtain

$$\int_{B_r} \big(1+|Dv_\varepsilon|^2\big)^\alpha \, dx \\ \leq \frac{c}{\rho^2} \left\{ \int_{B_{2R}} \big(1+|Dv_\varepsilon|^2\big)^{\frac{2-\mu}{2}} \, dx + \int_{B_{r+\rho} \sim B_r} \big(1+|Dv_\varepsilon|^2\big)^{\frac{q}{2}} \, dx \right\}^\chi ,$$

where the arguments used for the right–hand side are the same as in [BFM], i. e.: the interpolation procedure demonstrated in [ELM] (starting with the inequality given after (4.6) in [ELM]) is modified using (1.5):

$$\|Dv_\varepsilon\|_q \leq \|Dv_\varepsilon\|_s^\theta \, \|Dv_\varepsilon\|_{(2-\mu)\chi}^{1-\theta} .$$

This inequality holds with $\theta \in (0,1)$ defined according to $\frac{1}{q} = \frac{\theta}{s} + \frac{1-\theta}{(2-\mu)\chi}$. Note that the subsequent arguments of [ELM] require the bound $(1-\theta)q/(2-\mu) < 1$ which in case $n \geq 3$ is equivalent to (1.8). Now let $n = 2$ and define $\alpha = \chi(2-\mu)/2$. Then we have

$$\int_{B_r} \big(1+|Dv_\varepsilon|^2\big)^\alpha \, dx \leq \int_{B_{2R}} (\eta h_\varepsilon)^{2\chi} \, dx \leq c \left(\int_{B_{2R}} \big|D(\eta h_\varepsilon)\big|^t \, dx \right)^{\frac{2\chi}{t}} ,$$

where $t \in (1,2)$ is defined through $2\chi = 2t/(2-t)$. Using Hölder's inequality we get

$$\int_{B_r} \big(1+|Dv_\varepsilon|^2\big)^\alpha \, dx \leq c \left(\int_{B_{2R}} \big|D(\eta h_\varepsilon)\big|^2 \, dx \right)^\chi ,$$

and we can proceed as before with $n/(n-2)$ replaced by χ. Again we have to satisfy the requirement $(1-\theta)q/(2-\mu) < 1$ which for $n = 2$ is equivalent to $\chi > s/(s+2-\mu-q)$. But the latter inequality follows from our choice of χ, thus Lemma 3.4 is established also in case $n = 2$. □

Having established higher integrability of Du, the next proposition gives some preperations needed for the limit version of Caccioppoli's–type inequality.

Proposition 3.5. *Let* $h = (1+|Du|^2)^{(2-\mu)/4}$. *Then*

$(i.)$ $h \in W^1_{2,loc}(B_{2R})$,

$(ii.)$ $h_\varepsilon \rightharpoondown h$ *in* $W^1_{2,loc}(B_{2R})$ *as* $\varepsilon \to 0$,

$(iii.)$ $Dv_\varepsilon \to Du$ *almost everywhere on* B_{2R} *as* $\varepsilon \to 0$.

Proof of Proposition 3.5. We fix $0 < r < \hat{r} < 2R$, combine Lemma 3.1 and Proposition 3.2 to obtain

$$\|Dh_\varepsilon\|^2_{L^2(B_r,\mathbb{R}^n)} \leq c\big(1+\|Dv_\varepsilon\|^2_{L^q(B_{\hat{r}},\mathbb{R}^{nN})}\big) ,$$

hence, by Lemma 3.4, h_ε is uniformly bounded in $W^1_{2,loc}(B_{2R})$ and we may assume as $\varepsilon \to 0$

$$h_\varepsilon \rightharpoondown: \hat{h} \quad \text{weakly in } W^1_{2,loc}(B_{2R}) \text{ and almost everywhere.}$$

The proof of $\hat{h} = h$ together with the pointwise convergences exactly follows the lines of [FO], Lemma 4.1. □

Now we can formulate the limit version of Lemma 3.1.

Lemma 3.6. *There is a real number c such that for all $\eta \in C^1_0(B_{2R})$, $0 \le \eta \le 1$, and for all $Q \in \mathbb{R}^{nN}$*

$$\int_{B_{2R}} \eta^2 |Dh|^2 \, dx \le c \|D\eta\|^2_\infty \int_{B_{2R} \cap spt D\eta} \big|D^2 f(Du)\big| \, |Du - Q|^2 \, dx \, .$$

Proof of Lemma 3.6. Given Q, η as above, Proposition 3.5, lower semicontinuity, Lemma 3.1 and Proposition 3.2 together imply

$$\begin{aligned} \int_{B_{2R}} \eta^2 |Dh|^2 \, dx &\le \liminf_{\varepsilon \to 0} \int_{B_{2R}} \eta^2 |Dh_\varepsilon|^2 \, dx \\ &\le \liminf_{\varepsilon \to 0} c \|D\eta\|^2_\infty \int_{B_{2R} \cap spt D\eta} \big|D^2 f_\varepsilon(Dv_\varepsilon)\big| \, |Dv_\varepsilon - Q|^2 \, dx \\ &= \liminf_{\varepsilon \to 0} c \|D\eta\|^2_\infty \int_{B_{2R} \cap spt D\eta} \big|D^2 f(Dv_\varepsilon)\big| \, |Dv_\varepsilon - Q|^2 \, dx \, . \end{aligned} \tag{3.8}$$

Here, for the last equality, we made use of Lemma 2.1, (ii). Next, by the pointwise convergence almost everywhere stated in Proposition 3.5, (iii.), we have

$$\big|D^2 f(Dv_\varepsilon)\big| \, |Dv_\varepsilon - Q|^2 \to \big|D^2 f(Du)\big| \, |Du - Q|^2 \quad \text{a. e. as } \varepsilon \to 0 \, . \tag{3.9}$$

Finally, by Lemma 3.4 we know that $\big|D^2 f(Dv_\varepsilon)\big| \, |Dv_\varepsilon - Q|^2$ is uniformly bounded in $L^{1+\tau}_{loc}(B_{2R})$ for some $\tau > 0$, hence

$$\begin{aligned} \big|D^2 f(Dv_\varepsilon)\big| \, |Dv_\varepsilon - Q|^2 &\rightharpoondown: \vartheta \quad \text{in } L^{1+\tau}_{loc}(B_{2R}) \, , \\ \int_{B_{2R}} \big|D^2 f(Dv_\varepsilon)\big| \, |Dv_\varepsilon - Q|^2 \, dx &\to \int_{B_{2R}} \vartheta \, dx \end{aligned} \tag{3.10}$$

as $\varepsilon \to 0$. From (3.9), (3.10) we clearly get $\vartheta = |D^2 f(Du)| \, |Du - Q|^2$, which together with (3.8) gives the proof of Lemma 3.6. □

4 Blow–up

Now we fix a local minimizer u which by Lemma 3.4 is known to be of class $W^1_{q,loc}(\Omega,\mathbb{R}^N)$. The final step is to prove partial regularity of u via a blow–up procedure. As usual, the main tool is the decay estimate given in Lemma 4.1. The iteration of Lemma 4.1 leading to partial regularity is well known. Depending on the cases $q \geq 2$ and $q < 2$ an appropriate excess function has to be introduced: in the case $q \geq 2$ we let for balls $B_r(x) \Subset B_R \subset \Omega$

$$E^+(x,r) := \fint_{B_r(x)} |Du - (Du)_{x,r}|^2 \, dy + \fint_{B_r(x)} |Du - (Du)_{x,r}|^q \, dy \, ,$$

where $(g)_{x,r}$ denotes the mean value of the function g with respect to the ball $B_r(x)$. In the case $q < 2$ we define for all $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$,

$$V(\xi) := \big(1 + |\xi|^2\big)^{\frac{q-2}{4}} \xi \, .$$

The properties of V are studied for example in [CFM], in particular we refer to Lemma 2.1 of [CFM]. With these preliminaries we let for $q < 2$

$$E^-(x,r) := \fint_{B_r(x)} |V\big(Du(x)\big) - V\big((Du)_{x,r}\big)|^2 \, dy \, ,$$

a definition which makes sense since $q/2$ is the growth rate of V. In both cases we have

Lemma 4.1. *Fix $L > 0$. Then there exists a constant $C_*(L)$ such that for every $0 < \tau < 1/4$ there is an $\varepsilon = \varepsilon(L,\tau)$ satisfying: if $B_r(x) \Subset B_R$ and if we have*

$$\big|(Du)_{x,r}\big| \leq L \, , \quad E(x,r) \leq \varepsilon(L,\tau) \, ,$$

then

$$E(x,\tau r) \leq C_*(L) \tau^2 E(x,r) \, .$$

Here and in the following E denotes – depending on q – E^+ or E^- respectively.

Proof of Lemma 4.1. The proof is organized in four steps, always distinguishing the cases $q \geq 2$ and $q < 2$. If $q \geq 2$ then we mostly refer to [FO], the case $q < 2$ follows the lines of [CFM] and [EM2].

Step 1. (Blow–up and limit equation) To argue by contradiction, assume that $L > 0$ is fixed, the corresponding constant $C_*(L)$ will be chosen later on (see Step 4). If Lemma 4.1 is not true, then for some $0 < \tau < 1/4$, there are balls $B_{r_m}(x_m) \Subset B_R$ such that

(4.1) $\qquad \big|(Du)_{x_m,r_m}\big| \leq L \, , \quad E(x_m,r_m) =: \lambda_m^2 \overset{m\to\infty}{\longrightarrow} 0 \, ,$

(4.2) $\qquad E(x_m,\tau r_m) > C_* \tau^2 \lambda_m^2 \, .$

Now a sequence of rescaled functions is introduced by letting

$$a_m := (u)_{x_m,r_m}\,, \quad A_m := (Du)_{x_m,r_m}\,,$$
$$u_m(z) := \frac{1}{\lambda_m r_m}\big[u(x_m + r_m z) - a_m - r_m A_m z\big] \quad \text{if } |z| \le 1\,.$$

Passing to a subsequence, which is not relabeled, (4.1) implies

$$A_m \to: A \quad \text{in } \mathbb{R}^{nN}\,. \tag{4.3}$$

We also observe that

$$Du_m(z) = \lambda_m^{-1}\big[Du(x_m + r_m z) - A_m\big]\,, \quad (u_m)_{0,1} = 0\,, \quad (Du_m)_{0,1} = 0\,,$$

and concentrate for the moment on the case $q \ge 2$. Using (4.1) and (4.2) we have

$$\fint_{B_1} |Du_m|^2\,dz + \lambda_m^{q-2} \fint_{B_1} |Du_m|^q\,dz = \lambda_m^{-2} E^+(x_m, r_m) = 1\,, \tag{4.4}$$

$$\fint_{B_1} |Du_m - (Du_m)_{0,\tau}|^2\,dz + \lambda_m^{q-2} \fint_{B_1} |Du_m - (Du_m)_{0,\tau}|^q\,dz > C_* \tau^2\,. \tag{4.5}$$

With (4.4) we obtain as $m \to \infty$

$$u_m \rightharpoonup: \hat{u} \quad \text{in } W_2^1\big(B_1, \mathbb{R}^{nN}\big)\,, \tag{4.6}$$
$$\lambda_m Du_m \to 0 \quad \text{in } L^2\big(B_1, \mathbb{R}^{nN}\big) \text{ and almost everywhere}\,, \tag{4.7}$$
$$\lambda_m^{1-\frac{2}{q}} Du_m \rightharpoonup 0 \quad \text{in } L^q\big(B_1, \mathbb{R}^{nN}\big) \text{ if } q > 2\,. \tag{4.8}$$

Considering the case $q < 2$ we follow [CFM], Poposition 3.4, Step 1, to see

$$\fint_{B_1} \big|V\big(Du_m(z)\big)\big|^2\,dz \le c(L)\,, \tag{4.9}$$

hence the "$q/2$–growth" of V (compare [CFM], Lemma 2.1, (i)) implies the existence of a finite constant, independent of m, such that

$$\|Du_m\|_{L^q(B_1, \mathbb{R}^{nN})} \le c\,.$$

Thus, in the subquadratic situation (4.6)–(4.8) have to be replaced by

$$u_m \rightharpoonup: \hat{u} \quad \text{in } W_q^1\big(B_1, \mathbb{R}^{nN}\big)\,, \tag{4.10}$$
$$\lambda_m Du_m \to 0 \quad \text{in } L^q\big(B_1, \mathbb{R}^{nN}\big) \text{ and almost everywhere}\,. \tag{4.11}$$

In both cases the limit $\hat{u}$ satisfies a blow–up equation stated in

Proposition 4.2. *There is a constant C^*, only depending on L, such that for all $\varphi \in C^1_0(B_1,\mathbb{R}^N)$*

$$\int_{B_1} D^2 f(A)(D\hat{u}, D\varphi)\, dz = 0\,,$$

$$\fint_{B_\tau} \big|D\hat{u} - (D\hat{u})_\tau\big|^2\, dz \le C^*\,\tau^2\,. \tag{4.12}$$

Proof of Proposition 4.2. The proof of the limit equation for $q \ge 2$ is well known and can be taken from [EV], p. 236. The subquadratic case again is treated in [CFM], Step 2. Inequality (4.12) of Proposition 4.2 follows from the theory of linear elliptic systems (compare [Gi], Chapter 3) where the subquadratic case also involves Proposition 2.10 of [CFM]. □

Step 2. Proceeding in the proof of Lemma 4.1 we have to show the following proposition which will imply strong convergence in the third step.

Proposition 4.3. *Let $q \ge 2$ and $0 < \rho < 1$ or consider the case $q < 2$ together with $0 < \rho < 1/3$. Then*

$$\lim_{m\to\infty} \int_{B_\rho} \big(1 + |A_m + \lambda_m\, D\hat{u} + \lambda_m\, Dw_m|^2\big)^{-\frac{\mu}{2}} |Dw_m|^2\, dz \;=\; 0\,,$$

where we have set $w_m = u_m - \hat{u}$.

Remark 4.4. The restriction $\rho < 1/3$ in the case $q < 2$ is needed to apply the Sobolev–Poincaré type inequality, Theorem 2.4 of [CFM].

Proof of Proposition 4.3. Again $q \ge 2$ is the first case to consider, where the basic ideas are given for example in [EG1]. Here we argue exactly as in [FO], pp. 410, i. e. we use the minimality of u together with the convexity of f, and conclude for all $\varphi \in C^1_0\big(B_1,\mathbb{R}^N\big)$, $\varphi \ge 0$,

$$\begin{aligned}
&\int_{B_1}\int_0^1 \varphi\, D^2 f\big(A_m + \lambda_m\, D\hat{u} + s\,\lambda_m\, Dw_m\big)(Dw_m, Dw_m)\,(1-s)\, ds\, dz\\
&\quad = \lambda_m^{-2} \int_{B_1} \varphi\,\big\{f(A_m + \lambda_m\, Du_m) - f(A_m + \lambda_m\, D\hat{u})\big\}\, dz\\
&\qquad - \lambda_m^{-1} \int_{B_1} \varphi\, Df(A_m + \lambda_m\, D\hat{u}) : Dw_m\, dz\\
&\quad \le c\left\{\int_{B_1} |D\varphi|^2\, |w_m|^2\, dz + \lambda_m^{q-2} \int_{B_1} |D\varphi|^q\, |w_m|^q\, dz\right\}\\
&\qquad + \lambda_m^{-1} \int_{B_1} Df\Big(A_m + \lambda_m\,\big((1-\varphi)\, Du_m + \varphi\, D\hat{u}\big)\Big)\!:\big(D\varphi \otimes (\hat{u} - u_m)\big)\, dz\\
&\qquad - \lambda_m^{-1} \int_{B_1} \varphi\, Df(A_m + \lambda_m\, D\hat{u}) : Dw_m\, dz\,.
\end{aligned} \tag{4.13}$$

Clearly, (4.13) is the analogue to inequality (6.6) in [FO]. As demonstrated in [FO] we can discuss the last two integrals on the right–hand side of (4.13) which finally bounds the left–hand side of (4.13) by the quantity $c\left\{I_1 + I_2 + I_3\right\}$ where we have

$$I_1 := \int_{B_1} |D\varphi|^2 \, |w_m|^2 \, dz + \lambda_m^{q-2} \int_{B_1} |D\varphi|^q \, |w_m|^q \, dz \overset{m\to\infty}{\longrightarrow} 0 \, .$$

The limit behaviour follows from the weak convergence of u_m in $W_2^1\big(B_1, \mathbb{R}^{nN}\big)$ and from

$$\lambda_m^{1-\frac{2}{q}} \, w_m \to 0 \;\text{ in }\; L^q\big(B_1, \mathbb{R}^N\big) \quad \text{as } \; m \to \infty \, . \tag{4.14}$$

In fact, the latter convergence is obtained by (4.8) and by Poincaré's inequality which together with $(w_m)_{0,1} = 0$ implies (4.14). Further we have

$$\begin{aligned} I_2 &:= \int_{B_1} |Dw_m| \, |D\varphi| \, |w_m| \, dz + \lambda_m^{q-2} \int_{B_1} |Dw_m|^{q-1} \, |D\varphi| \, |w_m| \, dz \\ &\le \int_{B_1} |Dw_m| \, |D\varphi| \, |w_m| \, dz + c(D\varphi) \left(\int_{B_1} \lambda_m^{q-2} \, |Dw_m|^q \, dz \right)^{\frac{q-1}{q}} \\ &\times \left(\int_{B_1} \lambda_m^{q-2} \, |w_m|^q \, dz \right)^{\frac{1}{q}} \end{aligned}$$

and again we use (4.14) to see $I_2 \to 0$ as $m \to \infty$. The third part

$$I_3 := \Big| \int_{B_1} \int_0^1 D^2 f\big(A_m + s \, \lambda_m \, D\hat{u}\big) \big(D\hat{u}, D(\varphi w_m)\big) \, ds \, dz \Big|$$

is immediately seen to vanish as $m \to \infty$ and the proposition is proved if $q \ge 2$.

For $q < 2$ we now benefit from [EM2] (compare [EV]) since the proof of higher integrability given in [CFM], Step 3, is adapted to balanced structure conditions. Thus, let for $\xi \in \mathbb{R}^{nN}$

$$f_m(\xi) := \frac{f(A_m + \lambda_m \, \xi) - f(A_m) - \lambda_m \, Df(A_m) : \xi}{\lambda_m^2}$$

and define for $0 < \rho < 1/3$, $w \in W_{1,loc}^1(B_{1/3}, \mathbb{R}^N)$

$$I_\rho^m(w) := \int_{B_\rho} f_m(Dw) \, dz \, .$$

The first claim to prove is

$$\limsup_{m\to\infty} \Big\{ I_\rho^m(u_m) - I_\rho^m(\hat{u}) \Big\} \le 0 \qquad \text{for almost every } \; \rho \in (0, 1/3) \, . \tag{4.15}$$

To verify (4.15) we fix ρ as above, choose $0 < s < \rho$, $\eta \in C_0^\infty(B_\rho)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on B_s, $|\nabla \eta| \le c/(\rho - s)$ and define $\varphi_m = (\hat{u} - u_m)\,\eta$. Now, u_m obviously is a local minimizer of I_ρ^m and together with Lemma 3.3 of [CFM] this yields

$$
\begin{aligned}
I_\rho^m(u_m) - I_\rho^m(\hat{u}) &\le I_\rho^m(u_m+\varphi_m) - I_\rho^m(\hat{u})\\
&= \int_{B_\rho \sim B_s} \Big[f_m(Du_m + D\varphi_m) - f_m(D\hat{u})\Big]\, dz\\
&\le \frac{c(q,\Lambda,L)}{\lambda_m^2} \int_{B_\rho\sim B_s} \Big[\big|V(\lambda_m D\hat{u})\big|^2\\
&\quad + \big|V\big(\lambda_m(\hat{u}-u_m)\otimes D\eta + \lambda_m \eta D\hat{u}\\
&\quad + \lambda_m (1-\eta) Du_m\big)\big|^2\big)\Big]\, dz\\
&\le \frac{c}{\lambda_m^2} \int_{B_\rho\sim B_s} \Big[\big|V(\lambda_m D\hat{u})\big|^2 + \big|V(\lambda_m Du_m)\big|^2\\
&\quad + (\rho - s)^{-2} \big|V\big(\lambda_m(\hat{u}-u_m)\big)\big|^2\Big]\, dz\,.
\end{aligned}
$$

Next, a family of positive, uniformly bounded Radon measures μ^m on $B_{1/3}$ is introduced by letting

$$
\mu^m(S) := \int_S \frac{1}{\lambda_m^2} \Big[\big|V(\lambda_m D\hat{u})\big|^2 + \big|V(\lambda_m Du_m)\big|^2\Big]\, dz\,.
$$

We may assume that μ^m converges in measure to a Radon measure μ on $B_{1/3}$. Exactly as in [EM2], the Sobolev-Poincaré type inequality proved in [CFM], Theorem 2.4, gives for some $1 < \theta < 2$

$$
I_\rho^m(u_m) - I_\rho^m(\hat{u}) \le c \left[\mu^m(B_\rho \sim B_s) + (\rho-s)^{-2} \left(\int_{B_1} |u_m - \hat{u}|\, dz \right)^{2\theta}\right],
$$

hence, by taking first the limit $m \to \infty$ and then the limit $s \uparrow \rho$, we get (4.15) for any $0 < \rho < 1/3$ such that $\mu(\partial B_\rho) = 0$ which is true for a. a. ρ.
Once (4.15) is established for some $0 < \rho < 1/3$, the following identity is the starting point to derive an estimate for the left–hand side:

$$
\begin{aligned}
I_\rho^m(u_m) - I_\rho^m(\hat{u}) &= \lambda_m^{-2} \int_{B_\rho} \Big[f(A_m + \lambda_m Du_m) - f(A_m + \lambda_m D\hat{u})\\
&\quad - \lambda_m Df(A_m) : Dw_m \Big]\, dz\\
&= \lambda_m^{-1} \int_{B_\rho} \int_0^1 \Big[Df(A_m + \lambda_m D\hat{u} + t\lambda_m Dw_m)\\
&\quad - Df(A_m + \lambda_m D\hat{u})\Big] : Dw_m\, dt\, dz\\
&\quad + \lambda_m^{-1} \int_{B_\rho} \Big[Df(A_m + \lambda_m D\hat{u}) - Df(A_m)\Big] : Dw_m\, dz\\
&=: (I)_m + (II)_m\,.
\end{aligned}
$$

Local smoothness of $\hat{u}$ immediately implies $\lim_{m\to\infty}(II)_m = 0$. On account of

$$(I)_m = \int_{B_\rho}\int_0^1\int_0^1 t\, D^2 f(A_m + \lambda_m\, D\hat{u} + s\, t\, \lambda_m\, Dw_m)(Dw_m, Dw_m)\, ds\, dt\, dz$$
$$\geq c \int_{B_\rho} \big(1 + |A_m + \lambda_m\, D\hat{u} + \lambda_m\, Dw_m|^2\big)^{-\frac{\mu}{2}} |Dw_m|^2\, dz$$

and by (4.15) the proposition is proved for almost all, hence for any $\rho \in (0, 1/3)$.□

Step 3a. (Strong convergence for $q \geq 2$)

Proposition 4.5. *In the case $q \geq 2$ we have as $m \to \infty$*

$$(4.16)\qquad \begin{array}{ll} (i.) & Du_m \to D\hat{u} \ \textit{in}\ L^2_{loc}\big(B_1, \mathbb{R}^{nN}\big)\,; \\ (ii.) & \lambda_m^{1-\frac{2}{q}}\, Du_m \to 0 \quad \textit{in}\ L^q_{loc}\big(B_1, \mathbb{R}^{nN}\big) \quad \textit{if}\ q > 2\,. \end{array}$$

Proof of Proposition 4.5. Here we have to distinguish two subcases: For $\mu \leq 0$ the first convergence follows directly from Proposition 4.3. Using this fact, local smoothness of $\hat{u}$ and again Proposition 4.3, the next conclusion is

$$(4.17)\qquad \int_{B_\rho} \lambda_m^{-\mu}\, |Dw_m|^{2-\mu}\, dz \overset{m\to\infty}{\longrightarrow} 0 \quad \text{for all}\ \ 0 < \rho < 1\,.$$

The proceed further, we introduce the auxiliary functions ψ_m (see [FO]),

$$(4.18)\quad \psi_m := \lambda_m^{-1}\Big[\big(1 + |A_m + \lambda_m\, Du_m|^2\big)^{\frac{2-\mu}{4}} - \big(1 + |A_m|^2\big)^{\frac{2-\mu}{4}}\Big]\,,$$

and by Lemma 3.6, (4.6), (4.8), (1.7) we can estimate ($0 < \rho < 1$)

$$\int_{B_\rho} |D\psi_m|^2\, dz \,\leq\, c(\rho) \int_{B_1} \big|D^2 f(A_m + \lambda_m\, Du_m)\big|\, |Du_m|^2\, dz \,\leq\, c(\rho)\,.$$

If we now let $\Theta(Z) := (1 + |Z|^2)^{(2-\mu)/4}$, $Z \in \mathbb{R}^{nN}$, then

$$|\psi_m| = \lambda_m^{-1}\Big|\int_0^1 \frac{d}{dt}\,\Theta(A_m + t\,\lambda_m\, Du_m)\, dt\Big|$$
$$\leq c\Big|\int_0^1 Du_m : D\Theta(A_m + t\,\lambda_m\, Du_m)\, dt\Big|$$
$$\leq c \int_0^1 |Du_m|\, \big(1 + |A_m + t\,\lambda_m\, Du_m|^2\big)^{-\frac{\mu}{4}}\, dt$$
$$\leq c\Big(|Du_m| + \lambda_m^{-\frac{\mu}{2}}\, |Du_m|^{1-\frac{\mu}{2}}\Big)\,.$$

With this inequality we obtain

$$(4.19)\qquad \int_{B_\rho} |\psi_m|^2\, dz \,\leq\, c(\rho) \quad \text{for all}\ \ 0 < \rho < 1\,.$$

In fact, (4.19) is obvious for $\mu = 0$. If $\mu < 0$, then (4.19) is just a consequence of (4.17). Thus, we have proved that

$$\sup_m \|\psi_m\|_{W_2^1(B_\rho)} \le c(\rho) < \infty \quad \text{for all } 0 < \rho < 1 \tag{4.20}$$

and this will imply (4.16), (ii.): to this purpose we fix some real number $M \gg 1$ and let $U_m = U_m(M,\rho) := \{z \in B_\rho : \lambda_m |Du_m| \le M\}$. On one hand, local L^2–convergence and $q > 2$ prove

$$\begin{aligned}\int_{U_m} \lambda_m^{q-2} |Du_m|^q \, dz &\le \int_{U_m} \lambda_m^{q-2} |Dw_m|^q \, dz + \int_{U_m} \lambda_m^{q-2} |D\hat{u}|^q \, dz \\ &\le c \int_{U_m} \lambda_m^{q-2} \Big(|Du_m|^{q-2} \\ &\quad + |D\hat{u}|^{q-2} \Big) |Dw_m|^2 + \int_{U_m} \lambda_m^{q-2} |D\hat{u}|^q \, dz \\ &\to 0 \quad \text{as } m \to \infty \, . \end{aligned} \tag{4.21}$$

On the other hand, observe that for M sufficiently large and for $z \in B_\rho \sim U_m$

$$\begin{aligned} \psi_m(z) &\ge c \lambda_m^{-1} \lambda_m^{\frac{2-\mu}{2}} |Du_m(z)|^{\frac{2-\mu}{2}} \, , \quad \text{i. e.} \\ \lambda_m^{q-2+\frac{\mu q}{2-\mu}} \psi_m^{\frac{2q}{2-\mu}}(z) &\ge c \lambda_m^{q-2} |Du_m(z)|^q \, . \end{aligned}$$

Since (1.8) guarantees $2q/(2-\mu) < 2n/(n-2)$, since by (4.20) ψ_m is uniformly bounded in $L^{2n/(n-2)}$ and since $q - 2 + \mu q/(2-\mu) \ge 0$ follows from $q \ge 2 - \mu$, we can conclude

$$\int_{B_\rho \sim U_m} \lambda_m^{q-2} |Du_m|^q \, dz \to 0 \quad \text{for all } 0 < \rho < 1 \tag{4.22}$$

as $m \to \infty$. Summarizing the results, (4.21) and (4.22) prove Propostion 4.5 in the case $\mu \le 0$.
Now suppose that $\mu > 0$. Proposition 4.3 implies in the case at hand for any $0 < \rho < 1$

$$\int_{B_\rho} \big(1 + |\lambda_m Dw_m|^2\big)^{-\frac{\mu}{2}} |Dw_m|^2 \, dz \to 0 \quad \text{as } m \to \infty$$

which immediately gives

$$\int_{U_m} |Dw_m|^2 \, dz \to 0 \quad \text{as } m \to \infty \, . \tag{4.23}$$

Here U_m is defined as above for fixed M and ρ. Also as above we introduce ψ_m and observe that now $|\psi_m| \le c|Du_m|$ is obvious, i. e. (4.20) remains to be true in the case $\mu > 0$. If M is chosen sufficiently large, then

$$|\psi_m|^{\frac{4}{2-\mu}} \lambda_m^{\frac{2\mu}{2-\mu}} \ge |Du_m|^2 \quad \text{on } B_\rho \sim U_m \, ,$$

and since $4/(2-\mu) \le 2n/(n-2) \Leftrightarrow \mu \le 4/n$, the last inequality being true on account of $q \ge 2$, we get

$$\int_{B_\rho \sim U_m} |Dw_m|^2 \, dz \overset{m\to\infty}{\longrightarrow} 0 \quad \text{for all } 0 < \rho < 1. \tag{4.24}$$

With (4.23) and (4.24) the first claim of (4.16) also is proved in the case $\mu > 0$. (4.16), (ii.), for $\mu > 0$ follows exactly as for the case $\mu \le 0$ and the proof of the proposition is complete. □

Step 3b. (Strong convergence for $q < 2$)

Proposition 4.6. *If $q < 2$, then for any $0 < \rho < 1/3$*

$$\lim_{m\to\infty} \frac{1}{\lambda_m^2} \int_{B_\rho} \big|V(\lambda_m Dw_m)\big|^2 \, dz = 0 \, .$$

Proof of Proposition 4.6. In the subquadratic case, the auxiliary function ψ_m introduced in (4.18) is handled via Lemma 2.1, (vi.) of [CFM]. We have

$$\begin{aligned}
\int_{B_\rho} |D\psi_m|^2 \, dz &\le c \int_{B_1} \big(1 + |\lambda_m Du_m|^2\big)^{\frac{q-2}{2}} |Du_m|^2 \, dz \\
&\le \frac{c}{\lambda_m^2} \fint_{B_1} \big|V(\lambda_m Du_m)\big|^2 \, dz \\
&\le \frac{c}{\lambda_m^2} \fint_{B(x_m, R_m)} \big|V(Du - A_m)\big|^2 \, dx \\
&\le \frac{c(L)}{\lambda_m^2} \fint_{B(x_m, R_m)} \big|V(Du) - V(A_m)\big|^2 \, dx \ \le const.
\end{aligned}$$

for any $0 < \rho < 1$. In addition we have $|\psi_m| \le c|Du_m|$, hence $\psi_m \in W^1_{q,loc}(B_1)$, thus $\psi_m \in L^{q_1}_{loc}(B_1)$ with $q_1 := nq/(n-q)$. Iterating this argument we again have verified

$$\sup_m \|\psi_m\|_{W^1_2(B_\rho)} \le c(\rho) < \infty \quad \text{for all } 0 < \rho < 1 \, . \tag{4.25}$$

Assume now that $0 < \rho < 1/3$. With M and U_m as before, (4.23) is again a consequence of Proposition 4.3. Let us write ([CFM], Lemma 2.1)

$$\begin{aligned}
\frac{1}{\lambda_m^2} \int_{B_\rho} \big|V(\lambda_m Dw_m)\big|^2 \, dz \le \frac{c}{\lambda_m^2} &\int_{U_m} \big|V(\lambda_m Dw_m)\big|^2 \, dz \\
&+ \frac{c}{\lambda_m^2} \int_{B_\rho \sim U_m} \big|V(\lambda_m Du_m)\big|^2 \, dz \\
&+ \frac{c}{\lambda_m^2} \int_{B_\rho \sim U_m} \big|V(\lambda_m D\hat{u})\big|^2 \, dz \, .
\end{aligned}$$

Then, by (4.23)

$$\frac{1}{\lambda_m^2}\int_{U_m}\big|V(\lambda_m\,Dw_m)\big|^2\,dz \le \int_{U_m}|Dw_m|^2\big(1+\lambda_m^2\,|Dw_m|^2\big)^{\frac{q-2}{2}}\,dz$$
$$\le \int_{U_m}|Dw_m|^2\,dz \overset{m\to\infty}{\longrightarrow} 0\,.$$

The second term vanishes as $m\to\infty$ provided that

$$\int_{B_\rho\sim U_m}\lambda_m^{q-2}\,|Du_m|^q\,dz \to 0 \quad \text{as } m\to\infty\,.$$

To see this we recall the estimates for ψ_m stated after (4.21) being valid also in the case under consideration and with the same reasoning we obtain (4.22) where now we make use of the apriori bound (4.25). Finally, we use the local boundedness of $D\hat{u}$ to see

$$\frac{1}{\lambda_m^2}\int_{B_\rho\sim U_m}\big|V(\lambda_m\,D\hat{u})\big|^2\,dz \le \int_{B_\rho\sim U_m}|D\hat{u}|^2\,dz$$
$$\le \|D\hat{u}\|^2_{L^\infty(B_\rho,\mathbb{R}^{nN})}\,|B_\rho\sim U_m| \overset{m\to\infty}{\longrightarrow} 0$$

on account of $\lambda_m\,Du_m\to 0$ a. e. on B_1 as $m\to\infty$ (see (4.11)). This completes the proof of Proposition 4.6. □

Step 4. (Conclusion) Proposition 4.5 together with (4.5) gives in the case $q\ge 2$

$$⨍_{B_\tau}|D\hat{u}-(D\hat{u})_\tau|^2\,dz \ge C_*\,\tau^2\,,$$

thus we have a contradiction to (4.12) if we choose $C_*=2\,C^*$.

If $q<2$, then we estimate according to [CFM], p. 24,

$$\lim_{m\to\infty}\frac{E^-(x_m,\tau\,R_m)}{\lambda_m^2} \le \lim_{m\to\infty}\frac{c}{\lambda_m^2}⨍_{B_\tau}\Big\{\big|V(\lambda_m\,Dw_m)\big|^2$$
$$+\big|V\big(\lambda_m(D\hat{u}-(D\hat{u})_\tau)\big)\big|^2$$
$$+\big|V\big(\lambda_m\big((D\hat{u})_\tau-(Du_m)_\tau\big)\big)\big|^2\Big\}\,dz\,,$$

where the first integral is handled by Proposition 4.6. The last one vanishes when passing to the limit $m\to\infty$ since we may first estimate

$$⨍_{B_\tau}\big|V\big(\lambda_m\big((D\hat{u})_\tau-(Du_m)_\tau\big)\big)\big|^2\Big\}\,dz \le \lambda_m^2⨍_{B_\tau}\big|(D\hat{u})_\tau-(Du_m)_\tau\big|^2\,dz$$

and then use (4.10) for the right–hand side. The second integral again is estimated by (4.12). Thus, choosing C_* sufficiently large we also get the contradiction in the case $q<2$ and the lemma is proved. □

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